§4.4 Bases, Dimension

Definition A vector space V is called finite dimensional if there is a finite set $S = \{v_1, \dots, v_n\}$ that spans V.

Ex. we showed in the last section that $\text{span}\{(1,1,2),(1,0,1),(2,1,3)\} = R^3$. Hence R^3 is finite dimensional or has finite dimension.

Defn. If $S = \{v_1, \dots, v_n\}$ is a set of vectors in a finite dimensional vector space V, then S is called a basis for V if
(a) S spans V, i.e. $\text{span}\{S\} = V$,
(b) S is linearly independent.

In other words, a basis is a spanning set which has no redundancies.

Ex. show that the set $S = \{(1,2,3), (-1,2,4), (5,4,-3)\}$ is a basis for R^3.

we want to show first that (a) $\text{span}\{S\} = R^3$, i.e. every vector in R^3 can be written as a linear combination of the 3 vectors in S. Let (b_1, b_2, b_3) be an arbitrary (generic) vector in R^3. we want to show that there exist k_1, k_2, k_3 in R such that

$$k_1(1,2,3) + k_2(-1,2,4) + k_3(5,4,-3) = (b_1, b_2, b_3).$$

This translates into a system

$$\begin{aligned} k_1 - k_2 + 5k_3 &= b_1 \\ 2k_1 + 2k_2 + 4k_3 &= b_2 \\ 3k_1 + 4k_2 - 3k_3 &= b_3 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 5 & b_1 \\ 2 & 2 & 4 & b_2 \\ 3 & 4 & -3 & b_3 \end{array}\right]$$

$$Ak = b$$

we want to see if the system has a solution k for any given b.
One way to answer this question is to reduce

the augmented system $[A|b]$ into upper triangular form by Gauss Elimination and see if $\mathrm{rank}(A) = \mathrm{rank}(A|b)$.

Another way is to exploit the fact that the coefficient matrix A is square. There is a theory at hand already which says that the system $Ak = b$ has a solution for any given b iff A is invertible. iff $\det(A) \neq 0$.

we find $\det(A) = -30 \neq 0 \Rightarrow$ A invertible $\Rightarrow$ there is a solution k for any b $\Rightarrow \mathrm{span}(S) = \mathbb{R}^3$.

we need to check (b) namely if the 3 vectors $(1,2,3), (-1,2,4), (5,4,-3)$ are linearly independent. But, checking this fact is almost exactly what we did above except that now we want to see if the only solution of the homogeneous system $Ak=0$ is the trivial solution $k = 0 = (0,0,0)$.

Since we know now that A is invertible,

$$Ak = 0 \Rightarrow k = A^{-1} 0 = 0 \checkmark .$$

Thus S is a basis for $\mathbb{R}^3$.

<u>The standard basis for $\mathbb{R}^n$</u>

$e_1 = (1, 0, \dots, 0)$
$e_2 = (0, 1, \dots, 0)$
$\vdots \qquad \vdots$
$e_n = (0, 0, \dots, 1)$

Note that there are also the n rows or n columns of the $n \times n$ identity matrix.

let $(x_1, \dots, x_n)$ be a vector in $\mathbb{R}^n$. Then clearly

$$(x_1, \dots, x_n) = x_1 e_1 + x_2 e_2 + \dots + x_n e_n .$$

Thus $\{e_1, e_2, \dots, e_n\}$ spans $\mathbb{R}^n$.

we next check linear independence of $\{e_1, \dots, e_n\}$

Indeed suppose $\underbrace{k_1 e_1 + k_2 e_2 + \cdots + k_n e_n}_{(k_1, k_2, \ldots, k_n)} = \mathbf{0} = (0, 0, \ldots, 0)$

$$(k_1, k_2, \ldots, k_n) = (0, 0, \ldots, 0)$$

$$\Rightarrow k_1 = k_2 = \cdots = k_n = 0 .$$

we have shown that (a) $\text{span}\{e_1, e_2, \ldots, e_n\} = R^n$

(b) $\{e_1, \ldots, e_n\}$ are lin. independent

hence, the set $\{e_1, \ldots, e_n\}$ is a basis for R^n.

Remark This is not the only basis set for R^n. However, given the "simple" structure of $e_1, \ldots, e_n$, it is particularly easy to work with this basis.

