Chapter 4 : General Vector Spaces

A general vector space gets its motivation from physical vectors i.e. objects that possess magnitude ~~and~~ direction.

A vector space is a construct using ~~two sets~~ the interplay of two sets: A field of scalars and a set of objects called vectors.

Definition A Field is a set of elements called scalars that is equipped with two operations: addition, denoted $+$, and multiplication denoted $\cdot$. The elements of the set are required to satisfy the following conditions

(A) To every pair α and β of scalars, there corresponds a scalar $\alpha+\beta$ called the sum of α and β in such a way that

(1) $\alpha+\beta=\beta+\alpha$ addition is commutative
(2) $\alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma$ addition is associative.
(3) There exists a unique scalar called zero, denoted 0 such that $\alpha+0=\alpha$ for all α. (zero is the neutral element for addition)
(4) To every scalar α there corresponds a unique scalar $-\alpha$ such that $\alpha+(-\alpha)=0$. $-\alpha$ is called the additive inverse of α.

(B) To every pair α and β of scalars there corresponds a scalar $\alpha\beta$, called the product of α and β in such a way that

(1) $\alpha\beta=\beta\alpha$ mult. is commutative.
(2) $\alpha(\beta\gamma)=(\alpha\beta)(\gamma)$ mult. is associative
(3) There exists a nonzero scalar 1 (called one) such that $\alpha 1=\alpha$ for all α. (1 is the neutral element of multiplication
(4) To every nonzero scalar α, there corresponds a unique scalar α^{-1} or $1/\alpha$ such that $\alpha\frac{1}{\alpha}=1$.

$\frac{1}{\alpha}$ is the multiplicative inverse of α.

(C) $\alpha(\beta+\gamma)=\alpha\beta+\alpha\gamma$, mult. is distributive with respect to addition

Two important examples of fields are $\mathbb{R}$, the set of real numbers equipped with the usual operations of addition and multiplication and $\mathbb{C}$ the set of complex numbers equipped with the usual operations of addition and multiplication.

Another example of a field is $\mathbb{Q}$, the set of all rational numbers, i.e. fractions.

On the other hand, the set $\mathbb{Z}$ of all integers equipped with the usual operations of addition and multiplication is not a field. The only rule that fails to hold is (B)4.

$\mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$ are not by any means the only examples of fields. There are some that are very important in Mathematics and can be viewed as exotic or esoteric.

Ex. Let $Z_5 = \{0, 1, 2, 3, 4\}$.

Define "addition" + as follows

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

This is addition modulo 5

Define Multiplication $\cdot$ according to the table

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

multiplication modulo 5.

Ex. Verify that $Z_5, +, \cdot$ satisfies all the requirements or axioms A, B, C, of a field and thus is a field. Unlike $\mathbb{R}, \mathbb{Q}, \mathbb{C}$, $\mathbb{Z}_5$ is a finite field in that it has only a finite set of elements (scalars)

If $F, +, \cdot$ is a field, we can define two extra operations: subtraction $\alpha - \beta$ and division $\frac{\alpha}{\beta}$ as follows

Subtraction: -1 is the additive inverse of 1
we define $\alpha - \beta$ as $\alpha + (-\beta)$ $\quad -\beta$ is additive inverse of β

Division For any α and $\beta \neq 0$, we define $\frac{\alpha}{\beta}$ as $\alpha \cdot \frac{1}{\beta}$.

So in some sense the usual operations of subtraction and division are artificial in that they are derived operations.

Almost all the laws of elementary arithmetic are consequences of the axioms defining a field. In fact we can prove the following facts rather easily

(a) $0 + \alpha = \alpha$
(b) If $\alpha + \beta = \alpha + \gamma$, then $\beta = \gamma$ "cancellation law"
(c) $\alpha + (\beta - \alpha) = \beta$
(d) $\alpha \cdot 0 = 0 \cdot \alpha = 0$
(e) $(-1)\alpha = -\alpha$
(f) $(-\alpha)(-\beta) = \alpha\beta$
(g) If $\alpha\beta = 0$ then $\alpha = 0$ or $\beta = 0$.

Vector spaces

we assume that we are given a particular field, e.g. R; the scalars to be used are to be elements of F.

Definition A vector space is a set V of elements called vectors satisfying the following axioms

(A) To every pair x and y of vectors in V, there corresponds a vector $x+y$, called the sum of x and y, in such a way that

(1) $x+y=y+x$ commutativity of addition.
(2) $x+(y+z)=(x+y)+z$ associativity of addition
(3) There exists in V a unique vector denoted 0 such that $x+0=x$ for every vector x
(4) To every vector x in V there corresponds a unique vector $-x$ such that $x+(-x)=0$. $-x$ is called the additive inverse of x.

(B) To every pair α and x, where α is a scalar and x is a vector in V there corresponds a vector αx in V called the (scalar) product of α and x, in such a way that

(1) $\alpha(\beta x)=(\alpha\beta)x$ scalar multiplication is associative
(2) $1\cdot x=x$ for every vector x.

(C)
(1) $\alpha(x+y) = \alpha x+\alpha y$ scalar multiplication is distributive with respect to vector addition.
(2) $(\alpha+\beta)x=\alpha x+\beta x$ multiplication by a vector is distributive with respect to scalar addition.

we shall next give several examples of vector spaces. in all cases, the field of scalars is R.

