Chapter 2 : Determinants of square matrices

§2.0 permutations and determinants

The definition of a determinant that is given here relies on the concept of permutations

Defn. Given a positive integer n (the size of the matrix) consider the set $\{1,2,\dots,n\}$. A permutation is a one-to-one map from $\{1,2,\dots,n\}$ into $\{1,2,\dots,n\}$, i.e. a map π that assigns to each integer $i\in\{1,2,\dots,n\}$ an integer j_i from $\{1,2,\dots,n\}$. π is one-to-one, i.e

$$\text{if} \quad \pi(i)=\pi(j) \quad \text{then} \quad i=j.$$

Since the set $\{1,2,\dots,n\}$ is finite, it turns out that π is also onto.

Ex. $n=4$ $\quad \pi(1)=4,\ \pi(2)=3,\ \pi(3)=1,\ \pi(4)=2$

It is easy to see that for given n, there are exactly $n!$ distinct permutations

$n=2$

$1,2 \to 1,2$ identity permutation

$1,2 \to 2,1$

$n=3$

$1,2,3 \to 1,2,3$ identity permutation

$1,2,3 \to 1,3,2$

$1,2,3 \to 2,1,3$

$1,2,3 \to 2,3,1$

$1,2,3 \to 3,1,2$

$1,2,3 \to 3,2,1$

Thus as the name suggests, a permutation is a reordering of the integers $1,2,\dots,n$.

Composition of permutations Given 2 permutation, say π and σ we can define $\pi\sigma$ as follows:

$$(\pi\sigma)(i)=\pi(\sigma(i)),\ i=1,\dots,n$$

Ex. $n = 4$

	π	σ	$\pi\sigma$
1	4	2	2
2	2	3	1
3	1	4	3
4	3	1	4

$(\pi\sigma)(1) = \pi(\sigma(1)) = \pi(2) = 2$
$(\pi\sigma)(2) = \pi(\sigma(2)) = \pi(3) = 1$
$(\pi\sigma)(3) = \pi(\sigma(3)) = \pi(4) = 3$
$(\pi\sigma)(4) = \pi(\sigma(4)) = \pi(1) = 4$

	σ	π	$\sigma\pi$
1	2	4	1
2	3	2	3
3	4	1	2
4	1	3	4

$(\sigma\pi)(1) = \sigma(\pi(1)) = \sigma(4) = 1$
$(\sigma\pi)(2) = \sigma(\pi(2)) = \sigma(2) = 3$
$(\sigma\pi)(3) = \sigma(\pi(3)) = \sigma(1) = 2$
$(\sigma\pi)(4) = \sigma(\pi(4)) = \sigma(3) = 4$

This example also shows that composition (or product) of permutations is not commutative. However it is associative

$$\boxed{(\pi\sigma)\tau = \pi(\sigma\tau)}$$

Remark The integers $1, 2, \ldots, n$ could be replaced by any collection of n distinct objects without affecting any of the concepts above.

Ex. {red, blue, green} or {R, B, G}

R	R	B	B	G	G
B	G	R	G	R	B
G	B	G	R	B	R

we still get $6 = 3!$ distinct permutations.

The identity permutation is the simplest permutation and is defined by

$$\boxed{\epsilon(i) = i \quad , \quad i = 1, \ldots, n}$$

Indeed, it is easy to see that if π is any permutation

Then $\quad \epsilon \pi = \pi \epsilon = \pi$.

The inverse of a permutation. Let π be a permutation. Then, there is a unique inverse permutation denoted by π^{-1}. We define it as follows: For each i there is a unique j such that $\pi(i) = j$. We define $\pi^{-1}(j) = i$.

Ex. $n=4$

	π	π^{-1}
1	4	3
2	2	2
3	1	4
4	3	1

$\pi(1) = 4 \qquad \pi^{-1}(4) = 1$
$\pi(2) = 2 \qquad \pi^{-1}(2) = 2$
$\pi(3) = 1 \qquad \pi^{-1}(1) = 3$
$\pi(4) = 3 \qquad \pi^{-1}(3) = 4$

We have

$$\boxed{\pi \pi^{-1} = \pi^{-1} \pi = \epsilon} = \text{The identity}$$

Transpositions are simple permutations that consist of a single interchange. More precisely let $p \neq q$ be two distinct integers from $\{1, 2, \dots, n\}$. Then a transposition τ is defined by

$$\begin{cases} \tau(p) = q \\ \tau(q) = p \\ \tau(i) = i \qquad i \neq p,\ i \neq q. \end{cases}$$

We shall also use the notation (p, q) for the trans.

Ex. $(1, 3, 2)$, $(2, 1, 3)$, $(3, 2, 1)$ are transpositions

$(1, 2, 3)$, $(2, 3, 1)$, $(3, 1, 2)$ are not.

For a given n, there are $(n-1)n/2$ transpositions, far fewer, for n large, than the total number $n!$ of permutations.

It is also easy to see that if τ is a transposition, then $\tau^2 = \epsilon$, the identity permutation, in other words a transposition is its own inverse.

It is not difficult to show that every permutation other than ϵ is the product of transpositions.

Ex. $n = 5$ $\pi = (5, 3, 1, 2, 4) = (\pi(1), \pi(2), \pi(3), \pi(4), \pi(5))$

can be expressed as $\underset{\tau_4}{(1,5)}\,\underset{\tau_3}{(2,3)}\,\underset{\tau_2}{(3,5)}\,\underset{\tau_1}{(4,5)}\,\epsilon$

$\epsilon \xrightarrow{\tau_1 = (4,5)} \tau_1 \xrightarrow{\tau_2 = (3,5)} \tau_2\tau_1 \xrightarrow{\tau_3 = (2,3)} \tau_3\tau_2\tau_1 \xrightarrow{\tau_4 = (1,5)} \tau_4\tau_3\tau_2\tau_1 = \pi$

ϵ	τ_1	$\tau_2\tau_1$	$\tau_3\tau_2\tau_1$	$\tau_4\tau_3\tau_2\tau_1$	
1	1	1	1	5	
2	2	2	3	3	
3	3	5	5	1	$= \pi$
4	5	3	2	2	
5	4	4	4	4	

parity of a permutation It turns out that a permutation can be characterized as either even or odd. This relies on the following result

Theorem 2.0.1 A permutation π can be expressed as the product of transpositions in more than one way. However, the parity of the number of the transpositions in any such representations depends only on π, i.e. it is either odd or even. $\square$

Defn We assign the identity permutation ϵ the parity of even. For all other permutations, the parity of the permutation π is the parity of the number of transpositions in any representation of π as the product of transpositions.

