§1.4 Algebraic properties of matrices; Inverses

Theorem 1.4.1 properties of Matrix operations

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules are valid

(a) $A+B=B+A$ Commutativity of matrix addition
(b) $A+(B+C)=(A+B)+C$ Associative law of matrix addition
(c) $A(BC)=(AB)C$ Associative law of matrix multiplication
(d) $A(B+C)=AB+AC$ distributive law (left)
(e) $(B+C)A=BA+CA$ " " (right)
(f) $A(B-C)=AB-AC$ Same as (d)
(g) $(B-C)A=BA-CA$ " " (e)
(h) $a(B+C)=aB+aC$
(i) $a(B-C)=aB-aC$
(j) $(a+b)C=aC+bC$
(k) $(a-b)C=aC-bC$
(l) $a(bC)=(ab)C$
(m) $a(BC)=(aB)C=B(aC)$

Remarks (i) Matrix addition, matrix-matrix multiplication and scalar-matrix multiplication where defined between pairs of objects. The above laws Allow combining two or more operations

$$(A+B)C \overset{(e)}{=} AC+BC$$

Combining two or more of the same operation

$$ABC \overset{(c)}{=} A(BC) \overset{(c)}{=} (AB)C$$

(ii) Matrix subtraction $A-B$ is defined as $A-B=A+(-1)B$ i.e. in terms of matrix addition <u>and</u> scalar multiplication by (-1).

AB is $m\times p$
BC is $n\times s$

<u>proof of (c)</u> A $m\times n$, B $n\times p$, C $p\times s$

$(AB)C$ is defined as AB first then mult. with C
$A(BC)$ " " " BC first then mult. with A

$$((AB)C)_{ij} = \sum_{k=1}^{p} (AB)_{ik}\, C_{kj} = \sum_{k=1}^{p}\underbrace{\left(\sum_{\ell=1}^{n} A_{i\ell} B_{\ell k}\right)}_{(AB)_{ik}} C_{kj}$$

Commute two sums!

$$= \sum_{\ell=1}^{n}\sum_{k=1}^{p} A_{i\ell} B_{\ell k} C_{kj} = \sum_{\ell=1}^{n} A_{i\ell} \underbrace{\sum_{k=1}^{p} B_{\ell k} C_{kj}}_{(BC)_{\ell j}}$$

$$= \sum_{\ell=1}^{n} A_{i\ell} (BC)_{\ell j} = (A(BC))_{ij} \quad \checkmark$$

<u>Defn.</u> The <u>Identity matrix</u>. For any $n \geq 1$, the $n\times n$ identity matrix I_n (eye(n) in Matlab) or simply I is

$$I = \begin{bmatrix} 1 & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & \ddots \\ & & & & 1 \end{bmatrix}, \quad I_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

<u>Lemma</u> $AI = IA = A$ for any matrix A provided dimensions of A are compatible.

<u>Ex.</u> A is 4 by 3 then $I_4 A = A$
$AI_3 = A$.

<u>Defn.</u> The <u>zero</u> matrix. For m and n given

The m by n <u>zero matrix</u>, denoted O is given by

$$O_{ij} = 0 \qquad i=1,\dots,m,\ j=1,\dots,n$$

$$O = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{bmatrix}.$$

Theorem If c is a scalar and if the sizes of the matrices are such that the operations are valid, then

(a) $A+0=0+A=A$
(b) $A-0=A$
(c) $A-A=0$
(d) $0\cdot A=0$
(e) if $cA=0$, then $c=0$ or $A=0$ or both c and A are zero

proof (a) $(A+0)_{ij}=A_{ij}+0_{ij}=A_{ij}+0=A_{ij}$

(e) Suppose

$$cA=\begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

If $c=0$, then done with proof. If $c\neq 0$, then $a_{ij}=0$ $i=1,\dots,m$, $j=1,\dots,n$. ✓

Defn A matrix A is nonzero, written $A\neq 0$ if A has at least one nonzero entry. In other words if A is not the zero matrix!!

Remark It is possible for the product of two nonzero matrices to be zero.

Ex. $A=\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$, $AB=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\neq 0$ $\neq 0$

In view of this, the cancellation law valid for real numbers, is not valid in general for matrices.

For real numbers a, b, c if $ac=bc$ and $c\neq 0$ then $a=b$.

For matrices A, B, C, $AC=BC$, $C\neq 0$ matrix does not necessarily imply that $A=B$

<u>Ex.</u> Suppose $AB=0$ with $A\neq 0$, $B\neq 0$

let C be any matrix, same size as A

$$(A+C)B = \overset{0}{AB} + CB = CB$$

we have $(A+C)B = CB$ but $A+C \neq C$ (why?)

