Math 577 HW1 Spring 2018 solutions

#1)

(a) $f(a+h) = b_0 + b_1 h + |h|\,\varepsilon(h)$ ⊛

$h=0 \Rightarrow f(a) = b_0$, $\Rightarrow \dfrac{|f(a+h) - f(a) - b_1 h|}{|h|} = \varepsilon(h) \to 0$

we know $f'(a)$ exists and is unique. Hence $b_1 = f'(a)$.

(b) Again, let $h=0 \Rightarrow f(a) = b_0$.

⊛ $\Rightarrow \dfrac{|f(a+h) - f(a) - b_1 h|}{|h|} = \varepsilon(h) \to 0$

hence by defn. of derivative $f'(a)$ exists and $f'(a) = b_1$

#2) (a)

⊛ $f(a+h) = b_0 + b_1 h + b_2 h^2 + |h|^2 \varepsilon(h)$, $\varepsilon(h) \to 0$

f twice diff'ble. at a $\Rightarrow$ by Taylor

$$f(a+h) = f(a) + f'(a) + \tfrac{1}{2} f''(a) h^2 + |h|^2 \tilde{\varepsilon}(h), \tilde{\varepsilon}(h) \to 0$$

Subtracting this from ⊛

$$0 = (b_0 - f(a)) + (b_1 - f'(a))h + (b_2 - \tfrac{1}{2} f''(a)) h^2$$
$$+ |h|^2 (\varepsilon(h) - \tilde{\varepsilon}(h)).$$

Again, taking $h=0$, we successively get
$b_0 = f(a)$, $b_1 = f'(a)$, $b_2 = \frac{1}{2} f''(a)$.

(b) let $f(x) = x^3 \sin \frac{1}{x}$, $x \neq 0$ and $f(0) = 0$.

we have $f(0+h) = 0 + 0\cdot h + 0 \cdot h^2 + h^2\, h \sin\frac{1}{h}$, $\varepsilon(h) = h \sin \frac{1}{h} \to 0$.

On the other hand $f'(x) = 3x^2 \sin\frac{1}{x} - x \cos\frac{1}{x}$, $x \neq 0$

$= 0 \qquad x = 0$

but f' is not diff'ble. at 0.

#3) $h=(h_1,h_2)$, $\frac{1}{t}\left[f(0+t(h_1,h_2)-f(0)\right]$ $t\neq 0$, $(h_1,h_2)\neq(0,0)$

$$= \frac{t^5 h_1^5}{t[(th_2-t^2h_1^2)^2+t^8h_1^8]} = \frac{t^2 h_1^5}{(h_2-th_1^2)^2+t^5h_1^8}$$

If $h_1=0$, Then $h_2\neq 0 \Rightarrow \lim_{t\to 0} = 0$

If $h_1\neq 0$, and $h_2\neq 0$, Then $\lim_{t\to 0} = 0$

If $h_1\neq 0$ and $h_2=0$, Then $\lim_{t\to 0} \frac{t^2h_1^5}{t^2h_1^4+t^5h_1^8} = \lim_{t\to 0}\frac{h_1^5}{h_1^4+t^3h_1^8} = h_1$

Hence in all cases the limit exists $\Rightarrow f$ is G-diffble. at $(0,0)$.

To show f is not continuous at $(0,0)$, take limit of f along the path $y=x^2$. $x\neq 0$

$$\Rightarrow f(x,x^2) = \frac{x^5}{x^8}.$$ Thus $\lim_{\substack{y=x^2\\ x\to 0}} f(x,y)$ does not exist.

Hence f is not continuous at $(0,0)$.

#4) $$\lim_{t\to 0}\frac{\|0+th\|-\|0\|}{t} = \lim_{t\to 0}\frac{|t|\,\|h\|}{t} = \begin{cases}\|h\| & \text{if } t\to 0^+\\ -\|h\| & \text{if } t\to 0^-\end{cases}$$

#5) We will use the following result:

A invertible and $\|A^{-1}E\|<1 \Rightarrow A-E$ invertible

Write $B=A-E$, $E=A-B$. Then

$$\|A^{-1}E\| = \|A^{-1}(A-B)\| \le \|A^{-1}\|\,\|A-B\| \le \|A^{-1}\|\,\delta.$$

Hence for $\delta < \frac{1}{\|A^{-1}\|}$ we will have $\|A^{-1}E\|<1$.

#6) $U^T A V = \Sigma \iff A = U \Sigma V^T$.