Ex. We saw above that $\pi \leftrightarrow (5,3,1,2,4)$ can be expressed as

$$\pi = (1,5)(2,3)(3,5)(4,5).$$

This product has 4 transpositions so parity (π) is even. Indeed, no matter what representation of π we may come up with, it will always contain an even number of transpositions.

Definition of the determinant Given an $n \times n$ matrix A, the determinant of A denoted $\det(A)$ or $|A|$ is the real number

$$\det(A) = \sum_{\pi} \operatorname{sign}(\pi) \cdot a_{1,\pi(1)}\, a_{2,\pi(2)}, \dots, a_{n,\pi(n)}.$$

The sum is taken over all permutations π and $\operatorname{sign}(\pi) = -1$ if parity (π) is odd and $\operatorname{sign}(\pi) = 1$ if parity (π) is even.

Ex. $n = 2$.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

In this case, we have two permutations

π	# of transpositions	parity	sign of π
$(1,2)$	0	even	1
$(2,1)$	1	odd	-1

$$\Rightarrow \det(A) = +a_{11}\, a_{22} + (-1)\, a_{12} \cdot a_{21} = \boxed{a_{11} a_{22} - a_{12} a_{21}}$$

$n = 3$ There are 6 permutations

π	# of transpositions	parity (π)	sign (π)
$(1,2,3)$	0	even	$+$
$(1,3,2)$	1	odd	$-$

(2, 1, 3)	1	odd	−
(2, 3, 1)	2	even	+
(3, 1, 2)	2	even	+
(3, 2, 1)	3 or 1	odd	−

$$\det(A) = \overset{①}{+a_{11}a_{22}a_{33}} \overset{②}{+(-1)a_{11}a_{23}a_{32}} \overset{③}{+(-1)a_{12}a_{21}a_{33}}$$
$$\overset{④}{+(+1)a_{12}a_{23}a_{31}} \overset{⑤}{+(+1)a_{13}a_{21}a_{32}} \overset{⑥}{+(-1)a_{13}a_{22}a_{31}}$$

Special formula for 3x3 case

$$\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

Attach first 2 columns of A to A as shown

Multiply 3 elements along diagonals: Attach (+) to lower slanting diagonals
Attach (−) to upward slanting diagonals

$$\Rightarrow \det(A) = \overset{①}{a_{11}a_{22}a_{33}} + \overset{④}{a_{12}a_{23}a_{31}} + \overset{⑤}{a_{13}a_{21}a_{32}}$$
$$- \overset{⑥}{a_{13}a_{22}a_{31}} - \overset{②}{a_{11}a_{23}a_{32}} - \overset{③}{a_{12}a_{21}a_{33}}$$

General principles: For an $n \times n$ matrix A

1) A determinant is the sum of $n!$ terms
2) Each term is the product of one and only one term from each row
   and " " " " " " " " from each column.
3) The sign associated with the term depends on the parity of the permutation of the column indexes.
   Note that this is independent of the signs of the a's.
4) Exactly half the signs are + ⇒ the remaining terms are (−)

Remark As is often the case, computing something using the definition is quite computationally intensive. Later, we will learn much more efficient ways of computing determinants.

On the other hand, we can easily compute determinants of some special cases.

Theorem 2.0.2 A be an $n \times n$ matrix

a) If A has a row of zeros, then $\det(A) = 0$

b) If A has a column of zeros, then $\det(A) = 0$

c) If A is lower or upper triangular, then

$$\det(A) = a_{11} a_{22} \cdots a_{nn}.$$

proof

a) By definition, $\det(A) = \sum_{\pi} \operatorname{sign}(\pi)\, a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)}$.

Suppose row i of A consists of zeros. Then each term in the above sum contains $a_{i,\pi(i)} = 0$. Hence $\det(A) = 0$.

b) Suppose column j of A consists of zeros. Each term in the sum must contain an element from column j. The result follows.

c) Suppose A is lower triangular

we claim that all terms in the sum, with the exception of $a_{11} a_{22} \cdots a_{nn}$ must be zero. Indeed, consider the generic term

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & & 0 \\ \vdots & \vdots & \ddots & 0 \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\text{term} = a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)}$$

If $\pi(1) \neq 1$, then term $= 0$. So we must take $\pi(1) = 1$.

Now since $\pi(1)=1$, we cannot take $\pi(2)=1$.
Also, $\pi(2)=3,\ldots,n$ will imply $a_{2,\pi(2)}=0$.
Hence we must take $\pi(2)=2$.
For $\pi(3)$, we cannot take $\pi(3)=1,2$ ← already taken
Also, $\pi(3)=4,5,\ldots,n$ will imply $a_{3,\pi(3)}=0$.
Hence we must take $\pi(3)=3$.

This argument can be continued till we reach the conclusion that $a_{1,\pi(1)}a_{2,\pi(2)}\cdots a_{n,\pi(n)}=0$ unless

$\pi(i)=i$, $i=1,\ldots,n$, i.e. $\pi=\epsilon$, the identity permutation its parity is even. Hence

$$\det(A) = +a_{11}a_{22}\cdots a_{nn} + (\text{a whole bunch of zeros})$$

The proof for upper triangular A is similar.
we show

$$\pi(n)=n \quad \text{otherwise} \quad a_{n,\pi(n)}=0 \Rightarrow \text{take } \pi(n)=n$$

For $a_{n-1,\pi(n-1)}$, cannot take $\pi(n-1)=n$ since $\pi(n)=n$.