Standard basis for $R^{m\times n}$

This basis has $\boxed{mn}$ elements: For each pair (i,j) $i = 1, \ldots, m$; $j = 1, \ldots, n$, put 1 in entry (i,j) and zeros everywhere else

Ex. $R^{3\times 2}$.

$$\left\{ \underset{M_{11}}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{M_{12}}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{M_{21}}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{M_{22}}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{M_{31}}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{M_{32}}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$$

are the 6 elements of the standard basis for $R^{3\times 2}$. Again, we check that this set is indeed a basis for $R^{3\times 2}$

(a) let $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$ be a given generic element of $R^{3\times 2}$.

Then it is easy to see that we have

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = a_{11}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + a_{12}\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + a_{21}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + a_{31}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} + a_{32}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

This shows that every 3×2 matrix can be written as a linear combination of the 6 basis matrices. In other words we have shown that

$$\text{span}\{M_{11}, M_{12}, M_{21}, M_{22}, M_{31}, M_{32}\} = R^{m\times n}.$$

(b) To show linear independence, suppose

$$k_1 M_{11} + k_2 M_{12} + k_3 M_{21} + k_4 M_{22} + k_5 M_{31} + k_6 M_{32} = 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \\ k_5 & k_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow k_1 = k_2 = k_3 = k_4 = k_5 = k_6 = 0.$$

Thus the set is lin. indep. Hence $M_{11}, \dots, M_{32}$ form a basis for $R^{3\times 2}$.

Standard basis for P_n

The standard basis is $\{1, x, \dots, x^n\}$. Note it has $\underline{\underline{n+1}}$ elements.

(a) Let $a_0 + a_1 x + \dots + a_n x^n$ be a given element of P_n. Then clearly

$$a_0 + a_1 x + \dots + a_n x^n = a_0 \cdot 1 + a_1 x + \dots + a_n \cdot x^n$$

i.e.

$$\text{Span}\{1, x, \dots, x^n\} = P_n.$$

(b) Suppose $\underbrace{a_0 \cdot 1 + a_1 x + \dots + a_n x^n}_{P(x)} = 0$

Equality to zero for a polynomial $\underline{\underline{\text{means}}}$ —

$P(x) = 0$ for $\underline{\text{all}}$ x. There is a theorem in analysis which states that a polynomial can vanish for all x if and only if all of its coefficients are zero.

Theorem 4.4.1 Uniqueness of Basis Representations

If $S=\{v_1, .., v_n\}$ is a basis for a vector space V, then every vector v in V can be expressed in the form

$$v = k_1 v_1 + \cdots + k_n v_n$$

in a unique, i.e. exactly one way.

proof

Since $S=\{v_1, .., v_n\}$ is a basis for V, we have by definition of a basis

(a) $\text{span}\{S\} = V$
(b) S is linearly independent.

It follows from (a) that every vector v in V can be expressed as

$$v = k_1 v_1 + \cdots + k_n v_n.$$

We will show that this representation is unique, i.e. if we also had

$$v = d_1 v_1 + \cdots + d_n v_n$$

for some scalars $d_1, .., d_n$, then it would necessarily follows that

$$d_1 = k_1, d_2 = k_2, \ldots, d_n = k_n.$$

Indeed, subtracting the two expressions, we get

$$0 = v - v = k_1 v_1 + \cdots + k_n v_n - (d_1 v_1 + \cdots + d_n v_n)$$

$$= (k_1 - d_1) v_1 + \cdots + (k_n - d_n) v_n.$$

From (b) we know that S is linearly independent. Thus by the definition of linear independence, we must have

$$k_1 - d_1 = 0 \Rightarrow k_1 = d_1 ; \quad k_2 - d_2 = 0 \Rightarrow k_2 = d_2, \ldots, k_n - d_n = 0 \Rightarrow k_n = d_n.$$

Definition If $S=\{v_1, \ldots, v_n\}$ is a basis for the vector space V and

$$v = k_1 v_1 + k_2 v_2 + \cdots + k_n v_n ,$$

the scalars $k_1, k_2, \ldots, k_n$ are called the coordinates of v with respect to, or relative to the basis S.

Remark By the preceding theorem, the coordinates $k_1, k_2, \ldots, k_n$ are uniquely defined.

We may also think of $k_1, k_2, \ldots, k_n$ representing or forming a vector $(k_1, k_2, \ldots, k_n)$ in R^n.

We can use this observation to establish a one-to-one relationship between any finite dimensional vector space V and R^n. What we need is to define the concept of dimension which is done in the next section.

§ 4.5 Dimension of a Vectorspace

Let V be a finite dimensional vector space. What This means is That There is a finite set $S=\{v_1, v_2, \dots, v_n\}$ which spans V.

