Math 577 HW2 Spring 2018

Solutions

#1) Let $w_1, w_2 \in f(U)$. $w_1 = \ell v_1 + b$, $w_2 = \ell v_2 + b$
$v_1 \in U$, $v_2 \in U$

For $0 \le \alpha \le 1$, $w = \alpha w_1 + (1-\alpha) w_2 = \alpha(\ell v_1 + b) + (1-\alpha)(\ell v_2 + b)$

ℓ linear $\Rightarrow$ $= \ell(\alpha v_1 + (1-\alpha) v_2) + \underbrace{\alpha b + (1-\alpha) b}_{b}$.

Since U is convex, $\alpha v_1 + (1-\alpha) v_2 \in U$, hence $w \in f(U)$. ✓

#2) Assume f is convex. Let $(u_1, \alpha_1), (u_2, \alpha_2) \in epi(f)$.

$\Rightarrow \alpha_1 \ge f(u_1)$, $\alpha_2 \ge f(u_2)$. Let $t \in [0,1]$.

$t(u_1, \alpha_1) + (1-t)(u_2, \alpha_2) = (t u_1 + (1-t) u_2, t\alpha_1 + (1-t)\alpha_2)$

U convex $\Rightarrow t u_1 + (1-t) u_2 \in U$.

f convex $\Rightarrow f(t u_1 + (1-t) u_2) \le t f(u_1) + (1-t) f(u_2)$
$\le t\alpha_1 + (1-t)\alpha_2$

Thus $(t u_1 + (1-t) u_2, t\alpha_1 + (1-t)\alpha_2) \in epi(f)$. ✓

Assume $epi(f)$ is convex. Let $u_1, u_2 \in U$, $t \in [0,1]$.

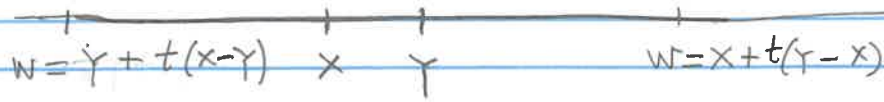
Clearly $(u_1, f(u_1)), (u_2, f(u_2)) \in epi(f)$.

$epi(f)$ convex $\Rightarrow t(u_1, f(u_1)) + (1-t)(u_2, f(u_2)) \in epi(f)$

i.e. $(t u_1 + (1-t) u_2, t f(u_1) + (1-t) f(u_2)) \in epi(f)$

$\Rightarrow t f(u_1) + (1-t) f(u_2) \ge f(t u_1 + (1-t) f(u_2))$. ✓

#3)



Let $x, y \in \mathbb{R}^n$. Let $t \geq 1$ and $w = x + t(y-x)$

$$\Rightarrow \quad y = \frac{1}{t} w + \frac{t-1}{t} x \,. \quad \frac{1}{t}, \frac{t-1}{t} \in [0,1].$$

f convex $\Rightarrow$ $f(y) \leq \frac{1}{t} f(w) + \frac{t-1}{t} f(x)$.

Now let $t \to \infty$. Since $f(w)$ is bounded from above

$\frac{1}{t} f(w) \to 0$, $\frac{t-1}{t} f(x) \to f(x)$. Hence $\boxed{f(y) \leq f(x)}$

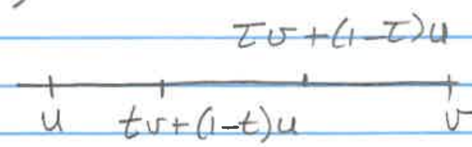
Now reverse the roles of x and y: $w = y + t(x-y)$.

Arguing as above we get $\boxed{f(x) \leq f(y)}$. ✓

#4) Let $0 < t < \tau < 1$. The key idea is to write $tv + (1-t)u$ as a convex combination of u and $\tau v + (1-\tau)u$.