Also, $\pi(n-1)=1,\ldots,n-2 \Rightarrow a_{n-1,\pi(n-1)}=0$.
Hence

$$\pi(n-1)=n-1 \quad \text{otherwise term}=0.$$

proceeding this way, we show that

$$\pi(i)=i,\ i=1,\ldots,n \quad \text{otherwise term}=0.$$

$$\Rightarrow \det(A) = a_{11}a_{22}\cdots a_{nn} + \text{zeros}. \quad \blacksquare$$

Another important result that can be established from the definition is

Theorem 2.0.3 $\det(A^T)=\det(A)$.

<u>proof.</u> Let $B = A^T$.

$$\det(B) = \sum_{\pi} \operatorname{sign}(\pi)\, b_{1,\pi(1)}\, b_{2,\pi(2)} \cdots b_{n,\pi(n)}$$

$$= \sum_{\pi} \operatorname{sign}(\pi)\, a_{\pi(1),1}\, a_{\pi(2),2} \cdots b_{\pi(n),n}$$

since π is one-to-one and onto, $\exists$ uniquely defined integers $j_1, \ldots, j_n$ such that

$$\pi(j_1) = 1,\ \pi(j_2) = 2,\ \ldots,\ \pi(j_n) = n$$

and hence

$$\pi^{-1}(1) = j_1,\ \pi^{-1}(2) = j_2,\ \ldots,\ \pi^{-1}(n) = j_n.$$

Reordering the terms in the product, we get

$$\det(B) = \sum_{\pi} \operatorname{sign}(\pi)\, a_{1,\pi^{-1}(1)}\, a_{2,\pi^{-1}(2)} \cdots a_{n,\pi^{-1}(n)}.$$

Given a permutation π, its inverse π^{-1} is uniquely defined and has the same parity. Hence

$$\det(B) = \sum_{\pi^{-1}} \operatorname{sign}(\pi^{-1})\, a_{1,\pi^{-1}(1)}\, a_{2,\pi^{-1}(2)} \cdots a_{n,\pi^{-1}(n)}.$$

The right hand side is exactly the definition of $\det(A)$. ∎

<u>Determinants of Elementary matrices</u>

<u>Theorem 2.0.4</u> For the elementary matrices we have

(a) $\det(M_i^c) = c$

(b) $\det(P_{ij}) = -1 \qquad i \neq j$

(c) $\det(E^M_{i,j}) = 1. \qquad i \neq j$

<u>proof.</u> (a) M_i^c is a diagonal matrix with

$$(M_i^c)_{kk} = \begin{cases} c & k = i \\ 1 & k \neq i \end{cases}$$

In particular, M_i^c is triangular, hence by a previous result,

$$\det(M_i^c) = \text{product of diagonal elements}$$
$$= \underbrace{1\cdot 1 \cdots 1}_{n-1}\cdot c = c. \checkmark$$

b).

$$P_{ij} = \begin{array}{c} \\ \left[\begin{array}{cccccc} 1 & & \vdots^{\,i} & & \vdots^{\,j} & \\ & \ddots & & & & \\ & & 1 & & & \\ & & 0 & \cdots & 1 & \\ & & & 1 \ddots & & \\ & & 1 & \cdots & 0 & \\ & & & & & 1 \ddots 1 \end{array}\right] \begin{array}{l} \\ \\ \\ \leftarrow i \\ \\ \leftarrow j \\ \\ \end{array} \end{array}$$

It is easy to see that the only possibly nonzero term in the sum defining $\det(P_{ij})$ is

$$a_{1,1} \cdots a_{i-1,i-1}\, a_{ij}\, a_{i+1,i+1} \cdots a_{j-1,j-1}\, a_{ji}\, a_{j+1,j+1} \cdots a_{nn}$$
$$= 1 \cdots 1 \cdot 1 \cdot 1 \cdot 1 \cdots 1 \cdot 1 \cdot 1 \cdots 1$$

This corresponds to the permutation π

$$\pi(i) = j$$
$$\pi(j) = i$$
$$\pi(k) = k, \quad k \neq i,\ k \neq j.$$

This is a transposition with parity odd. Hence

$$\det(P_{ij}) = -1\cdot 1\cdot 1 \cdots 1 = -1 \quad \checkmark$$

c) If $i > j$ then $E^{\mu}_{i,j}$ is upper triangular
If $i < j$ " " " lower ".

In both cases, $E^{\mu}_{i,j}$ is triangular with 1's along the diagonal. The result follows. $\blacksquare$

As far as determinants are concerned, two of the most important results in Linear Algebra are:

Theorem 2.0.5 A square matrix A is invertible if and only if $\det(A) \neq 0$.

Theorem 2.0.6 Let A and B be square matrices. Then

$$\det(AB) = \det(A)\det(B).$$

The proof of these results directly from the definition of the determinant is quite difficult. We shall therefore do this using elementary matrices.

proposition 2.0.7 Let A be a square matrix. Then if E is an elementary matrix, we have

$$\det(EA) = \det(E)\det(A) = \det(AE).$$

proof

we shall verify $\det(EA) = \det(E)\det(A)$ for each of the 3 types of elementary matrices. Once this is done, it will follow that

$$\begin{aligned} \det(AE) &= \det((AE)^T) && \text{Thm } 2.0.3 \\ &= \det(E^T A^T) \\ &= \det(E^T)\det(A^T) && E^T \text{ is also an elementary matrix} \\ &= \det(E)\det(A) && \text{Thm } 2.0.3 \end{aligned}$$

Consider the case $E = M_i^c$

$$M_i^c A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ ca_{i1} & ca_{i2} & \cdots & ca_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

$$\det(M_i^c A) = \sum_\pi \operatorname{sign}(\pi)\, a_{1,\pi(1)} \cdots a_{i-1,\pi(i-1)} (c\, a_{i,\pi(i)}) \cdots a_{n,\pi(n)}$$
$$= c \sum_\pi \operatorname{sign}(\pi)\, a_{1,\pi(1)} \cdots a_{n,\pi(n)}$$
$$= c \det(A) = \det(M_i^c) \det(A).$$