$V=R^n$ For integer $n\geq 1$, let V be the set of all ordered n-tuples of real numbers:

$$V=\{x=(x_1,x_2,\dots,x_n) \mid x_i\in R, i=1,\dots,n\}$$

The $X+Y$ of X and Y is defined by <u>componentwise addition</u>:

$$X+Y \equiv (x_1+y_1, \ldots, x_n+y_n).$$

Similarly, scalar - vector multiplication is also defined componentwise

$$\alpha X = (\alpha x_1, \ldots, \alpha x_n).$$

The zero vector is $0 = (0, \ldots, 0)$. It is straight forward to verify that all the axioms A, B, C are satisfied.

<u>Ex.</u> <u>$R^{m\times n}$ or M_{mn}</u> Let $R^{m\times n}$ be the set of all $m\times n$ matrices with real elements.

The sum $A+B$ of A and B is defined by

$$(A+B)_{ij} = A_{ij} + B_{ij} \qquad \begin{array}{l} i=1,\ldots,m \\ j=1,\ldots,n \end{array}$$

scalar-vector (matrix) multiplication is defined by

$$(\alpha A)_{ij} = \alpha A_{ij} \qquad \begin{array}{l} i=1,\ldots,m \\ j=1,\ldots,n \end{array}.$$

The zero "vector" i.e. matrix is $m\times n$ matrix O all of whose elements are zero:

$$O = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

Can verify that all axioms of a vector space are satisfied.

<u>Ex.</u> $V = C[a,b] = \{$ set of all real-valued functions that are continuous on $[a,b]\}$.

<u>Addition</u> $(f+g)(x) = f(x) + g(x)$

From Calculus, we know that the sum of two continuous is continuous. Hence $f+g$ is well defined i.e. it belongs to $C[a,b]$.

Scalar-vector multiplication is also defined as

$$(\alpha f)(x) = \alpha f(x)$$

Again, we know that if f is continuous on $[a,b]$, then the function αf is also continuous on $[a,b]$.

The 0 vector is the function that is identically zero in $[a,b]$. Again, all axioms (A), (B), (C) can be proven to hold.

<u>Ex.</u> <u>A set which is not a vector space</u>

Let $V = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$

Define addition $+$ by $X + Y = (x_1 + y_1, x_2 + y_2)$

Define scalar multiplication by $\alpha X = (\alpha x_1, 0)$.

We can show that all 4 axioms of (A) are satisfied. (B1), (C1) and (C2) are satisfied. However, (B2) is not satisfied in general. Indeed,

$$1 \cdot X = (1 \cdot x_1, 0) = (x_1, 0) \neq (x_1, x_2) = X.$$

<u>Ex.</u> An unusual (non intuitive) vector space.

Let V be the set of all <u>positive</u> real numbers

$x \oplus y \equiv xy$ vector addition is numerical multiplication

$\alpha \odot x = x^{\alpha}$ scalar-vector multiplication is exponentiation

All axioms of a vector space are satisfied:

(A) (1) $X \oplus Y = XY = YX = Y \oplus X$

(2) $X \oplus (Y \oplus Z) = X \oplus (Y Z) = X (Y Z)$

$$\|$$

$(X \oplus Y) \oplus Z = (X Y) \oplus Z = (X Y) Z$

(3) $X \oplus 1 = X 1 = X$ i.e. The zero vector of addition is the number 1.

(4) $X \oplus \frac{1}{X} = X \frac{1}{X} = 1$ i.e. The additive inverse of x is $\frac{1}{x}$.

(B) $\alpha X = X^{\alpha}$ is defined for any real number α and any vector X i.e. any positive real number X also X^{α} is positive and hence belongs to V.

(1) $\alpha(\beta X) = \alpha(X^{\beta}) = (X^{\beta})^{\alpha} = X^{\alpha\beta}$ by a rule of exponentiation

$(\alpha\beta) X = X^{\alpha\beta}$ ✓

(2) $1 . X = X^{1} = X$ ✓

(C)

(1) $\alpha(X \oplus Y) = \alpha(X Y) = (X Y)^{\alpha} = X^{\alpha} Y^{\alpha}$ rule of exponents

$= X^{\alpha} \oplus Y^{\alpha}$

$= (\alpha X) \oplus (\alpha Y)$ ✓

(2) $(\alpha + \beta) X = X^{(\alpha+\beta)} = X^{\alpha} . X^{\beta}$ rule of exponents

$= (\alpha X)(\beta X) = (\alpha X) \oplus (\beta X)$.

The following theorem contains some facts that hold in a general vector space and are easily derived from the axioms.