Of course if A is singular, then $\inf_{B \in S} \|A-B\|_2 = 0 = \sigma_n(A)$.
So suppose A is invertible.

<u>Claim</u> $\|A-B\|_2 \geq \sigma_n$ for any $B \in S$.

$\exists\, \hat{x}$, $\|\hat{x}\|_2 = 1$ such that $B\hat{x} = 0$

$$\|A-B\|_2 = \sup_{\|x\|_2=1} \|(A-B)x\|_2 \geq \|(A-B)\hat{x}\|_2 = \|A\hat{x}\|_2$$

$\|A\hat{x}\|_2^2 = \|U \Sigma V^T \hat{x}\|_2^2 = \|\Sigma V^T \hat{x}\|_2^2$, U orthogonal = isometry

let $\hat{z} = V^T\hat{x}$. $\Rightarrow \|A\hat{x}\|_2^2 = \|\Sigma \hat{z}\|_2^2 = \sum_{i=1}^{n} \sigma_i \hat{z}_i^2 \geq \sigma_n \|\hat{z}\|_2^2$

$\Rightarrow \|A\hat{x}\|_2^2 \geq \sigma_n \|\hat{z}\|_2^2 = \sigma_n \|V\hat{x}\|_2^2 = \sigma_n \|\hat{x}\|_2^2 = \sigma_n^2$ ✓

Finally, let $B = U \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_{n-1} & \\ 0 & & 0 \end{bmatrix} V^T$, B is singular

$\|A - B\|_2 = \left\| U \begin{bmatrix} 0 & & 0 \\ & 0 & \\ 0 & & \sigma_n \end{bmatrix} V^T \right\|_2 = \sigma_n$, U, V orthogonal. ✓

#7) If A is invertible, take $A_\epsilon = A$ ✓

Let $A = U \Sigma V^T$. Let $A_\epsilon = U(\Sigma + \delta I)V^T$ (δ to be chosen later)

Since the singular values $\sigma_1, \dots, \sigma_n$ form a finite set, there exists $c > 0$ such that $\sigma_i + \delta \neq 0$, $i = 1, \dots, n$, $\forall \delta$ with $|\delta| < c$

$\Rightarrow \|A - A_\epsilon\| = \|U(-\delta I)V^T\| \leq |\delta|\, \|U\|\, \|V^T\|$

Now choose $|\delta| < \min\left\{c, \frac{\epsilon}{\|U\|\,\|V^T\|}\right\}$. ✓

<u>Alternatively,</u> take $A_\epsilon = A - \epsilon I$ for ϵ small, A_ϵ is invertible since eigenvalues of A_ϵ are $\lambda - \epsilon$, λ eig. of A.

$\|A - A_\epsilon\| = |\epsilon|\, \|I\|$

#8) (i) $\langle x, x\rangle_A = x^T A x > 0 \quad \forall x \neq 0$ since A is s.p.d.

(ii) $\langle x, y\rangle_A = x^T A y = (x^T A y)^T = y^T A^T x = y^T A x = \langle y, x\rangle_A$

(iii) Clearly $\langle x, y\rangle_A$ is <u>bilinear</u> !!

#9) $\langle x, BAy\rangle_A = x^T A B A y = y^T A^T B^T A^T x = y^T A B A x$

$= \langle y, BAx\rangle_A = \langle BAx, y\rangle_A$ by (ii) in #8

$\langle x, BAy\rangle_{B^{-1}} = x^T B^{-1} B A y = x^T A y = y^T A^T x = y^T A x$

$= y^T B^{-1} B A x = \langle y, BAx\rangle_{B^{-1}} = \langle BAx, y\rangle_{B^{-1}}$ ✓

#10)

$f(x+th) - f(x) = (x+th)^T A (x+th) - b^T(x+th) + c$

$- x^T A x + b^T x - c$

$= t h^T A x + t x^T A h + t^2 h^T A h - t b^T h$

Hence

$\lim_{t\to 0} \frac{1}{t}\left[f(x+th) - f(x)\right] = h^T A x + x^T A h - b^T h =$

$\Rightarrow \quad f'(x)h = (x^T A^T + x^T A - b^T)h \Rightarrow f'(x) = (A + A^T)x - b$

$f''(x)(h,k) = \lim_{t\to 0} \frac{1}{t}\left[f'(x+tk)h - f'(x)\right] = h^T (A + A^T) k$

$\Rightarrow f''(x) = A + A^T.$

#11)