Next, we care $E = P_{ij}$

$$B \equiv P_{ij} A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow i \\ \\ \leftarrow j \\ \\ \\ \end{array}$$

$$\det(P_{ij} A) = \sum_\pi \operatorname{sign}(\pi)\, b_{1,\pi(1)} \cdots b_{i-1,\pi(i-1)}\, b_{i,\pi(i)} \cdot b_{i+1,\pi(i+1)} \cdots b_{j-1,\pi(j-1)}\, b_{j,\pi(j)}\, b_{j+1,\pi(j+1)} \cdots b_{n,\pi(n)}.$$

$$= \sum_\pi \operatorname{sign}(\pi)\, a_{1,\pi(1)} \cdots a_{i-1,\pi(i-1)}\, a_{j,\pi(i)} \cdot a_{i+1,\pi(i+1)} \cdots a_{j-1,\pi(j-1)}\, a_{i,\pi(j)}\, a_{j+1,\pi(j+1)} \cdots a_{n,\pi(n)}$$

$$= \sum_\pi \operatorname{sign}(\pi)\, a_{1,\pi(1)} \cdots a_{i-1,\pi(i-1)}\, a_{i,\pi(j)} \cdot a_{i+1,\pi(i+1)} \cdots a_{j-1,\pi(j-1)}\, a_{j,\pi(i)}\, a_{j+1,\pi(j+1)} \cdots a_{n,\pi(n)}$$

upon reordering.

$$= \sum_\pi \operatorname{sign}(\pi)\, a_{1,\pi'(1)} \cdots a_{n,\pi'(n)}.$$

Here, the permutation π' is related to π by
$\pi' = \tau_{ij}\pi$ or equivalently $\pi = \tau_{ij}\pi'$

where τ is the transposition $i \leftrightarrow j$.

Clearly $\operatorname{sign}(\pi') = -\operatorname{sign}(\pi)$. Also, to each permutation π there corresponds a unique permutation $\pi' = \tau_{ij}\pi$. Hence

$$\det(P_{ij}A) = -\sum_{\pi'} \operatorname{sign}(\pi')\, a_{1,\pi'(1)} \cdots a_{n,\pi'(n)}$$

$$= -\det(A) = \det(P_{ij})\det(A).$$

Finally, we consider the case $E = E_{ij}^{\mu}$. We only consider the case $i > j$.

$$B \equiv E_{ij}^{\mu} A = \begin{array}{c} \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{j1}+\mu a_{i1} & a_{j2}+\mu a_{i2} & \cdots & a_{jn}+\mu a_{in} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{array}\right] \end{array} \begin{array}{l} \\ \\ \leftarrow j \\ \leftarrow i \\ \\ \\ \end{array}$$

$$\det(E_{ij}^{\mu}A) = \sum_{\pi} \operatorname{sign}(\pi)\, a_{1,\pi(1)} \cdots a_{j-1,\pi(j-1)} \left(a_{j,\pi(j)} + \mu a_{i,\pi(j)}\right)$$

$$a_{j+1,\pi(j+1)} \cdots a_{i,\pi(i)} \cdots a_{n,\pi(n)}.$$

$$= \sum_{\pi} \operatorname{sign}(\pi)\, a_{1,\pi(1)} \cdots a_{n,\pi(n)}$$

$$+ \mu \sum_{\pi} \operatorname{sign}(\pi)\, a_{1,\pi(1)} \cdots a_{j-1,\pi(j-1)}\, a_{i,\pi(j)} \cdot$$

$$a_{j+1,\pi(j+1)} \cdots a_{i,\pi(i)} \cdots a_{n,\pi(n)}.$$

The first sum is that of $\det(A)$. Also, the second sum is the determinant of a matrix whose i-th row is identical to the j-th row. By corollary that follows this proof, it must be zero. ∎

<u>Corollary</u> If a square matrix A has two (or more) identical rows, then $\det(A)=0$. Similarly, if A has two are more identical columns, then $\det(A)=0$.

<u>proof</u>

Suppose rows i ad j are identical. Here, of course, the intent is that $i \neq j$. Clearly

$P_{ij} A = A$, since rows i ad j of A are identical, so interchanging them will not change A.

We saw in the proof of the preceding theorem that

$$\det(A) = -\det(P_{ij} A) = -\det(A).$$

which shows that $\det(A)=0$.

If A has two identical columns, then A^T has two identical rows. Thus $\det(A^T)=0$ by the above. However $\det(A^T)=\det(A)$, hence $\det(A)=0$. ∎

<u>Theorem 2.0.5</u> A square matrix is invertible if and only if $\det(A)\neq 0$.

<u>proof</u>

We use the already established fact that

A is invertible if and only if A is the product of elementary matrices, i.e.

$$A = E_1 E_2 \cdots E_m.$$

Now

$$\det(A) = \det(E_1 (E_2 \cdots E_m)) = \det(E_1)\det(E_2 \cdots E_m)$$

$$= \cdots = \det(E_1) \cdots \det(E_m).$$

The determinants in this product are c, -1, or 1 depending on type. Hence $\det(A)\neq 0$.

Theorem 2.0.6 $\det(AB) = \det(A)\det(B)$

proof

we know that AB is invertible if and only if both A and B are invertible. Equivalently

AB singular $\iff$ A and/or B are singular.

In this case by Theorem 2.0.5,

$$0 = \det(AB) = \underset{0}{\det(A)} \cdot \det(B) \qquad \text{only } A \text{ singular}$$

$$0 = \det(AB) = \det(A) \cdot \underset{0}{\det(B)} \qquad \text{only } B \text{ singular}$$

$$0 = \det(AB) = \underset{0}{\det(A)} \cdot \underset{0}{\det(B)} \qquad A \text{ and } B \text{ singular}$$

we see that in all these 3 cases $\det(AB) = \det(A)\det(B)$.