Theorem. 4.1.1 Let u, v be vectors in a vector space V and let α be a scalar. Then

(a) $0 u = \mathbf{0}$ The zero on the left is the scalar 0, on the right, the 0 vector

(b) $\alpha \mathbf{0} = \mathbf{0}$

(c) $(-1) u = -u$

(d) If $\alpha u = \mathbf{0}$, then $\alpha = 0$ or $u = \mathbf{0}$.

proof (a) $\quad 0u = (0+0)u \quad \Leftarrow \quad 0 = 0+0$ scalar zeros

$$= 0u + 0u \quad C2.$$

Add $-0u$ to both sides

$$\underbrace{0u + (-0u)}_{\mathbf{0}} = (0u+0u) + (-0u)$$
$$= 0 + \underbrace{0u + (-0u)}_{0} \quad \text{associativity}$$
$$= \mathbf{0} + \mathbf{0}$$
$$= \mathbf{0} \quad \checkmark$$

(b)

$$\alpha \mathbf{0} = \alpha(\mathbf{0} + \mathbf{0}) \qquad \mathbf{0} + \mathbf{0} = \mathbf{0}$$
$$= \alpha \mathbf{0} + \alpha \mathbf{0} \qquad C1.$$

Add $-\alpha \mathbf{0}$ to both sides and use Cancellation.

(c) $\quad (-1)u + u = (-1)u + (1)u \qquad B2$

$$= (-1+1)u \qquad C2$$
$$= 0u \qquad -1+1=0,$$
$$= \mathbf{0} \qquad \text{by part(a).}$$

Now $(-1)u + u = \mathbf{0}$ means that $(-1)u$ is the additive inverse of u, which is unique and is denoted by $-u$. ✓

(d) Suppose $\alpha u = \mathbf{0}$. If $\alpha = 0$, then there is nothing left to prove. So suppose $\alpha \neq 0$. Then α has a multiplicative inverse $\frac{1}{\alpha}$ such that $\alpha \cdot \frac{1}{\alpha} = 1$.

Hence

$$\alpha u = \mathbf{0} \Rightarrow \frac{1}{\alpha}(\alpha u) = \frac{1}{\alpha}\mathbf{0}$$

Now

$$\frac{1}{\alpha}(\alpha u) = \left(\frac{1}{\alpha}\alpha\right) u = 1 \cdot u = u \quad \text{by } B2$$

Also,

$$\frac{1}{\alpha}\mathbf{0} = \mathbf{0} \quad \text{by part (b) above.}$$

Hence $\quad u = \mathbf{0}$. ∎

§4.2 Subspaces

Defn A subset W of a vector space V is called a subspace of V if W is itself a vector space under the same operations of addition and scalar multiplication defined on V.

In order to verify that a subset W of V is a subspace of V, we need to verify the same 10 axioms of the vector space V.

(*) $u+v$ belongs to W if u and $v \in W$
$u+v=v+u$ for $u,v \in W$: inherited from V
$u+(v+w)=(u+v)+w$, $u,v,w \in W$: inherited from V
(*) The zero vector $\mathbf{0}$ belongs to W
(*) for $v \in W$, $-v \in W$
(*) $k \in \mathbb{R}$, $v \in W$, then $kv \in W$
$\alpha(\beta v)=(\alpha\beta)v$, $\alpha,\beta \in \mathbb{R}$, $v \in W$: inherited from V
$(\alpha+\beta)v=\alpha v+\beta v$, $\alpha,\beta \in \mathbb{R}$, $v \in W$: inherited from V
$\alpha(v+w)=\alpha v+\alpha w$, $\alpha \in \mathbb{R}$, $w \in W$: inherited from V
$1v=v$ $1 \in \mathbb{R}$, $v \in W$: inherited from V.

So we need only verify the 4 (*) conditions. It also turns out that we really need only verify 2 of these 4 conditions.

Theorem 4.2.1 If W is a set of one or more vectors in a vector space V, then W is a subspace of V if and only if the following conditions are satisfied

(a) If u and v belong to W, then $u+v$ belongs to W.
We say W is closed under addition

(b) If k is a scalar and v belongs to W, then kv belongs to W. We say W is closed under scalar multiplication.

proof. If W is a subspace of V, then all 10 axioms above are satisfied, including the 4 starred ones, and these include (a) and (b).

There are 2 of the ⊛ conditions, hence

Conversely, suppose (a) and (b) hold. We need to show that the remaining 2 starred conditions are satisfied. There are:

(i) $\mathbf{0}$ belongs to W.
(ii) if $v \in W$, then $-v$ belongs to W.

We will see that (i) and (ii) both follow from (b). Indeed, by Theorem 4.1.1 (a) $0v = \mathbf{0}$ $\forall v \in V$. So if we assume that $kv \in W$ if $k \in \mathbb{R}$ and $v \in W$, then $\mathbf{0} = 0v$ must belong to W.

Finally, we saw in Theorem 4.1.1 (c) that $-u = (-1)u$ $\forall u \in V$. So if we assume that kv belongs to W whenever $k \in \mathbb{R}$ and $v \in W$, then $-v$ must belong to W whenever v belongs to W. ∎

Examples of subspaces.

1) The zero subspace. Let V be any vector space. The set $W = \{0\}$ consisting of the zero vector of V is a subspace of V.

2) Lines through the origin in $\mathbb{R}^2$.

A line in $\mathbb{R}^2$ is characterized by the equation

$$ax + by + c = 0, \quad \text{for some constants } a, b, c.$$

We have a line through the origin iff $c = 0$.

$$W = \{(x,y) \in \mathbb{R}^2 \mid ax + by = 0\}.$$

We need to show as in Theorem 4.2.1 that the line W is closed under addition and scaler multiplication.