$f(A+tH) - f(A) = (A+tH)^2 - A^2 = A^2 + tAH + tHA + t^2 H^2 - A^2$

$= t(AH + HA) + t^2 H^2$

Hence

$$\boxed{f'(A)H = AH + HA}$$

#12) For t small, $I + tA^{-1}H$ is invertible and

$$\boxed{(I + tA^{-1}H)^{-1} = I - tA^{-1}H + O(t^2)} \quad (*)$$

$$\Rightarrow \quad (A+tH)^{-1} - A^{-1} = \left(A(I+tA^{-1}H)\right)^{-1} - A^{-1}$$
$$= (I+tA^{-1}H)^{-1}A^{-1} - A^{-1}$$
$$= \cancel{A^{-1}} - tA^{-1}HA^{-1} + O(t^2) - \cancel{A^{-1}}$$
$$= -tA^{-1}HA^{-1} + O(t^2).$$

Hence
$$\lim_{t\to 0} \frac{1}{t}\left[(A+tH)^{-1} - A^{-1}\right] = -A^{-1}HA^{-1} = f'(A)H$$

The same argument works without t, i.e. for $\|H\|$ small $(A+H)^{-1}$ exists.

$$f''(A)(H,K) = \lim_{t\to 0}\frac{1}{t}\left[-(A+tK)^{-1}H(A+tK)^{-1} + A^{-1}HA^{-1}\right]$$

$$= \lim_{t\to 0}\frac{1}{t}\left[-\left[I - tA^{-1}K + O(t^2)\right]A^{-1}H\left[I - tA^{-1}K + O(t^2)\right]A^{-1} + A^{-1}HA^{-1}\right]$$

$$= \lim_{t\to 0}\frac{1}{t}\left[\cancel{-A^{-1}HA^{-1}} + tA^{-1}KA^{-1}HA^{-1} + tA^{-1}HA^{-1}KA^{-1} + O(t^2) + \cancel{A^{-1}HA^{-1}}\right]$$

$$= A^{-1}KA^{-1}HA^{-1} + A^{-1}HA^{-1}KA^{-1}. \checkmark$$

#13) Let $f: V^n \to Y$ be an n-linear map. Then

$$f(v_1 + tw_1, v_2 + tw_2, \ldots, v_n + tw_n) = f(v_1, \ldots, v_n)$$
$$+ t\{f(w_1, v_2, \ldots, v_n) + f(v_1, w_2, v_3, \ldots, v_n) + \cdots + f(v_1, \ldots, v_{n-1}, w_n)\} + O(t^2).$$

If $a_1, \ldots, a_n$ and $h_1, \ldots, h_n$ denote the columns of A and H respectively

$$\det(A+tH) = \det(A) + t\{\det(h_1, a_2, \ldots, a_n) + \cdots + \det(a_1, \ldots, a_{n-1}, h_n)\} + O(t^2)$$

Hence $\lim_{t \to 0} \frac{1}{t}\left[\det(A+tH) - \det(A)\right]$

$$= \det(h_1, a_2, \dots, a_n) + \dots + \det(a_1, \dots, a_{n-1}, h_n) \equiv S$$

This is the sum of n determinants where by each column of A is replaced by a column of H.

Now $\det(a_1, \dots, a_{k-1}, h_k, a_{k+1}, \dots, a_n) = \sum_{i=1}^{n} h_{ki} C_{ki}, \quad k = 1, \dots, n$

where $\{C_{ki}\}$ are cofactors of A. Hence

$$S = \sum_{k=1}^{n} \sum_{i=1}^{n} h_{ki} C_{ki} = \sum_{k=1}^{n} \sum_{i=1}^{n} h_{ki} (\mathrm{adj}(A))_{ik}$$

($\mathrm{adj}(A)$ is the transpose of the matrix of cofactors)

$$S = \sum_{i=1}^{n} \sum_{k=1}^{n} (\mathrm{adj}(A))_{ik} h_{ki} = \sum_{i=1}^{n} (\mathrm{adj}(A) H)_{ii}$$

$$= \mathrm{tr}(\mathrm{adj}(A) H).$$

#14) $f(X) = \ln \det(X^{-1}) = \ln\left(\frac{1}{\det(X)}\right) = -\ln(\det(X))$

Now $f = \phi \circ \psi$ where $\psi : R^{n \times n} \to R \quad \psi(X) = \det(X)$

$\phi : R \to R \quad \phi(t) = \ln t$

By the chainrule

$$\phi'(t) = \frac{1}{t}$$

$$f'(X) = \phi'(\psi(X)) \psi'(X) \qquad \psi'(X) H = \mathrm{tr}(\mathrm{adj}(X) H)$$

$$\Rightarrow f'(X) H = -\frac{1}{\psi(X)} \mathrm{tr}(\mathrm{adj}(X) H)$$

$$= -\mathrm{tr}\left(\frac{\mathrm{adj}(X)}{\det(X)} H\right) = \boxed{-\mathrm{tr}(X^{-1} H)}$$

Using $\mathrm{adj}(X) \cdot X = \det(X) \cdot I \Rightarrow X^{-1} = \frac{\mathrm{adj}(X)}{\det(X)}$.