Now suppose A and B are both invertible

$$A = E_1 \cdots E_m, \quad B = E_1' E_2' \cdots E_\ell'$$

$$\det(AB) = \det(E_1 \cdots E_m E_1' \cdots E_\ell')$$

$$= \underbrace{\det(E_1) \cdots \det(E_m)}_{\det(A)} \underbrace{\det(E_1') \cdots \det(E_\ell')}_{\det(B).}$$

$$= \det(A) \cdot \det(B). \qquad \blacksquare$$

Corollary If A is invertible, then $\det(A^{-1}) = 1/\det(A)$.

proof

$$I = AA^{-1} \Rightarrow 1 = \det(I) = \det(AA^{-1})$$
$$= \det(A)\det(A^{-1})$$

$$\Rightarrow \det(A^{-1}) = 1/\det(A) \qquad \blacksquare$$

Remark It is, in general, not true that

$$\det(A+B) = \det(A) + \det(B).$$

<u>Remark</u> we defined the determinant as a real-valued function acting on matrices. As such $\det(\cdot)$ is not a linear map. It turns out that there is another way of looking at determinants namely viewing them as functions acting on the n rows of A considered as separate entities or independent variables. In other words

$$\det(A) = \det(\text{row}1, \text{row}2, \dots, \text{row}\,n).$$

Alternatively, we may view $\det(A)$ as a function of the n-columns of A

$$\det(A) = \det(\text{col}\,1, \text{col}\,2, \dots, \text{col}\,n).$$

With this view point, the determinant acquires certain interesting properties whose proofs can be found in the preceding pages.

One important result is the following:

viewed as a function of its rows, the determinant is linear in each row, i.e. for any given row index i, α, β scalars, V, W $1 \times n$ matrices

$$\det(\text{row}\,1, \dots, \text{row}(i-1), \alpha V + \beta W, \text{row}_{i+1}, \dots, \text{row}\,n)$$

$$= \alpha \det(\text{row}_1, \dots, \text{row}_{i-1}, V, \text{row}_{i+1}, \dots, \text{row}\,n)$$

$$+$$

$$\beta \cdot \det(\text{row}\,1, \dots, \text{row}_{i-1}, W, \text{row}_{i+1}, \dots, \text{row}\,n).$$

<u>Ex.</u>

let $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & -7 & 9 \\ 2 & 4 & 5 \end{bmatrix}$.

suppose we write $\text{row}\,2 = (5 \ \ -7 \ \ 9) = \overbrace{2}^{\alpha}\overbrace{(1 \ \ -2 \ \ 3)}^{V}$

$$+ \underbrace{3}_{\beta}\underbrace{(1 \ \ -1 \ \ 1)}_{W}$$

Then, $\det(A) = 2 \det \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 3 \\ 2 & 4 & 5 \end{bmatrix} + 3 \det \begin{bmatrix} 1 & 2 & -3 \\ 1 & -1 & 1 \\ 2 & 4 & 5 \end{bmatrix}$.

Another property we encountered earlier is that if two rows are interchanged, then the determinant changes sign. In view of these two properties, we say that a determinant is a skew-symmetric, n-linear form. ~~of its rows~~.

Similarly, since $\det(A^T) = \det(A)$, the determinant is linear in each column and is skew-symmetric in that if we interchange two columns then the determinant changes sign.

Minors, cofactors and determinants using cofactor expansion

Recall the definition of the determinant

$$\det(A) = \sum_{\pi} \text{sign}(\pi)\, a_{1,\pi(1)}\, a_{2,\pi(2)} \cdots, a_{n,\pi(n)}.$$

It is quite inefficient as a computational tool with a work estimate proportional to $n!$ for an $n\times n$ matrix. However, it is quite useful in deriving important results about properties of determinants.

We shall next indicate an alternative way for calculating determinants, which is equally inefficient for large n but has some nice theoretical properties. Moreover, it can be an efficient way for calculating the determinant if A has some nice properties, e.g. contains many zeros.

Defn. Let A be an $n\times n$ matrix. For any choice of the pair (i,j), $1\le i,j\le n$, the minor

M_{ij} is the determinant of the ~~the~~ submatrix obtained by deleting row i and column j of A

Ex.

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 5 & 7 \\ 2 & -2 & 3 \end{bmatrix}$$

Let $i=2$, $j=1$. The submatrix obtained by deleting row 2 and column 1 is the 2×2 matrix

$$S_{21} = \begin{bmatrix} -1 & 3 \\ -2 & 3 \end{bmatrix}.$$

Thus the corresponding minor M_{21} is

$$\det\begin{bmatrix} -1 & 3 \\ -2 & 3 \end{bmatrix} = (-1)(3) - (-2)(3) = 3.$$

Similarly $S_{13} = \begin{bmatrix} 4 & 5 \\ 2 & -2 \end{bmatrix}$ and $M_{13} = \det(S_{13}) = -18$.

Thus there are n^2 minors for an $n \times n$ matrix.

Similarly, to each pair (i,j), we can associate a real number C_{ij}, called the i-j th <u>cofactor</u>. This is done as follows: Looking at

$$\det(A) = \sum_{\pi} \operatorname{sign}(\pi)\, a_{1,\pi(1)} \cdot a_{2,\pi(2)} \cdots a_{n,\pi(n)},$$

since the sum is over all permutations π of $\{1, 2, \ldots, n\}$, we see that every term in the sum contains exactly one entry from row i.

Hence, we can express the determinant as the sum of all terms containing a_{i1} + the sum of all terms that contain a_{i1} + --- + sum of all terms that contain a_{in}. Equivalently

$$\boxed{\det(A) = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}} \quad (*)$$

The quantity C_{ij} is uniquely defined and is called the i-j th <u>cofactor</u> or the <u>cofactor</u> associated with a_{ij}.