We shall next show That V has a finite basis. This is not immediately obvious and requires proof. Of course we know This is true for R^n, $R^{m\times n}$ and P_n since we exhibited "standard" bases for each of These 3 vector spaces. So our proof will address The case of a <u>general</u> vector space V.

<u>Theorem 4.5.0</u> Let V be a finite dimensional vectorspace. Then V has a finite basis.

<u>proof</u> By definition, V has a finite spanning set $S=\{v_1, v_2, \dots, v_n\}$, i.e. $\text{span}(S)=V$. As a first step, assume That all The vectors in S are nonzero.

If S is linearly independent, Then we are done, since S would be a basis.

So suppose S is linearly dependent. (Then it must contain at least 2 vectors.) The strategy is to Keep removing "redundancies" from S until we arrive at a linearly independent <u>subset</u> of S which also spans V.

If S is linearly dependent, Then there is at least one vector, say v_1, which is a linear combination of The others, i.e.

$$v_1 = c_2 v_2 + \dots + c_n v_n .$$

We will show next That $\text{span}(S-\{v_1\})=\text{span}(S)$. Indeed, let $v \in V$. Then

$$v = k_1 v_1 + \dots + k_n v_n \quad \Leftarrow \text{span}(S)=V$$

$$= k_1 \underbrace{(c_2 v_2 + \dots + c_n v_n)}_{v_1} + k_2 v_2 + \dots + k_n v_n$$

$$= (k_2 + k_1 c_2) v_2 + (k_3 + k_1 c_3) v_3 + \dots + (k_n + k_1 c_n) v_n .$$

Thus, the reduced set $S-\{v_1\}$ can span V.

Now, if the reduced set $S-\{v_1\}$ is linearly independent, then it is a basis and the proof is complete. If it still linearly dependent, then it must contain a "redundent" element which can be removed as was done above, i.e. we get a new reduced set $S-\{v_1, v_2\}$ which also spans V. This way, after a finite number of steps we obtain a finite subset S' of S, which is nonempty, linearly independent and spans V. Thus S' is the sought after basis for V. $\boxtimes$

We are now ready to prove that all bases of a finite dimensional vector space have the same number of elements (vectors). This will be a corollary of the following result.

Theorem 4.5.2 Let $S=\{v_1,..,v_n\}$ be a basis for the finite dimensional vector space V. Then

(a) Any subset of V that contains more than n elements must be linearly dependent.

(b) Any subset of V with fewer than n elements cannot span V.

proof

(a) Let $\{w_1, .., w_m\}$ with $m>n$ be any subset of V that contains more than n elements. Since S is a basis for V, we can write each w as a linear combination of $v_1,..,v_n$, i.e.

$$\begin{aligned}
w_1 &= a_{11} v_1 + a_{12} v_2 + \cdots + a_{1n} v_n \quad \text{for some scalars } a_{11},..,a_{1n}\\
w_2 &= a_{21} v_1 + a_{22} v_2 + \cdots + a_{2n} v_n \quad \text{" " " } a_{21},..,a_{2n}\\
&\vdots\\
w_m &= a_{m1} v_1 + a_{m2} v_2 + \cdots + a_{mn} v_n \quad \text{" " " } a_{m1},..,a_{mn}
\end{aligned}$$

i.e.

$$w_i = \sum_{\ell=1}^{n} a_{\ell i} v_\ell \quad , \quad i=1,...,m.$$

We will construct a linear combination $k_1 w_1 + \cdots + k_m w_m$ which is equal to zero, but with not all $k_1,..,k_m$ equal to zero, thus implying that $w_1,..,w_m$ is linearly dependent.

Now
$$k_1 w_1 + \cdots + k_m w_m = \sum_{i=1}^{m} k_i w_i = \sum_{i=1}^{m} k_i \sum_{\ell=1}^{n} a_{\ell i} v_\ell$$
$$= \sum_{\ell=1}^{n} \left(\sum_{i=1}^{m} a_{\ell i} k_i \right) v_\ell .$$

Consider the linear system $\sum_{i=1}^{m} a_{\ell i} k_i = 0, \ \ell = 1, \ldots, n$.

We will show that it has a nontrivial solution. Indeed, in matrix form it is $AK = 0$. Since $m > n$, we have more unknowns (m) than equations (n) and thus this system has a <u>non-zero</u> solution $K = (k_1, k_2, \ldots, k_m)$. Thus, $k_1 w_1 + \cdots + k_m w_m = 0$, that is $\{w_1, \ldots, w_m\}$ is linearly dependent.