(a) Let $u, v \in W \Rightarrow a u_1 + b u_2 = 0$
$a v_1 + b v_2 = 0$

adding $\Rightarrow a(u_1 + v_1) + b(u_2 + v_2) = 0 \Rightarrow$

$$u+v=(u_1+v_1, u_2+v_2)\in W.$$

(b) let $k\in R$, $u\in W \Rightarrow au_1+bu_2=0$

mult. by $k \Rightarrow kau_1+kbu_2=0$

$\Rightarrow a\cdot(ku_1)+b(ku_2)=0$

$\Rightarrow ku=k(u_1,u_2)\in W.$

<u>Note</u> W is not a subspace if $c\neq 0$.

<u>Ex.</u> planes through the origin in R^3 are subspaces of R^3.

planes in R^3 are given by the equation $ax+by+cz+d=0$

$W=\{u=(u_1,u_2,u_3)\in R \mid au_1+bu_2+cu_3=0\}$

is a subspace.

<u>Ex.</u> $V=R^{2\times 2}$ equipped with usual addition and scalar multiplication of matrices

$W=$ set of all 2×2 upper triangular matrices

Let A & $B\in W$. Then clearly $A+B$ belongs to W since $A+B$ is upper triangular

$k\in R$, $A\in W \Rightarrow kA$ is upper triangular.

Hence the set W is a subspace of $R^{2\times 2}$.

<u>Ex. The Null space of a matrix.</u>

Let A be an $m\times n$ matrix. Recall that A represents a linear transformation T from R^n to R^m. We define the Null space or Kernel of A by

$$\text{Null}(A) = \text{Ker}(A) = \{ x \in R^n \mid Ax = 0 \}.$$

$\boxed{\text{Null}(A) \text{ is a subspace of } R^n}$. This is Theorems 4.2.4 and 4.2.5

(a) let $x, y \in \text{Null}(A)$. This means $Ax = 0$ and $Ay = 0$

$$\Rightarrow \underbrace{Ax + Ay}_{A(x+y)} = 0 + 0 = 0$$

$$A(x+y) = 0 \quad \Rightarrow x + y \in \text{Null}(A)$$

(b) let $x \in \text{Null}(A)$ and $k \in R$

$$\Downarrow$$

$$Ax = 0 \Rightarrow k(Ax) = k0 = 0$$

$$A(kx) = 0 \quad \Rightarrow kx \in \text{Null}(A).$$

<u>Ex.</u> <u>The Range of a matrix</u> or <u>column space</u> of a matrix is defined as

$$\text{Range}(A) = \{ y \in R^m \mid y = Ax \text{ for some } x \in R^n \}.$$

$\boxed{\text{Range}(A) \text{ is a subspace of } R^m.}$

(a) let $y, z \in \text{Range}(A) \Rightarrow Ax = y$ for some $x \in R^n$
$Aw = z$ for some $w \in R^n$

Add: $Ax + Aw = y + z$

$$A(x+w) = y + z \Rightarrow y + z \in \text{Range}(A)$$

Now $x + w \in R^n$, hence $y+z$ is the image of $x+w$)

(b) $y \in \text{Range}(A)$, $k \in R$

$$\Downarrow$$

$$y = Av \text{ for some } v \in R^n$$

$$ky = kAv = A(kv) \Rightarrow ky \in \text{Range}(A)$$

Thus Range(A) is a subspace of R^m.

Theorem 4.2.2 The intersection of two (or more) subspaces of a vector space V is a subspace of V.

proof

let W_1 and W_2 be two subspaces of V. Suppose $W_1 \cap W_2$ is nonempty.

(a) let $u, v \in W_1 \cap W_2$. want to show $u+v \in W_1 \cap W_2$

$$u \in W_1 \cap W_2 \Rightarrow u \in W_1 \text{ and } u \in W_2$$
$$v \in W_1 \cap W_2 \Rightarrow v \in W_1 \text{ and } v \in W_2$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\underbrace{\boxed{u+v \in W_1}}_{\text{since } W_1 \text{ is a subspace}} \text{ and } \underbrace{\boxed{u+v \in W_2}}_{\text{since } W_2 \text{ is a subspace}} \Rightarrow u+v \in W_1 \cap W_2.$$

(b) let $k \in R$, $u \in W_1 \cap W_2$. want to show $ku \in W_1 \cap W_2$.

$$u \in W_1 \cap W_2 \Rightarrow \begin{cases} u \in W_1 \Rightarrow ku \in W_1, & W_1 \text{ is a subspace} \\ \text{and} \\ u \in W_2 \Rightarrow ku \in W_2, & W_2 \text{ is a subspace} \end{cases}$$

$$\Downarrow$$

$$ku \in W_1 \cap W_2$$

Defn (a) Let $\{v_1, \dots, v_m\}$ be a set of vectors in a vector space V.

a linear combination of $v_1, \dots, v_m$ is a sum of the form

$$k_1 v_1 + k_2 v_2 + \cdots + k_m v_m. \in V.$$

(b) The span of $\{v_1, \dots, v_m\}$ is the set of all linear combinations of $v_1, \dots, v_m$, i.e.

$$\text{span}\{v_1, \dots, v_m\} = \{v \in V \mid v = k_1 v_1 + \cdots + k_m v_m,\ k_1, \dots, k_m \in R\}.$$

Theorem 4.2.3 If $S=\{w_1, w_2, \dots, w_r\}$ is a nonempty collection of vectors in a vector space V, then

(a) span$\{S\}$ is a subspace of V.