Note $\mathrm{tr}(AB) = \mathrm{tr}(BA) \Rightarrow f'(X) H = -\mathrm{tr}(H X^{-1})$

#15) $f(x,y,z) = \sin(xyz) - z \qquad f: R^3 \to R$

$R^3 = R^2 \times R$. So we can hope to solve for one variable in terms of the other two

$$\frac{\partial f}{\partial x} = yz\cos(xyz)\Big|_{(\frac{\pi}{2},1,1)} = 0$$

So we cannot solve for x in terms of y,z in a nbhd. of the point $(\frac{\pi}{2},1,1)$

$$\frac{\partial f}{\partial y} = xz\cos(xyz)\Big|_{(\frac{\pi}{2},1,1)} = 0 \text{. Same for } y.$$

$$\frac{\partial f}{\partial z} = xy\cos(xyz) - 1\Big|_{(\frac{\pi}{2},1,1)} = -1 \text{, so } \exists \text{ a nbhd.}$$

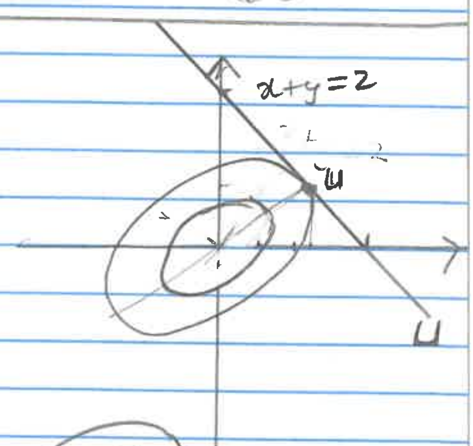
O_1 of $(\frac{\pi}{2},1)$ and a nbhd. $1 \in O_2$ such that

$$z = g(x,y) \qquad z \in O_2,\ (x,y) \in O_1.$$

$$g'(\tfrac{\pi}{2},1) = \begin{bmatrix} \partial z/\partial x \\ \partial z/\partial y \end{bmatrix}_{(\frac{\pi}{2},1)} = -(-1)^{-1}\begin{bmatrix} yz\cos(xyz) \\ xz\cos(xyz) \end{bmatrix}_{(\frac{\pi}{2},1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

#16) $f'(u) + \Lambda\phi'(u) = (4x+2y,\ 2x+6y)$

$$+ \lambda(1,1) = (0,0)$$



$$\left.\begin{aligned} 4x + 2y + \lambda &= 0 \\ 2x + 6y + \lambda &= 0 \end{aligned}\right\} \Rightarrow 4x + 2y = 2x + 6y$$

$$\Rightarrow 2x = 4y \Rightarrow x = 2y$$

$$x + y = 2 \Rightarrow 3y = 2 \Rightarrow x = 4/3 \quad y = 2/3$$

The level curves of f are ellipses as shown centered at 0. So the point $u = (4/3, 2/3)$ corresponds to a local and global minimum of f with respect to U.

Also note the following: U is a subspace of R^2, and thus convex.

Furthermore $f(x,y) = (x,y)\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

A is s.p.d. hence f is strictly convex on R^2. Also

$$f'(4/3, 2/3) = \frac{20}{3}\langle 1, 1\rangle$$ and is orthogonal to W.

$f'(u)(v-u)=0 \ \forall v \in W$

By part 3 of Thm 7.4-4, f has a local and global minimum at $u=(4/3, 2/3)$.