Also, $(*)$ is called the <u>cofactor expansion of det(A) according to row i</u>

There is a cofactor expansion of det(A) according to <u>each</u> row

$$\begin{aligned} \det(A) &= a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n} \\ &= a_{21} C_{21} + a_{22} C_{22} + \cdots + a_{2n} C_{2n} \\ &\;\;\vdots \qquad\qquad\quad \vdots \qquad\qquad\quad \vdots \\ &= a_{n1} C_{n1} + a_{n2} C_{n2} + \cdots + a_{nn} C_{nn}. \end{aligned}$$

Collectively, all these are represented by $(*)$

Ex. $n=3$, $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

we already calculated the determinant of A

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$$= a_{11}\underbrace{(a_{22}a_{33} - a_{23}a_{32})}_{C_{11}} + a_{12}\underbrace{(a_{23}a_{31} - a_{21}a_{33})}_{C_{12}} + a_{13}\underbrace{(a_{21}a_{32} - a_{22}a_{31})}_{C_{13}}$$

$$= a_{21}\underbrace{(a_{13}a_{32} - a_{12}a_{33})}_{C_{21}} + a_{22}\underbrace{(a_{11}a_{33} - a_{13}a_{31})}_{C_{22}} + a_{23}\underbrace{(a_{12}a_{31} - a_{11}a_{32})}_{C_{23}}$$

$$= a_{31}\underbrace{(a_{12}a_{23} - a_{13}a_{22})}_{C_{31}} + a_{32}\underbrace{(a_{13}a_{21} - a_{11}a_{23})}_{C_{32}} + a_{33}\underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{C_{33}}$$

These are the 3 cofactor expansions according to rows 1, 2 and 3 respectively.

Similarly, there are cofactor expansions according to each column. Collectively, these are written as

$$\boxed{\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj},\ j=1,\dots,n}$$

So far, we see that there are exactly n^2 cofactors, one for each (i,j) or a_{ij}. Also, we can obtain the determinant of A by expanding according to each row, using the cofactors of the same row or according to each column, using the cofactors of the same column.

The next result shows what happens if we use an expansion according to a row, or column, but use the cofactors from a different row or column.

<u>Theorem</u> Let $r \neq s$. Then

(i) $a_{r1}C_{s1} + a_{r2}C_{s2} + \cdots + a_{rn}C_{sn} = 0$

(ii) $a_{1r}C_{1s} + a_{2r}C_{2s} + \cdots + a_{nr}C_{ns} = 0$.

<u>proof</u> (i) Consider the matrix B obtained from A by replacing row s by row r. <u>Note: we are not interchanging rows r and s.</u>

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{r1} & a_{r2} & \cdots & a_{rn} \\ \vdots & & & \\ a_{s1} & a_{s2} & \cdots & a_{sn} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{matrix} \\ \\ \leftarrow r \\ \\ \leftarrow s \\ \\ \\ \end{matrix}, \quad B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{r1} & a_{r2} & \cdots & a_{rn} \\ \vdots & & & \\ a_{r1} & a_{r2} & \cdots & a_{rn} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{matrix} \\ \\ \leftarrow r \\ \\ \leftarrow s \\ \\ \\ \end{matrix}$$

Note that the cofactors $C^B_{s1}, C^B_{s2}, \ldots, C^B_{sn}$ of B corresponding to row s are identical to the cofactors $C^A_{s1}, C^A_{s2}, \ldots, C^A_{sn}$ of A since all rows of A and B other than s are identical.

Also, since B has identical rows r and s, $\det(B) = 0$. Hence

$$0 = \det(B) = b_{s1}C^B_{s1} + b_{s2}C^B_{s2} + \cdots + b_{sn}C^B_{sn}$$

$$= a_{r1}C^A_{s1} + a_{r2}C^A_{s2} + \cdots + a_{rn}C^A_{sn}. \checkmark$$

(ii) The proof is identical. Work with columns rather than rows. ☒

The preceding results can be summarized into a remarkable formula, called the <u>adjoint formula</u>.

Defn. Given an $n \times n$ matrix A, its adjoint, $\mathrm{adj}(A)$ is defined as the $n \times n$ matrix

$$\mathrm{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

i.e. it is the transpose of the matrix of cofactors of A.

Theorem The following identity holds

$$A\,\mathrm{adj}(A) = \begin{bmatrix} \det(A) & & \\ & \det(A) & 0 \\ & 0 & \ddots \\ & & & \det(A) \end{bmatrix} = \det(A) \cdot I.$$

proof.
For $i, j = 1, \dots, n$,

$$(A\,\mathrm{adj}(A))_{ij} = \sum_{k=1}^{n} A_{ik} (\mathrm{adj}(A))_{kj}$$

$$= \sum_{k=1}^{n} a_{ik} C_{jk}.$$

If $i = j$, then this is $\sum_{k=1}^{n} a_{ik} C_{ik}$ which is the cofactor expansion of A according to row i.

If $i \neq j$, then this is $\sum_{k=1}^{n} a_{ik} C_{jk}$ which is the cofactor expansion of A according to row i but using cofactors from a different row. Hence we get 0. ∎

So far we have developed some formulas showing how determinants can be expressed as expansions in terms of cofactors. However, these formulas are as of yet of little practical use. The next result will provide a remedy by establishing a link between cofactors and minors.

Theorem $\quad C_{ij} = (-1)^{i+j} M_{ij} \quad , \quad i,j = 1,\dots,n.$

proof

Recall that the cofactor C_{ij} is precisely the coefficient in $\det(A)$ of all the terms that contain a_{ij}. In other words

$$\det(A) = a_{ij} C_{ij} \; + \text{ all terms not containing } a_{ij}.$$

Thus

$$C_{ij} = \sum_{\substack{\pi \\ \pi(i)=j}} \overset{\operatorname{sign}(\pi)}{\swarrow} a_{1,\pi(1)} \cdots a_{i-1,\pi(i-1)} a_{i+1,\pi(i+1)} \cdots a_{n,\pi(n)}.$$

The sum is taken over all permutations of $\{1,2,\dots,n\}$ with $\pi(i)=j$. We write

$$\pi = (j_1,\dots,j_{i-1}, j, j_{i+1},\dots,j_n).$$

Let $p = (j_1,\dots,j_{i-1}, j_{i+1},\dots,j_n)$. p is "part" of π and is a permutation of $\{1,2,\dots,n\} - \{j\}$. Note that p and π are uniquely defined in terms of the other. Furthermore, we see that

$$(*) \quad C_{ij} = \sum_{p=(j_1,\dots,j_{i-1},j_{i+1},\dots,j_n)} \operatorname{sign}(\pi)\, a_{1,j_1} \cdots a_{i-1,j_{i-1}} a_{i+1,j_{i+1}} \cdots a_{n,j_n},$$

where the sum is over all permutations of $\{1,2,\dots,n\} - \{j\}$.