(b) span$\{S\}$ is the smallest subspace of V that contains the set S.

proof

(a) we need to show that span$\{W\}$ is closed under addition and scalar multiplication.

let $w \in \text{span}\{S\} \Rightarrow w = k_1 w_1 + k_2 w_2 + \dots k_r w_r$
let $z \in \text{span}\{S\} \Rightarrow z = \alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_r w_r$

add $\qquad w+z = (k_1+\alpha_1)w_1 + (k_2+\alpha_2)w_2 + \dots + (k_r+\alpha_r)w_r$

$(k_1+\alpha_1)w_1 + \dots + (k_r+\alpha_r)w_r$ is also a linear combination of $w_1, \dots, w_r$ and hence belongs to span$\{S\}$.

let $k \in R$, $v \in \text{span}\{S\} \Rightarrow v = \beta_1 w_1 + \beta_2 w_2 + \dots + \beta_r w_r$

$$kv = k(\beta_1 w_1 + \beta_2 w_2 + \dots + \beta_r w_r)$$

$$= (k\beta_1)w_1 + (k\beta_2)w_2 + \dots + (k\beta_r)w_r$$

$k\beta_1, \dots, k\beta_r$ are scalars, $(k\beta_1)w_1 + \dots + (k\beta_r)w_r$ is a linear combination of $w_1, \dots, w_r$; hence $kv \in \text{span}\{S\}$.

(b) we can accomplish this by showing that if W is any subspace of V that contains the set S, then span$(S) \subseteq W$.

let $v \in \text{span}(S)$. This means v is a linear combination of $w_1, \dots, w_r$, i.e.

$$v = \alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_r w_r.$$

Since $w_1, \dots, w_r$ all belong to W, and W is a subspace, i.e. W is closed under addition and scalar multiplication, v must belong to W. we have $v \in \text{Span}(S) \Rightarrow v \in W$ means span$(S) \subseteq W$.

One of the important issues in Linear Algebra is the following: Given a vector space V (or a subspace W of V) find a set of vectors $S = \{v_1, \ldots, w_r\}$ such that

$$\text{span}\{S\} = V. \quad (\text{or } \text{span}(S) = W)$$

Ex. Determine whether $\text{span}\{v_1, v_2, v_3\} = \mathbb{R}^3$, where

$$v_1 = (1,1,2), \; v_2 = (1,0,1), \; v_3 = (2,1,5).$$

We want to see if every vector (a, b, c) in $\mathbb{R}^3$ can be written as a linear combination of v_1, v_2, v_3.
some

Equivalently, we want to see if we can find scalars x_1, x_2, x_3 such that

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = (a, b, c)$$

i.e.

$$x_1(1,1,2) + x_2(1,0,1) + x_3(2,1,5) = (a, b, c)$$

i.e.

$$(x_1 + x_2 + 2x_3,\; x_1 + 0 + x_3,\; 2x_1 + x_2 + 5x_3) = (a, b, c)$$

i.e.

$$\begin{aligned} x_1 + x_2 + 2x_3 &= a \\ x_1 + + x_3 &= b \\ 2x_1 + x_2 + 5x_3 &= c \end{aligned}$$

~~In other words $\text{span}\{v_1, v_2, v_3\} = \mathbb{R}^3$ if and only if~~

The system above has a solution for every vector (a, b, c).

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & 0 & 1 & b \\ 2 & 1 & 5 & c \end{array}\right].$$

This question can be settled by Gaussian Elimination

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & 0 & 1 & b \\ 2 & 1 & 5 & c \end{array}\right] \xrightarrow[(-2)r_1+r_3]{(-1)r_1+r_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & -1 & 1 & c-2a \end{array}\right]$$

$$\Big\downarrow (-1)r_2+r_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & 0 & 2 & c-2a-b+a \end{array}\right]$$

This system has a unique solution for any given right hand side (a,b,c). Hence, we conclude that

$$\text{span}\{(1,1,2),(1,0,1),(2,1,5)\} = R^3.$$

<u>Ex.</u> Repeat with V_3 replaced by $(2,1,3)$.

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & -1 & -1 & c-2a \end{array}\right] \xrightarrow{(-1)r_2+r_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & 0 & 0 & c-2a-b+a=c-a-b. \end{array}\right]$$

we see that <u>now</u> the system is solvable only if

$$\boxed{c-a-b=0}.$$

The set of vectors (a,b,c) satisfying $c-a-b=0$ i.e $c=a+b$, is a <u>proper</u> subset of R^3.

i.e. there are vectors in R^3 that do not satisfy this condition. Hence

$$\text{span}\{(1,1,2),(1,0,1),(2,1,3)\} \neq R^3.$$

Actually, $\text{span}\{(1,1,2),(1,0,1),(2,1,3)\} \equiv W$

$W = \{(a,b,c) \in R^3 \mid c=a+b\}$ is a subspace of R^3

as can be shown directly.

§4.3 Linear Independence

The concept of linear independence is of central importance.