(b) Suppose we have a set $\{w_1, \ldots, w_m\}$ with $m < n$, i.e. with less elements than n. We want to show that $\{w_1, \ldots, w_m\}$ cannot span V. Suppose, to the contrary, that it did, i.e. $\operatorname{span}\{w_1, \ldots, w_m\} = V$. Since the basis vectors $v_1, \ldots, v_n$ belong to V, we can represent each v_i as a linear combination of $w_1, \ldots, w_m$. So we can write

$v_1 = a_{11} w_1 + a_{21} w_2 + \cdots + a_{m1} w_m$ for some scalars $a_{11}, \ldots, a_{m1}$
$v_2 = a_{12} w_1 + a_{22} w_2 + \cdots + a_{m2} w_m$ " " " $a_{12}, \ldots, a_{m2}$
$\vdots$
$v_n = a_{1n} w_1 + a_{2n} w_2 + \cdots + a_{mn} w_m$ " " " $a_{1n}, \ldots, a_{mn}$.

i.e.
$$v_i = \sum_{\ell=1}^{m} a_{\ell i} w_\ell , \qquad i = 1, \ldots, n .$$

Consider the linear combination $k_1 v_1 + \cdots + k_n v_n$; we have
$$k_1 v_1 + \cdots + k_n v_n = \sum_{i=1}^{n} k_i v_i = \sum_{i=1}^{n} k_i \sum_{\ell=1}^{m} a_{\ell i} w_\ell$$
$$= \sum_{\ell=1}^{m} \left(\sum_{i=1}^{n} a_{\ell i} k_i \right) w_\ell .$$

The homogeneous system $\sum_{i=1}^{n} a_{\ell i} k_i = 0, \ \ell = 1, \ldots, m$

has n unknowns $k_1, \ldots, k_n$ and m equations. Since $m < n$, there is a non zero solution $k_1, \ldots, k_n$ leading to $k_1 v_1 + \cdots + k_n v_n = \sum_{\ell=1}^{m} 0 \cdot w_\ell = 0$. Thus, $\{v_1, \ldots, v_n\}$ is linearly dependent, contradicting the fact that $\{v_1, \ldots, v_n\}$ is a <u>basis</u>.

Another way of showing this proceeds as follows:
Suppose $m < n$ and $\operatorname{span}\{w_1, \ldots, w_m\} = V$.

If $\{w_1, \ldots, w_m\}$ is linearly independent, then $\{w_1, \ldots, w_m\}$ is a basis for V. But from (a), $\{v_1, \ldots, v_n\}$ must be linearly dependent since it has more elements that $\{w_1, \ldots, w_m\}$ which is now a basis.

If $\{w_1, \ldots, w_m\}$ is linearly dependent, then as we did in a previous proof, we can remove elements from it until we arrive to a linearly independent subset of $\{w_1, \ldots, w_m\}$ which still spans V. Again, $\{v_1, \ldots, v_n\}$ must be then linearly dependent since it contains more elements than this basis. ▣

Theorem 4.5.1 All bases in a finite dimensional vector space have the same number of elements.

proof.

Suppose $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ are two bases for V.

If $m > n$, then $\{w_1, \ldots, w_m\}$ is linearly dependent by part (a) of the previous Theorem, so cannot be basis.

If $m < n$, then $\{v_1, \ldots, v_n\}$ is linearly dependent by part (a) of the previous Theorem, so cannot be a basis.

The only way out of this conundrum is to have $m = n$. ▣

In view of this result, we have the following

Definition The dimension of a finite dimensional vector space V is the number of elements/vectors in any basis of V.

We next give a series of useful results whose proofs use arguments which we have seen already.

Theorem 4.5.3 Plus/Minus Theorem.

Let S be a nonempty set of vectors in a vector space V.

(a) If S is linearly independent and $v \in V$ is vector that does not belong to $\mathrm{span}(S)$, then the augmented set $\{S\} \cup \{v\}$ is linearly independent.

(b) If v belongs to S and is expressible as a linear combination of other vectors in S, then

$$\mathrm{span}(S - \{v\}) = \mathrm{span}(S).$$

proof

(a) Let $S = \{v_1, \ldots, v_n\}$. Consider the linear combination

$$k_1 v_1 + k_2 v_2 + \cdots + k_n v_n + k_{n+1} v = 0.$$

There are two cases: $k_{n+1} \neq 0$ or $k_{n+1} = 0$.

