MATH 472 COMPUTING PROJECT # 1

February 26, 2019, Due March 12, 2019

The $n \times n$ **Hilbert** matrix H^n defined by

$$H^n_{ij} = \frac{1}{i+j-1}, \quad i,j = 1,\dots,n. \tag{1}$$

The Hilbert matrix H^n arises when approximating a function by a polynomial of degree less than or equal to n using the Least-Squares method. They are extremely ill-conditioned even for small n. In fact the following asymptotic estimate is known

$$\kappa_\infty(H^n) = O((1+\sqrt{2})^{4n}/\sqrt{n}). \tag{2}$$

(1) Obviously H^n is symmetric. Using the fact that $H^n_{ij} = \int_0^1 x^{i+j-2}\,dx,\ i,j = 1,\dots,n$, show that H^n is positive definite.

(2) The inverse of H^n is explicitly known:

$$(H^n)^{-1}_{ij} = (-1)^{i+j}(i+j-1)\binom{n+i-1}{n-j}\binom{n+j-1}{n-i}\binom{i+j-2}{i-1}^2. \tag{3}$$

Write a program to compute $\kappa_\infty(H^8)$ using formulas (1) and (3). Compare it to the estimate provided by (2).

The task here is to compute the inverse of H^8 using two methods: Choleski and QR. The purpose is to 1) getting familiarity with the implementation of these methods and 2) to gauge the effect of roundoff errors on the accuracy of the answers and 3) to compare the two methods and to see which one is less/more sensitive to roundoff errors. Indeed, Hilbert matrices are very ill-conditioned so we expect large errors.

(1) Write a program to compute the Choleski factorization LL^T of H^8. You may use the program I gave in class. Then write a program to do Forward and Back substitution. You should implement these yourself and not use a software package. So include a copy of your codes with your project. With these codes compute the inverse by solving 8 linear systems $H^8\mathbf{x}_i = \mathbf{e}_i,\ i = 1,\dots,8,\ \mathbf{x}_i$ being the ith column of the inverse.

The output should be (i) $(H^8)^{-1}$ using formula (3), (ii) the computed approximate inverse $H^8_{approx-inv}$ and (iii) $\|(H^8)^{-1} - H^8_{approx-inv}\|_\infty$. It is recommended that you use double precision in all your calculations since it is possible for single precision to be overwhelmed by the ill-conditioning.

(2) Repeat the above but using the QR factorization of H^8. For the factorization you can use Matlab or other software. For the solution phase use the Back substitution code use in the Choleski method.

To summarize, you should submit the following

(1) The proof of positive definiteness of H^n.

(2) $\kappa_\infty(H^8)$

(3) The codes for the Choleski factorization, the Forward and Back substitution
(4) $(H^8)^{-1}$ using formula (3)
(5) The computed approximate inverse $H^8_{approx-inv}$ using Choleski
(6) $\|(H^8)^{-1} - H^8_{approx-inv}\|_\infty$ using Choleski
(7) The computed approximate inverse $H^8_{approx-inv}$ using QR.
(8) $\|(H^8)^{-1} - H^8_{approx-inv}\|_\infty$ using QR.