We shall next show that there is a close relationship between the signs of π and p which depends only on $|j-i|$

we will show that

$$\text{sign}(\pi) = \text{sign}(p)\cdot(-1)^{|j-i|}.$$

Indeed, suppose the identity permutation ϵ_{n-1} on $\{1,\dots,n\}-\{j\}$ is $\epsilon_{n-1}=(k_1,k_2,\dots,k_{i-1},k_{i+1},\dots,k_n)$ with $k_1<k_2<\dots<k_{i-1}<k_{i+1}<\dots<k_n$ and the identity permutation on $\{1,2,\dots,n\}$ is $\epsilon_n=(1,2,\dots,n)$.

Suppose there are q transpositions $\tau_1,\dots,\tau_q$ such that

$$\tau_q\cdots\tau_1 p=(k_1,k_2,\dots,k_{i-1},k_{i+1},\dots,k_n)=\epsilon_{n-1}$$

The same transpositions applied to π yield

$$\tau_q\cdots\tau_1\pi=(k_1,k_2,\dots,k_{i-1},j,k_{i+1},\dots,k_n).$$

Note that j is occupying the i-th position. Since the integers $k_1,\dots,k_{i-1},k_{i+1},\dots,k_n$ are in increasing order, exactly $|j-i|$ transpositions are needed to bring $\tau_q\cdots\tau_1\pi$ into the ordering $\epsilon_n=(1,2,\dots,n)$. This proves the claimed relationship between the signs of p and π.

Now from (*) and the above

$$C_{ij}=(-1)^{|j-i|}\sum_{p=(j_1,\dots,j_{i-1},j_{i+1},\dots,j_n)}\text{sign}(p)\,a_{1,j_1}\cdots a_{i-1,j_{i-1}}a_{i+1,j_{i+1}}\cdots a_{n,j_n}.$$

But the sum above is precisely the determinant of the submatrix of A obtained by removing row i and column j. The observation that $(-1)^{|j-i|}=(-1)^{i+j}$ completes the proof. ∎

Ex. Evaluate det(A) with $A=\begin{bmatrix}2&1&2\\0&3&-1\\1&2&1\end{bmatrix}$

using cofactor expansion along the first row.

$$\det(A)=a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}$$

$$C_{11}=(-1)^{1+1}\det\begin{bmatrix}3&-1\\2&1\end{bmatrix}=5$$ remove row 1 and col 1

$$C_{12}=(-1)^{1+2}\det\begin{bmatrix}0&-1\\1&1\end{bmatrix}=-1$$ remove row 1 and col. 2

$$C_{13}=(-1)^{1+3}\det\begin{bmatrix}0&3\\1&2\end{bmatrix}=-3$$ remove row 1 and col 3

$$\Rightarrow \det(A)=2(5)+1(-1)+2(-3)=3$$

Ex. Evaluate det(A), $A=\begin{bmatrix}3&1&-1&2\\0&2&0&0\\4&1&0&2\\-2&1&3&4\end{bmatrix}$.

we are not told which of the 8 possible expansions to use.
However, it would be advantageous to use the 2nd row since the presence of zeros there removes the need to compute the corresponding cofactors.

$$\det(A)=\overset{=0}{a_{21}}C_{21}+a_{22}C_{22}+\overset{=0}{a_{23}}C_{23}+\overset{=0}{a_{24}}C_{24},$$

Thus only C_{22} needs to be computed

$$C_{22}=(-1)^{2+2}\det\begin{bmatrix}3&-1&2\\4&0&2\\-2&3&4\end{bmatrix}\doteq\det(B)$$ remove row 2 and column 2.

call it B

To evaluate this 3x3 determinant, we use cofactor expansion along row 2

$$\det(B) = b_{21}\tilde{C}_{21} + \cancel{b_{22}}^{0}\tilde{C}_{22} + b_{23}\tilde{C}_{23}.$$

Note that $\tilde{C}_{21}, \tilde{C}_{22}, \tilde{C}_{23}$ are cofactors of $\underline{\underline{B}}$.

$$C_{21} = (-1)^{2+1}\det\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} = 10$$ remove row 2 and col. 1 of B

$$C_{23} = (-1)^{2+3}\det\begin{bmatrix} 3 & -1 \\ -2 & 3 \end{bmatrix} = -7$$

$$\Rightarrow \det(B) = 4(10) + 2(-7) = 26$$

$$\Rightarrow \det(A) = a_{22}C_{22} = 2\det(B) = 2(26) = 52$$

Remark The work estimate for computing determinants via cofactor expansions is "similar" ($O(n!)$) to that of the definition. In deed, it is basically a recursive algorithm: For an $n\times n$ matrix, n

determinants of size $(n-1)\times(n-1)$ must be computed. Each of these require the evaluation of $(n-1)$ determinants of size $(n-2)\times(n-2)$.

we have $W(n) = n\,W(n-1) = n(n-1)W(n-2)$
$= \cdots = n!\,W(1)$.

Nevertheless, it offers a structured algorithm for the "hand" computation of determinants of very small size ($n \leq 4$).

Evaluation of determinants via triangularization.

The combination of the following facts about determinants leads to an extremely efficient algorithm:

1) The determinant of a triangular matrix is the product of its diagonal elements.

2) Any square matrix can be transformed into an upper triangular matrix via a sequence of elementary row operations. Equivalently, there exist elementary matrices $E_1, \ldots, E_s$ such that

$$(*) \qquad E_s \cdots E_1 A = U, \qquad U \text{ upper triangular}$$

3) The determinant of the product is the product of determinants.