It refers to a set of vectors $\{v_1, \ldots, v_r\}$ in a vector space V. Recall that with such a set of vectors, we can form their span $\operatorname{span}\{v_1, \ldots, v_r\}$ which is the set of all possible linear combinations of $v_1, \ldots, v_r$. We showed in the previous section that $\operatorname{span}\{v_1, \ldots, v_r\}$ is a vector subspace of V.

At this point, we ask the following question:

"Is every $v_1, \ldots, v_r$ essential/necessary in building $\operatorname{span}\{v_1, \ldots, v_r\}$?"

"Is there some element(s) of the set $\{v_1, \ldots, v_r\}$ which is redundant, i.e. can be written/expressed as a linear combination of the remaining vectors?"

Ex. Consider the set of 3 vectors $\{(1,1,2), (1,0,1), (2,1,3)\}$ in R^3.

We see/observe that the third vector, $(2,1,3)$, can be expressed as a linear combination of the other two; indeed.

$$(2,1,3) = 1\cdot(1,1,2) + 1\cdot(1,0,1).$$

In some sense, the vector $(2,1,3)$ is "redundant". It is a linear combination of the other two.

Hence

$$\operatorname{span}\{(1,1,2), (1,0,1), (2,1,3)\} = \operatorname{span}\{(1,1,2), (1,0,1)\}.$$

We say that the set $\{(1,1,2), (1,0,1), (2,1,3)\}$ is not linearly independent; equivalently, we say that it is linearly dependent.

We shall next give a formal definition of linear independence. It is designed to "detect" redundancy is given set of vectors.

<u>Defn.</u> Given the set of vectors $\{v_1, .., v_r\}$ in a vector space V, we say that it is a <u>linearly independent</u> set if the linear combination $k_1 v_1 + k_2 v_2 + \cdots + k_r v_r$ is zero, then necessarily $k_1 = k_2 = \cdots = k_r = 0$.

If the set is not linearly independent, we say that it is <u>linearly dependent</u>. In this case, it is possible to find a set of scalars $k_1, k_2, .., k_r$ <u>not all zero</u> such that $k_1 v_1 + k_2 v_2 + \cdots + k_r v_r = 0$

An instructive way of interpreting this definition is:

If the set $\{v_1, .., v_r\}$ is linearly independent, then if we omit a single vector from the list, then the span of the reduced set is a <u>proper</u> subspace of $\text{span}\{v_1, .., v_r\}$.

If the set $\{v_1, .., v_r\}$ is linearly dependent, then there must be at least one member of the set whose omission from the set does not alter $\text{span}\{v_1, .., v_r\}$.

After this long winded discussion of the concept, we ask: How does one determine whether a given set of vectors is linearly independent or dependent. It turns out that there is a straightforward way of finding the answer using the matrix techniques we learned earlier.

<u>Plan</u>: Given the set $\{v_1, .., v_r\}$

(i) write $k_1 v_1 + \cdots + k_r v_r = 0$, $k_1, .., k_r$ scalars

(ii) Translate the above equation into a system of equations in the <u>unknowns</u> $k_1, .., k_r$. The system will be <u>homogeneous</u> and will be will be <u>square</u>, i.e. r equations in the r unknowns $k_1, \dots, k_r$.

(iv) If the only solution (recall homogeneous systems always have solutions) is the zero or trivial solution, then the set $\{v_1, \dots, v_r\}$ is linearly independent. Otherwise, there will be a nonzero solution, i.e. not all k's zero. In this case the set is linearly dependent.

<u>Ex.</u> Determine if the set $\{(-3,0,4), (5,-1,2), (1,1,3)\}$ is linearly independent in R^3.

(i) $k_1(-3,0,4) + k_2(5,-1,2) + k_3(1,1,3) = (0,0,0)$

$\Downarrow$

(ii) $(-3k_1 + 5k_2 + k_3,\ 0k_1 - k_2 + k_3,\ 4k_1 + 2k_2 + 3k_3) = (0,0,0)$

$$\Rightarrow \begin{aligned} -3k_1 + 5k_2 + k_3 &= 0 \\ -k_2 + k_3 &= 0 \\ 4k_1 + 2k_2 + 3k_3 &= 0 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} -3 & 5 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 4 & 2 & 3 & 0 \end{array}\right]$$

$$\Big\downarrow P_{23}$$

$$\left[\begin{array}{ccc|c} -3 & 5 & 1 & 0 \\ 0 & 26/3 & 7 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \xleftarrow{\frac{4}{3}r_1 + r_2} \left[\begin{array}{ccc|c} -3 & 5 & 1 & 0 \\ 4 & 2 & 3 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$$\Big\downarrow \tfrac{3}{26}r_2 + r_3$$

$$\left[\begin{array}{ccc|c} -3 & 5 & 1 & 0 \\ 0 & \frac{26}{3} & 7 & 0 \\ 0 & 0 & \frac{47}{26} & 0 \end{array}\right] \Rightarrow k_3 = k_2 = k_1 = 0$$

Thus, set is linearly independent.

Remark The system that is obtained in checking independence is of the form

$$AK=0$$

where A is $r\times r$. We know that the solution vector K will be zero if and only if A is invertible.

So if you can determine somehow (correctly of course!) that A is invertible, then you don't have to go through the chore of solving the system!.

Ex. Determine if the set $\{2-x+4x^2, 3+6x+2x^2, 2+10x-4x^2\}$ is linearly independent in P_2.