$$tv + (1-t)u = \alpha u + (1-\alpha)\big(\tau v + (1-\tau)u\big)$$

Solving for α, we get



$\alpha = \frac{\tau - t}{\tau}$, $1 - \alpha = \frac{t}{\tau} \in (0,1]$. So write

$$tv + (1-t)u = \frac{t}{\tau}\big(\tau v + (1-\tau)u\big) + \frac{\tau - t}{\tau} u \,.$$

$$\Rightarrow \quad g(t) = \frac{f(tv + (1-t)u) - f(u)}{t} = \frac{f\left(\frac{t}{\tau}(\tau v + (1-\tau)u) + \frac{\tau - t}{\tau} u\right) - f(u)}{t}$$

f convex,

$$\leq \frac{\frac{t}{\tau} f(\tau v + (1-\tau)u) + \frac{\tau - t}{\tau} f(u) - f(u)}{t}$$

$$= \frac{f(\tau v + (1-\tau)u) - f(u)}{\tau} = g(\tau) \quad ✓$$

#5] Let $x, y \in U$ and $\alpha \in [0,1]$

$$f(\alpha x + (1-\alpha)y) = \max\{f_1(\alpha x + (1-\alpha)y), f_2(\alpha x + (1-\alpha)y)\}$$

$$\leq \max\{\alpha f_1(x) + (1-\alpha) f_1(y), \alpha f_2(x) + (1-\alpha) f_2(y)\}$$

$\max\{A+B, C+D\} \leq \max\{A, C\} + \max\{B, D\}$

$$\leq \max\{\alpha f_1(x), \alpha f_2(x)\} + \max\{(1-\alpha) f_1(y), (1-\alpha) f_2(y)\}$$

$\alpha, 1-\alpha \in [0,1] \Rightarrow$ $= \alpha \max\{f_1(x), f_2(x)\} + (1-\alpha)\max\{f_1(y), f_2(y)\}$

$$= \alpha f(x) + (1-\alpha) f(y)$$ ✓

#6] we have $f'(u)(v-u) \geq 0 \quad \forall v \in U$

(i) $0 \in C \Rightarrow f'(u)(-u) \geq 0$

$u \in U \Rightarrow 2u \in U \Rightarrow f'(u)(\underbrace{2u-u}_{u}) \geq 0$ $\Big\} \Rightarrow f'(u)u = 0$

$\Rightarrow f'(u)(v-u) = f'(u)v \geq 0.$

(ii) Let $w \in S$. Then $u + tw = b + z + tw$, $z \in S$

$\Rightarrow u + tw \in C \quad \forall t \in \mathbb{R}.$

Let $t = 1 \Rightarrow 0 \leq f'(u)(u + w - u) = f'(u)w$ ($f'(u)$ linear)

$t = -1 \Rightarrow 0 \leq f'(u)(u - w - u) = -f'(u)w$

$\Rightarrow f'(u)w = 0$ ✓

#7] Assume w.l.o.g. That $\beta \equiv \sum_{i=1}^{n-1} \alpha_i \neq 0$ or 1; otherwise

There is nothing to prove. Then write (note $0 < \beta < 1$)

$$\sum_{i=1}^{n} \alpha_i u_i = \beta \sum_{i=1}^{n-1} \left(\frac{\alpha_i}{\beta}\right) u_i + \alpha_n u_n.$$

Since $\sum_{i=1}^{n-1} \frac{\alpha_i}{\beta} = 1$, by the induction hypothesis

$$z = \sum_{i=1}^{n-1} \frac{\alpha_i}{\beta} u_i \in U. \text{ Also } \beta + \alpha_n = 1, \quad 0 < \alpha_n, \beta < 1$$

$$\Rightarrow \quad \beta z + \alpha_n u_n \in U. \quad \checkmark$$

Also

$$f\left(\sum_{i=1}^{n} \alpha_i u_i\right) = f\left(\beta \sum_{i=1}^{n-1} \frac{\alpha_i}{\beta} u_i + \alpha_n u_n\right)$$

$\beta + \alpha_n = 1$

$$\leq \beta f\left(\sum_{i=1}^{n} \frac{\alpha_i}{\beta} u_i\right) + \alpha_n u_n \quad f \text{ convex}$$

$$\leq \beta \sum_{i=1}^{n-1} \frac{\alpha_i}{\beta} f(u_i) + \alpha_n u_n \quad \text{induction hyp.}$$

$$= \sum_{i=1}^{n} \alpha_i u_i. \quad \checkmark$$

#8)

$$\ln\left(a^\theta b^{1-\theta}\right) = \theta \ln a + (1-\theta) \ln b$$

$$\leq \ln(\theta a + (1-\theta) b) \quad \ln \text{ is } \underline{\text{concave}}$$

$$\Rightarrow \quad a^\theta b^{1-\theta} \leq \theta a + (1-\theta) b \quad \ln \text{ is increasing.}$$

