MATH 472 COMPUTING PROJECT # 3

April 22, 2019, Due May 2, 2019

The object of this project is to implement and study the convergence rates of the Steepest Descent and Conjugate Gradient iterative methods. These methods will be applied to the system $A\mathbf{x} = \mathbf{b}$ where A is the $n \times n$ tridiagonal matrix $A = [-1 \quad 2 \quad -1]$ and we want to apply these methods to different system sizes $n = 25, 50, 100, 200$.

For a given n, let $h = \frac{1}{n+1}$. The vector $\mathbf{b}$ is given by

$$b_i = 2h^2, \; i = 1, \ldots, n,$$

and the solution $\mathbf{x}$ is given by

$$x_i = ih(1 - ih), \; i = 1, \ldots, n.$$

Implement each of the three methods paying special attention **not the store the matrix in general form and to avoid multiplication by zeros**. In other words you need to exploit the sparsity of the matrix.

For each of the 8 tasks

(1) Start the iteration with $\mathbf{x}^{(0)} = \mathbf{0}$.

(2) Stop the iteration when $\|\mathbf{x}^{(k)} - \mathbf{x}\|_A < 10^{-6}\|\mathbf{x}^{(0)} - \mathbf{x}\|_A$. This norm is defined in the notes. There is a place in the algorithm where this test should be applied. Print the number k of iterations in Table 1.

(3) Unlike the Jacobi, Gauss-Seidel and SOR methods, the Steepest Descent and Conjugate Gradient methods are not *stationary* in the sense that there is no iteration matrix T whose spectral radius controls the convergence rate of the method and which we wish to estimate experimentally. For the two methods at hand, the convergence rates are governed by the condition number $\kappa(A) = \lambda_{\max}/\lambda_{\min}$ of the matrix A where $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalues of A respectively. For the two methods at hand, the rates of convergence are given by

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}\|_A}{\|\mathbf{x}^{(\mathbf{0})} - \mathbf{x}\|_{\mathbf{A}}} \leq \left(\frac{\kappa - 1}{\kappa + 1}\right)^k$$

for the Steepest Descent method and

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}\|_A}{\|\mathbf{x}^{(0)} - \mathbf{x}\|_A} \leq \sqrt{2}\left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\right)^k,$$

for the Conjugate Gradient method.

Organize your output as

n	Steepest Descent	Conjugate Gradient
25		
50		
100		
200		

TABLE 1. Number of iterations

n	Steepest Descent	Conjugate Gradient
25		
50		
100		
200		

TABLE 2. Condition numbers

Upon successful termination, your code should give you the number of iterations k as well as $\mathbf{x}^{(k)}$. You have then to calculate the norms $\|\mathbf{x}^{(k)} - \mathbf{x}\|_A$ and $\|\mathbf{x}^{(0)} - \mathbf{x}\|_A$. The condition number κ can then be calculated from the one of the two formulas given above matching the method used.

Remark It is known that $\kappa(A) = O(h^{-p})$ for some positive integer p. You should be able to determine p using the experimentally obtained estimates of κ. As another check on your code, there should be good agreement between the values of κ obtained from the two methods.