(i) $k_1(2-x+4x^2)+k_2(3+6x+2x^2)+k_3(2+10x-4x^2)=0$

Note It is important to know what is meant by the equality to zero in this context: We mean that the left-hand-side must be equal to zero for any x.

(ii) To obtain the system, we gather terms in equal powers of x

$$\Rightarrow 2k_1+3k_2+2k_3+(-k_1+6k_2+10k_3)x+(4k_1+2k_2-4k_3)x^2=0$$

We need to also use a principle from Calculus which states that a polynomial is equal to zero for all x, if and only if all its coefficients must be zero.

$$\Rightarrow \begin{aligned} 2k_1+3k_2+2k_3&=0\\ -k_1+6k_2+10k_3&=0\\ 4k_1+2k_2-4k_3&=0\end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 2 & 0\\ -1 & 6 & 10 & 0\\ 4 & 2 & -4 & 0\end{array}\right]$$

$$\Big\downarrow \begin{array}{l}\tfrac{1}{2}r_1+r_2\\ -2r_1+r_3\end{array}$$

Thus set is linearly independent.

$$\left[\begin{array}{ccc|c} 2 & 3 & 2 & 0\\ 0 & \frac{15}{2} & 11 & 0\\ 0 & 0 & \frac{88}{15} & 0\end{array}\right] \xleftarrow{\frac{8}{15}r_2+r_3} \left[\begin{array}{ccc|c} 2 & 3 & 2 & 0\\ 0 & \frac{15}{2} & 11 & 0\\ 0 & -4 & 0 & 0\end{array}\right]$$

Ex. Determine all values of α for which the set

$$\left\{\begin{bmatrix}1 & 0\\ 1 & \alpha\end{bmatrix}, \begin{bmatrix}-1 & 0\\ \alpha & 1\end{bmatrix}, \begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix}\right\}$$ is linearly dependent in $\mathbb{R}^{2\times 2}$.

(i) $$k_1\begin{bmatrix}1 & 0\\ 1 & \alpha\end{bmatrix} + k_2\begin{bmatrix}-1 & 0\\ \alpha & 1\end{bmatrix} + k_3\begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \leftarrow \text{zero vector in } \mathbb{R}^{2\times 2}$$

(ii) $$\Rightarrow \begin{bmatrix}k_1 & 0\\ k_1 & k_1\alpha\end{bmatrix} + \begin{bmatrix}-k_2 & 0\\ k_2\alpha & k_2\end{bmatrix} + \begin{bmatrix}2k_3 & 0\\ k_3 & 3k_3\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix}k_1 - k_2 + 2k_3 & 0\\ k_1 + k_2\alpha + k_3 & k_1\alpha + k_2 + 3k_3\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{aligned} k_1 - k_2 + 2k_3 &= 0\\ k_1 + k_2\alpha + k_3 &= 0\\ k_1\alpha + k_2 + 3k_3 &= 0\end{aligned} \Rightarrow \left[\begin{array}{ccc|c}1 & -1 & 2 & 0\\ 1 & \alpha & 1 & 0\\ \alpha & 1 & 3 & 0\end{array}\right]$$

$\boxed{\text{No need for} \quad 0 = 0}$

or $A k = 0$

Of course we can apply Gaussian Elimination to the above system. However, let us follow a different approach! We know the solution will be zero if and only if A is invertible $\Leftrightarrow$ $\det(A) \neq 0$.
So let's calculate $\det(A)$

$$\left[\begin{array}{ccc|cc}1 & -1 & 2 & 1 & -1\\ 1 & \alpha & 1 & 1 & \alpha\\ \alpha & 1 & 3 & \alpha & 1\end{array}\right] \rightarrow \begin{aligned}&(1)(\alpha)(3) + (-1)(1)(\alpha) + (2)(1)(1)\\ &- \left[(2)(\alpha)(\alpha) + (1)(1)(1) + (-1)(1)(3)\right] = \det(A)\end{aligned}$$

$$\Rightarrow \det(A) = -2\alpha^2 + 2\alpha + 4 = -2(\alpha^2 - \alpha - 2) = -2(\alpha+1)(\alpha-2)$$

$$\Rightarrow \boxed{\det(A) = 0 \Leftrightarrow \alpha = -1 \text{ or } \alpha = 2}.$$ Thus

set is linearly dependent for $\alpha = -1$ or $\alpha = 2$, and
" " " independent for $\alpha \neq -1$ and $\alpha \neq 2$.

The following results are easily derived from the definition.

Theorem 4.3.2

(a) Any finite set of vectors that contains 0 is linearly dependent

(b) A set with exactly one vector is linearly independent iff that vector is not zero.

(c) A set of exactly two vectors is linearly dependent iff one of the two vectors is a scalar multiple of the other

(d) If the set $S = \{v_1, .., v_r\}$ is linearly independent, then so is any nonempty subset S

(e) If the set $S = \{v_1, .., v_r\}$ is linearly dependent, then the set obtained by augmenting S by vectors not in S is also linearly dependent.

(f) If $S = \{v_1, .., v_r\}$ is linearly independent and v_{r+1} is any (nonzero) vector not in span(S), then the augmented set $S' = \{v_1, .., v_r, v_{r+1}\}$ is linearly independent.

proof

(a) Let $S = \{v_1, .., v_r, 0\}$. Then we have

$$0 \cdot v_1 + \cdots + 0 v_r + 1\,0 = 0.$$

Since the collection of scalars $0, \cdots, 0, 1$ is not all zero, the set S is linearly dependent.