#9) Let $v, w \in H$, $0 \leq \alpha \leq 1$. $\prod_{i=1}^{n} v_i \geq 1$, $\prod_{i=1}^{n} w_i \geq 1$

$$\prod_{i=1}^{n} (\alpha v_i + (1-\alpha) w_i) \geq \prod_{i=1}^{n} \left(v_i^\alpha w_i^{1-\alpha}\right) \quad v_i, w_i > 0 \text{ a.g.m.i}$$

$$= \underbrace{\left(\prod_{i=1}^{n} v_i\right)^\alpha}_{\geq 1} \underbrace{\left(\prod_{i=1}^{n} w_i\right)^{1-\alpha}}_{\geq 1} \geq 1. \quad \checkmark$$

#10)

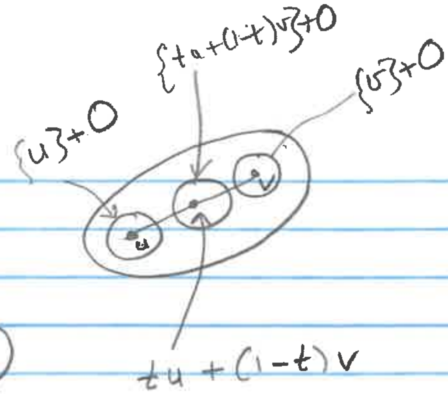
Let $v, w \in \overline{U}$ (closure).

$v_n \in U$, $v_n \to v$, $w_n \in U$, $w_n \to w$. $0 \leq \alpha \leq 1$

$$\alpha v + (1-\alpha) w = \alpha (v - v_n) + (1-\alpha)(w - w_n) + \underbrace{\alpha v_n + (1-\alpha) w_n}_{z_n}$$

$v - v_n \to 0$, $w - w_n \to 0 \Rightarrow z_n \to \alpha v + (1-\alpha) w$

since $z_n \in U$, $\alpha v + (1-\alpha) w$ belongs to $\overline{U}$. $\checkmark$



Let $u, v \in \text{int}(U)$. We can find an open ball O centered at the origin with radius $r > 0$ such that the "translate" open ball $\{u\}+O$ is contained in U and the open ball $\{v\}+O$ is contained in U.

Now for $0 \leq t \leq 1$, the translated open ball $\{tu+(1-t)v\}+O$ is centered at $tu+(1-t)v$. We need to show that this ball is contained in U. This is true since every point in it is of the form $tx+(1-t)y$ for some $x \in \{u\}+O$ and $y \in \{v\}+O$. ✓

#11)

Let $x, y \in \bigcap_{\alpha \in A} U_\alpha$. Then $x, y \in U_\alpha \quad \forall \alpha \in A$.

$0 \leq t \leq 1$.

$\Rightarrow \quad tx + (1-t)y \in U_\alpha \quad \forall \alpha \in A$ since U_α is convex

$\Rightarrow \quad tx + (1-t)y \in \bigcap_{\alpha \in A} U_\alpha$.

#12)

Let U be a closed, convex set.

Clearly $U \subseteq \bigcap_{a \in A} H_a$, H_a closed half-space containing U.

To show equality, let $w \in \bigcap_{a \in A} H_a$ and assume $w \notin U$.

Then by a strict separation result, $\exists\ \alpha \in R$ and $z \in V$ such that

$$(w, z) > \alpha > (v, z) \quad \forall\ v \in U. \quad (*)$$

Consider the closed hyperplane $H = \{v \in V, (z, v) \leq \alpha\}$.

By $(*)$, H contains U so $H = H_a$ for some $a \in A$.

Thus $w \in H$ since $w \in \bigcap_{a \in A} H_a$; but this contradicts the fact that $w \notin H$ since by $(*)$ $(w, z) > \alpha$. ✓

f is ~~coercive~~ strictly convex and U is convex $\Rightarrow \exists !$ minimizer.