#17) $f(x,y,z) = (x-3)^2 + (y-2)^2 + z^2$

$$\phi_1(x,y,z) = x^2+y^2-1, \ \phi_2(x,y,z) = y+z-1$$

$$f: R^3 \to R, \quad f'(x,y,z) = \langle 3(x-2), 2(y-2), 2z\rangle$$

$$\phi'(x,y,z) = \begin{bmatrix} 2x & 2y & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Note rows of ϕ' are linearly independent. Hence by the L.-M. Theorem, $\exists\ \lambda_1, \lambda_2$ such that

$$\langle 3x-6, 2y-4, 2z\rangle + (\lambda_1, \lambda_2)\begin{bmatrix} 2x & 2y & 0 \\ 0 & 1 & 1 \end{bmatrix} = \langle 0, 0, 0\rangle.$$

There are 3 equations. Combining them with the constraints, we get the system

$$\begin{cases} 3x - 6 + 2x\lambda_1 = 0 \\ 2y - 4 + 2y\lambda_1 + \lambda_2 = 0 \\ 2z + \lambda_2 = 0 \\ x^2 + y^2 - 1 = 0 \\ y + z - 1 = 0 \end{cases}$$

We can try to find solutions of this system using Newton's or some other method. However

(1) There may be problems with convergence.

(2) How do we know we found all solutions

(3) Once a solution is found, how do we know if it corresponds to a local/global minimum or maximum, or neither.

As an alternative, we can try to find The restriction of f to U. In This case it is convenient to use parametric representations of $\phi_1=0$ and $\phi_2=0$

$$\phi_1 \Longleftrightarrow \langle \cos t, \sin t, s \rangle \qquad t, s \in R$$
$$\phi_2 \Longleftrightarrow \langle \tau, \sigma, 1-\sigma \rangle \qquad \tau, \sigma \in R$$

Hence $\quad U \Longleftrightarrow \langle \cos t, \sin t, 1-\sin t \rangle \quad 0 \leq t \leq 2\pi.$

Thus

$$g(t) = f\big|_U = (\cos t - 3)^2 + (\sin t - 2)^2 + (1-\sin t)^2$$
$$= 15 - 6\cos t - 6\sin t + \sin^2 t$$

To find The Extrema of g we use Newton's method on $g'(t)$. We find:

Local and global minimum at

$$\langle x, y, z \rangle = \langle .783575, .621297, .378703 \rangle$$
$$\text{with } f = 6.956777$$

Local and global maximum at

$$\langle x, y, z \rangle = \langle -.621297, -.783575, 1.783575 \rangle$$
$$\text{with } f = 24.043223$$

#18)

$$f'(x,y,z) = \langle y, x+z, y \rangle$$

$$\phi_1'(x,y,z) = \langle y, x, 0 \rangle, \ \phi_2'(x,y,z) = \langle 0, 2y, 2z \rangle$$

$$(y, x+z, y) + (\lambda_1, \lambda_2) \begin{bmatrix} y & x & 0 \\ 0 & 2y & 2z \end{bmatrix} = (0,0,0)$$

$$\begin{cases} y+\lambda_1 y=0 \\ x+z+\lambda_1 x+2\lambda_2 y=0 \\ y+2\lambda_2 z=0 \\ xy-1=0 \quad \Rightarrow \boxed{x\neq 0,\ y\neq 0} \\ y^2+z^2-1=0 \end{cases}$$

x, y have the same sign.

$0=y+\lambda_1 y=y(1+\lambda_1) \Rightarrow \boxed{\lambda_1=-1}$ since $y\neq 0$.

2nd eqn. $\overset{1+\lambda_1=0}{\Rightarrow}$ $z+2\lambda_2 y=0 \Rightarrow z^2+2\lambda_2 yz=0$

3rd eqn. $\Rightarrow$ $y+2\lambda_2 z=0 \Rightarrow y^2+2\lambda_2 yz=0$ $\Big\} \Rightarrow \boxed{z^2=y^2}$

$z^2=y^2$ and $y^2+z^2-1=0 \Rightarrow \boxed{y=\pm\frac{1}{\sqrt2},\ z=\pm\frac{1}{\sqrt2}}$

we have 4 extrema

$\left(\sqrt2, \frac{1}{\sqrt2}, -\frac{1}{\sqrt2}\right)$ and $\left(-\sqrt2, -\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)$ local/global min

$f=1-\frac12=\frac12$

$\left(\sqrt2, \frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)$ and $\left(-\sqrt2, -\frac{1}{\sqrt2}, \frac{-1}{\sqrt2}\right)$ local/global max

$f=1+\frac12=3/2.$

$U \leftrightarrow \left\langle \frac{1}{\cos t}, \cos t, \sin t \right\rangle \quad 0\le t\le 2\pi$

$\Rightarrow f|_U = 1+\cos t\sin t$