(b) Suppose $k_1 v_1 = 0$. Then we know $k_1 = 0$ or $v_1 = 0$. If $k_1 = 0$ and $v_1 \neq 0$, then $\{v_1\}$ is linearly independent. If $k_1 \neq 0$, then $v_1 = 0 \Rightarrow \{v_1\}$ is linearly dependent.

We thus see that $v_1 \neq 0$ if and only if $\{v_1\}$ is lin. indep.

(c) Let $S = \{v_1, v_2\}$. Suppose S is linearly dependent. Then $\exists\ k_1, k_2$ not both zero such that

$$k_1 v_1 + k_2 v_2 = 0.$$

We are saying one of the two scalars k_1, k_2 is not zero. Without loss of generality, assume $k_1 \neq 0$. Then

$$v_1 = -\frac{k_2}{k_1} v_2 \quad \text{which is a scalar multiple of } v_2.$$

Conversely, we want to show that if one of the two vectors v_1, v_2 is a scalar multiple of the other, then the set $S=\{v_1, v_2\}$ is linearly dependent.

First, note that if $v_1 = 0$ and/or $v_2 = 0$, then the set S is linearly dependent by (a) S is linearly dependent so we are done with the proof. Hence assume v_1 and v_2 are both nonzero in addition to the assumption that one of the two is a scalar multiple of the other. Say

$$v_2 = k\, v_1 .$$

Obviously k cannot be zero since otherwise $v_2 = 0 \cdot v_1 = 0$ which is contrary to the assumption made above.
Hence

$$v_2 = k v_1 \Rightarrow k\, v_1 - v_2 = 0.$$ Since the scalars $k, -1$ are not all zero, the set S is lin. dependent.

(d) Suppose the set $S = \{v_1, v_2, \ldots, v_r\}$ is linearly independent. Let

$$S' = \{v_1', v_2', \ldots, v_{r'}'\} \qquad r' < r$$

be any proper and nonempty subset of S. Suppose

$$k_1' v_1' + k_2' v_2' + \cdots + k_{r'}' v_{r'} = 0 .$$

Now "complete" this equation by adding zeros to

$$k_1' v_1' + k_2' v_2' + \cdots + k_{r'}' v_{r'}' + 0 \cdot v_{r'+1} + \cdots + 0 v_r = 0 .$$

Note that the augmented sum is a linear combination of the elements of S. Since S is linearly independent, all the scalars in the linear combination must be zero

$$\Rightarrow k_1' = \cdots = k_{r'}' = 0,$$ which shows that S' is linearly independent.

(e) Suppose the set $S = \{v_1, \ldots, v_r\}$ is linearly dependent. This means there exist scalars $k_1, \ldots, k_r$, not all zero such that

$$k_1 v_1 + k_2 v_2 + \cdots + k_r v_r = 0.$$

Now if $v_{r+1}, \ldots, v_p$ are vectors not in span(S), we again can augment the above equation to

$$\underbrace{k_1 v_1 + \cdots + k_r v_r}_{0} + 0 \cdot v_{r+1} + \cdots + 0 \cdot v_p = 0$$

The set of scalars $k_1, k_2, \ldots, k_r, 0, \ldots, 0$ obviously contains some nonzero terms (coming from $k_1, \ldots, k_r$). Hence the set $\{v_1, \ldots, v_r, v_{r+1}, \ldots, v_p\}$ is linearly dependent.

(f) Suppose the set $S = \{v_1, \ldots, v_r\}$ is linearly independent and $v_{r+1} \notin \text{span}\{v_1, \ldots, v_r\}$. We want to show

the set $\{v_1, \ldots, v_r, v_{r+1}\}$ is linearly independent.

Suppose otherwise, i.e. that this set is linearly dependent. Then, there must exist $r+1$ scalars $k_1, \ldots, k_r, k_{r+1}$ not all zero such that

$$k_1 v_1 + \cdots + k_r v_r + k_{r+1} v_{r+1} = 0.$$

We consider the two cases;

Case I $\boxed{k_{r+1} \neq 0}$. Then $v_{r+1} = -\left(\frac{k_1}{k_{r+1}}\right) v_1 - \left(\frac{k_2}{k_{r+1}}\right) v_2 - \cdots - \left(\frac{k_r}{k_{r+1}}\right) v_r$

which says that v_{r+1} is a linear combination of $v_1, \ldots, v_r$. This obviously contradicts the assumption made above that $v_{r+1} \notin \text{span}\{v_1, \ldots, v_r\}$.

Case II $\boxed{k_{r+1} = 0}$. We assumed that the set $\{k_1, \ldots, k_r, k_{r+1}\}$ must have at least one nonzero member. Since $k_{r+1} = 0$, there must be at least one nonzero scalar among $\{k_1, \ldots, k_r\}$. But then $k_1 v_1 + \cdots + k_r v_r = 0$. Since there is a nonzero among $k_1, \ldots, k_r$, the set $\{v_1, \ldots, v_r\}$ is linearly dependent. But this is a contradiction. $\blacksquare$