#13) since $\mathrm{rank}(B) = m < n$, i.e. the rows of B are linearly independent, we can use the L-M. Theorem to get

$$\underbrace{Au - b}_{\nabla f(u)} + B^T \Lambda = 0 \qquad \Lambda \in \mathbb{R}^m$$

combining this with $Bu = 0$, we get

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ \Lambda \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}.$$

Multiply from the left by the (block) elimination matrix

$$\begin{bmatrix} I & 0 \\ -BA^{-1} & I \end{bmatrix} \Rightarrow \begin{bmatrix} A & B^T \\ 0 & -BA^{-1}B^T \end{bmatrix} \begin{bmatrix} u \\ \Lambda \end{bmatrix} = \begin{bmatrix} b \\ -BA^{-1}b \end{bmatrix}.$$

Since $\mathrm{rank}(B) = m$, and A^{-1} is also spd. it is easily show that the $m \times m$ matrix $BA^{-1}B^T$ is s.p.d.. Hence the system has a unique solution:

$$\boxed{\Lambda = (BA^{-1}B^T)^{-1} BA^{-1} b}$$

and

$$\boxed{u = A^{-1}(b - B^T \Lambda)}$$ use the above for Λ

#14) $\mathbb{R}^m = \mathrm{Null}(A^T) \oplus \mathrm{Range}(A)$, A sym. $\Rightarrow A^T = A$

$\{x \in \mathbb{R}^n, Ax = b\} = \emptyset \Rightarrow b \notin \mathrm{Range}(A)$

$\Rightarrow b = b_1 + b_2$, $b_1 \in \mathrm{Null}(A)$, $b_2 \in \mathrm{Range}(A)$, also $b_1 \neq 0$

let $x = \alpha b_1$, $\alpha \in \mathbb{R} \Rightarrow f(x) = \frac{\alpha^2}{2} b_1^T \cancel{A b_1}^{0} - (b_1^T + b_2^T)\alpha b_1$

$= -\alpha \|b_1\|_2^2 \to -\infty$ as $\alpha \to +\infty$. ✓

#15 (a) $\exists x \in R^n, x \geq 0, x \neq 0$ such that $Ax \leq 0$
(b) $\exists y \in R^m, y > 0$ such that $A^T y > 0$

Suppose (a) is true. Suppose that (b) is also true. Then

$(y, Ax) \leq 0$ since $y > 0$ and $Ax \leq 0$.
$\|$
$(A^T y, x) > 0$ since $A^T y > 0$ and $x \geq 0, x \neq 0$.

This is a contradiction. Hence (a) True $\Rightarrow$ (b) False.

We now show that (b) False $\Rightarrow$ (a) True.

Let $K = \{A^T y, \ y \in R^m_{++}\}$. K is a convex subset of R^n.

(b) False implies that $K \cap R^n_{++} = \emptyset$

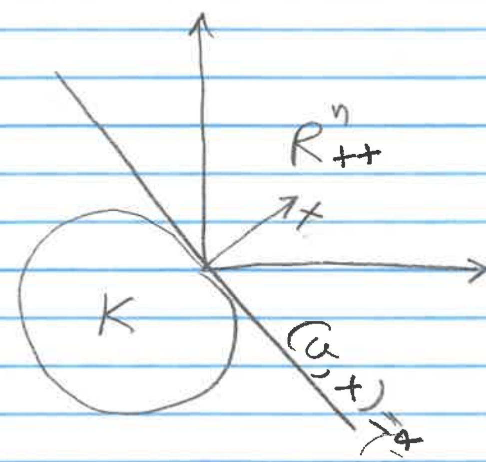
By a separation result,

$\exists x \in R^n, x \neq 0$ and $\alpha \in R$

Such that

$$\underset{*}{(w, x) \leq \alpha} \underset{**}{\leq (v, x)}$$

$\forall w \in K, \ \forall v \in R^n_{++}$.



Since $\{0\}$ belongs to the closure of K and R^n_{++},

we must have $\boxed{\alpha = 0}$. Next, we show that $x \geq 0$.

Indeed, let the sequence $\{y_n^{(i)}\}$ in R^n_{++} such that $y_n^{(i)} \to e_i$

Then $(y_n^{(i)}, x) \to (e_i, x) = x_i \geq 0$ by (**).

This is true for $i = 1, \dots, n$. Thus $x \geq 0$. Finally we show that $Ax \leq 0$.

Again, let $\{y_n^{(i)}\}$ be a sequence in $\mathbb{R}_{++}^m$ with $\lim_{n\to\infty} y_n^{(i)} = e_i$. We have

$$0 \overset{*}{\geq} (A^T y_n^{(i)}, x) = (y_n^{(i)}, Ax) \to (e_i, Ax) = (Ax)_i .$$

Hence $(Ax)_i \leq 0$, $i=1,\dots,m \Rightarrow Ax \leq 0$.