If $k_{n+1} \neq 0$, then we ~~write~~ set $v = -\frac{k_1}{k_{n+1}} v_1 - \frac{k_2}{k_{n+1}} v_2 - \cdots - \frac{k_n}{k_{n+1}} v_n$

which says that $v \in \mathrm{span}(S)$. But this contradicts the original assumption. So this case cannot happen.

If on the other hand $k_{n+1} = 0$, then $k_1 v_1 + \cdots + k_n v_n = 0$. Since S is linearly independent, $k_1 = k_2 = \cdots = k_n = 0$. Together with $k_{n+1} = 0$, this shows that the set $S \cup \{v\}$ is linearly independent.

(b) Let $S = \{v_1, \ldots, v_n\}$ and suppose without loss of generality that v_1 is a linear combination of $v_2, \ldots, v_n$. i.e.

$$v_1 = k_2 v_2 + \cdots + k_n v_n.$$

Now let $v \in \mathrm{span}(S)$. $\Rightarrow$ $v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$

$$= c_1 (k_2 v_2 + \cdots + k_n v_n) + c_2 v_2 + \cdots + c_n v_n$$

$$= (c_1 k_2 + c_2) v_2 + (c_1 k_3 + c_3) v_3 + \cdots + (c_1 k_n + c_n) v_n$$

which means that $v_1 \in \mathrm{span}\{v_2, \ldots, v_n\} = \mathrm{span}(S - \{v_1\})$. ∎

Theorem 4.5.4 Let V be an n-dimensional vector space, and let $S = \{v_1, \ldots, v_n\}$ be a subset of V with $\underline{n}$ elements.

(a) If S is linearly independent, then $\text{span}(S) = V$, i.e. S is a basis for V.
(b) If S spans V, then S is linearly independent, thus S is a basis for V.

Remark This result's usefelness comes from the fact that if a subset S of V has the same number of elements as the dimension of V, then to check/verify whether S is a basis or not one needs to check only one of the two requirements

$$\text{span}(S) = V \quad , \quad S \text{ is linearly independent.}$$

Theorem 4.5.5 Let S be finite set of vectors in a finite-dimensional vector space V.
(a) If S spans V but is not linearly independent, then S can be reduced to a basis by removing appropriate vectors from S.

(b) If S is linearly independent but does not span V, Then S can be enlarged to a basis for V by inserting appropriate vectors into S.

proof. (a) we saw this done already.

(b) suppose $\text{span}(S) \neq V$. Then there must be a vector $v_1 \in V$ which does not belong to $\text{span}(S)$. Then by Theorem 4.5.3 plus, $S \cup \{v_1\}$ is linearly independent. If $\text{span}(S \cup \{v_1\}) = V$, then the proof is complete. Otherwise there exists $v_2 \in V$ which does not belong to $\text{span}(S \cup \{v_1\})$. Then by Theorem 4.5.3 plus, $S \cup \{v_1\} \cup \{v_2\}$ is linearly independent. Again, if $\text{span}(S \cup \{v_1\} \cup \{v_2\}) = V$, done. etc...
This argument will eventually terminate after a finite number of steps (why?) leading to a basis for V. $\blacksquare$

Theorem 4.5.6 If W is a subspace of a finite-dimensional vector space V, then

(a) W is finite dimensional.
(b) $\dim W \le \dim V$
(c) $W = V$ if and only $\dim W = \dim V$.

proof (a) let $S = \{v_1, \ldots, v_n\}$ be a basis for V.

since W is (also) a subset of V, $W \subseteq \text{span}(S)$. Thus W is finite dimensional.

(b) W is by (a) finite dimensional, hence has a basis $\{w_1, \ldots, w_m\}$. It is clear that we cannot have $m > n$ for otherwise by Thm 4.5.2 (a), $\{w_1, \ldots, w_m\}$ must be linearly dependent, so cannot be a basis for W or anything else.

(c) Suppose $W = V$, then clearly $\dim W = \dim V$. Conversely, suppose $\dim W = \dim V$. We want to show $W = V$. Suppose not. This means W is a proper subset of V, implying there exist $v \in V$, $v \notin W$. Now let $\{w_1, \ldots, w_m\}$ be a basis for W. The set $\{w_1, \ldots, w_m\}$ is linearly independent. Also, W is a subspace of V and $v \notin W$ means $v \notin \text{span}\{w_1, \ldots, w_m\}$. By Theorem 4.5.3 plus the augmented set $\{w_1, \ldots, w_m, v\}$ is also linearly independent; but since it has $m+1 > n$, it must be linearly dependent by Theorem 4.5.2(a). A clear contradiction. ∎