Applying 3) to $(*)$, we get

$$\boxed{\det(E_s) \cdots \det(E_1) \det(A) = \det(U)}$$

$$\Rightarrow \boxed{\det(A) = \frac{\det(U)}{\det(E_1) \cdots \det(E_s)}}$$

Note $\boxed{\det(M_i^c) = c, \quad \det(P_{ij}) = -1, \quad \det(E_{i,j}^{\mu}) = 1}$

Ex. Compute the determinant of

$$A = \begin{bmatrix} 3 & 1 & -1 & 2 \\ 0 & 2 & 0 & 0 \\ 4 & 1 & 0 & 2 \\ -2 & 1 & 3 & 4 \end{bmatrix}$$

via triangularization.

$$\begin{bmatrix} 3 & -1 & -1 & 2 \\ 0 & 2 & 0 & 0 \\ 4 & 1 & 0 & 2 \\ -2 & 1 & 3 & 4 \end{bmatrix}$$

$\downarrow$ $-\frac{4}{3}r_1 + r_3$ type 3, $\det(E_1) = 1$

$\quad$ $\frac{2}{3}r_1 + r_4$ type 3, $\det(E_2) = 1$

$$\begin{bmatrix} 3 & 1 & -1 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & -1/3 & 4/3 & -2/3 \\ 0 & 5/3 & 7/3 & 16/3 \end{bmatrix}$$

$\downarrow$ $\frac{1}{6}r_2 + r_3$ type 3, $\det(E_3) = 1$

$\quad$ $-\frac{5}{6}r_2 + r_4$ " " $\det(E_4 = 1$

$$\begin{bmatrix} 3 & 1 & -1 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4/3 & -2/3 \\ 0 & 0 & 7/3 & 16/3 \end{bmatrix}$$

$\downarrow$ $-\frac{7}{4}r_3 + r_4$ $\det(E_5) = 1$ type 3

$$\begin{bmatrix} 3 & 1 & -1 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4/3 & -2/3 \\ 0 & 0 & 0 & \frac{39}{6} \end{bmatrix} = U$$

$$E_5 E_4 E_3 E_2 E_1 A = U \Rightarrow \det(A) = \frac{\det(U)}{\det(E_1)\cdots\det(E_5)}$$

$$= \frac{(3)(2)(4/3)(39/6) = 52}{(1)(1)(1)(1)(1)} = 52$$

Cramer's Rule: Consider the system $Ax=b$, where A is an $n\times n$ invertible matrix. We know already that

$\det(A)\neq 0$ and the system has a unique solution x for each b.

It turns out that the components of the solution x can be computed using formulas involving determinants.

Theorem (Cramer's rule) Let $Ax=b$ be a linear system with A invertible. Then the components $x_1, \dots x_n$ of x are given by

$$(*)\quad x_i = \frac{1}{\det(A)} \det \overbrace{\begin{bmatrix} a_{11} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \cdots & a_{1n} \\ \vdots & & \vdots & b_2 & \vdots & & \vdots \\ a_{n1} & & a_{n,i-1} & b_n & a_{n,i+1} & \cdots & a_{nn} \end{bmatrix}}^{B_i}, \quad i=1,\dots,n.$$

Here in the numerator we have the determinant of the matrix B_i obtained by replacing the i-th column of A by the $n\times 1$ matrix b.

proof

This is an easy consequence of the adjoint formula

$$A \,\mathrm{adj}(A) = \det(A)\cdot I.$$

Now if A is invertible, $\det(A)\neq 0$, hence we write

$$A\left[\frac{1}{\det(A)}\mathrm{adj}(A)\right] = I.$$

In other words, if A is invertible, then its inverse is given by

$$A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$$

$$= \frac{1}{\det(A)}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

where C_{ij} are the cofactors of A.

Now $\quad Ax = b \Rightarrow x = A^{-1}b = \frac{1}{\det(A)}\,\mathrm{adj}(A)\, b$

$$\Rightarrow \quad x = \frac{1}{\det(A)} \begin{bmatrix} b_1 C_{11} + b_2 C_{21} + \cdots + b_n C_{n1} \\ b_1 C_{12} + b_2 C_{22} + \cdots + b_n C_{n2} \\ \vdots \\ b_1 C_{1n} + b_2 C_{2n} + \cdots + b_n C_{nn} \end{bmatrix}.$$

In particular, x_i is given by

$$x_i = \frac{1}{\det(A)} \left[b_1 C_{1i} + b_2 C_{2i} + \cdots + b_n C_{ni} \right].$$

Now using cofactor expansion of $\det(B_i)$ according to the i-th column, we have

$$\det(B_i) = b_1 C_{1i}^{B_i} + b_2 C_{2i}^{B_i} + \cdots + b_n C_{ni}^{B_i}.$$

However, it is easy to see that the cofactor $C_{ji}^{B_i}$ of B_i is equal to the cofactor C_{ji} of A, since they both involve the same elements. Indeed A and B_i agree everywhere except column i.

Hence

$$\det(B_i) = b_1 C_{1i} + b_2 C_{2i} + \cdots + b_n C_{ni}$$

$$\Rightarrow \quad \frac{1}{\det(A)} \det(B_i) = \frac{1}{\det(A)} \left[b_1 C_{1i} + b_2 C_{2i} + \cdots + b_n C_{ni} \right]$$

$$= x_i. \checkmark \qquad \blacksquare$$

Ex. Solve the system $Ax = b$, $A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$, $b = \begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix}$

$$x_1 = \frac{\det \begin{bmatrix} -3 & 1 & 2 \\ 4 & 3 & -1 \\ 2 & 2 & 1 \end{bmatrix}}{\det \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & -1 \\ 1 & 2 & 1 \end{bmatrix}} = \frac{-17}{3} = -\frac{17}{3}$$

$$x_2 = \frac{\det\begin{bmatrix} 2 & -3 & 2 \\ 0 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}}{\det\left[\ A\ \right]} = \frac{7}{3}, \quad x_3 = \frac{\det\begin{bmatrix} 2 & 1 & -3 \\ 0 & 3 & 4 \\ 1 & 2 & 2 \end{bmatrix}}{\det(A)} = \frac{9}{3} = 3$$

so $x = \begin{pmatrix} -17/ \\ 7/3 \\ 3 \end{pmatrix}$. check $Ax = \begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix}$ ✓.