<u>Theorem 1.4.3</u> If R is the rref of a <u>square</u> $n\times n$ matrix then <u>either</u> R has a row of zeros <u>or</u> it is the identity matrix

<u>proof.</u> There are two mutually exclusive possibilities:

Either R has a row of zeros or it does not.
we have to show that in the 2nd case, i.e. no zero rows $R=I$. Indeed, if there are no zero rows, every row has a leading 1. It is clear that the leading ones are along the diagonal. Also, since each column containing a leading one has zeros with the exception of the leading one (property (4)), $R=I$

<u>$n=2$</u> $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, <u>$n=3$</u> $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$, <u>$n=4$</u> $\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$.

<u>Defn.</u> let A be a <u>square</u> matrix. Suppose there exists a square matrix B of the same sizes such that

$$AB = BA = I,$$

then we say that A is <u>invertible</u> or <u>nonsingular</u> and call B the <u>inverse of A</u>.

<u>Remark</u> clearly the definition is symmetric, in the sense that A is invertible iff B is

Ex.

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} (2)(3)+(-5)(1)=1 & 2(5)+(-5)(2)=0 \\ (-1)(3)+(3)(1)=0 & (-1)(5)+3(2)=1 \end{bmatrix} = I_2$$

$$BA = \begin{bmatrix} 3(2)+5(-1)=1 & 3(-5)+5(3)=0 \\ 1(2)+2(-1)=0 & 1(-5)+2(3)=1 \end{bmatrix} = I_2 \quad \checkmark$$

Remark Turns out verifying $AB = I$ or $BA = I$ is enough.

Theorem 1.4.4 Uniqueness of the inverse. Suppose B and C satisfy $AB = BA = I$ and $AC = CA = I$. Then $B = C$.

proof.

$$AB = I \Rightarrow C(AB) = CI = C$$

$$\underbrace{(CA)}B = C$$

$$B = I \cdot B = C \quad \checkmark$$

Theorem 1.4.6 The product of any finite number of square matrices of the same size is invertible if every matrix in the product is invertible. Furthermore, the inverse of the product is the product of inverses in reverse order.

proof.

Consider the product $B = A_1 A_2 \cdots A_m$. Assume A_i's are all invertible, i.e. $A_i A_i^{-1} = I$. Then

$$(A_1 A_2 \cdots A_m)(A_m^{-1} \cdots A_2^{-1} A_1^{-1})$$

$$= A_1 A_2 \cdots \underbrace{A_m A_m^{-1}}_{I} \cdots A_2^{-1} A_1^{-1} \quad \text{associativity}$$

$$= A_1 A_2 \cdots \underbrace{A_{m-1} A_{m-1}^{-1}}_{I} \cdots A_2^{-1} A_1^{-1}$$

$$\vdots$$

$$= A_1 A_1^{-1} = I \quad \checkmark$$

$\blacksquare$

Remark If one or more A_i's are singular, then so is the product.

Theorem 1.4.7 If A is invertible, then

(a) A^{-1} is invertible and $(A^{-1})^{-1} = A$

(b) For any integer $n \geq 1$, A^n is invertible and $(A^n)^{-1} = (A^{-1})^n$.

(c) If $k \neq 0$ scalar, then kA is invertible and $(kA)^{-1} = \frac{1}{k} A^{-1}$.

proof (a) $A^{-1} A = I$ says also $(A^{-1})^{-1} = A$!

(b) follows from Thm 1.4.6, take $A_i = A$, $i = 1, \ldots, n$

(c) $(kA)(\frac{1}{k} A^{-1}) = kA \frac{1}{k} A^{-1} = \underbrace{k \cdot \frac{1}{k}}_{1} \underbrace{A A^{-1}}_{I} = I$ ✓.

Theorem 1.4.8 If A is invertible, then so is A^T and
$(A^T)^{-1} = (A^{-1})^T$.

proof.

$$A^T (A^{-1})^T = ((A^{-1})(A))^T \quad \text{Thm } 1.4.8\,(e)$$
$$= I^T = I.$$

§1.5 Elementary matrices.

Recall the elementary row operations:

Type 1 Multiply row i of A by a constant $c \neq 0$

Type 2 Interchange/permute rows i and j of A.

Type 3 Add μ times row i of A to row j of A $\boxed{i \neq j}$

To each elementary row operation we can associate a corresponding $m \times m$ matrix called an elementary matrix.

This is done by applying the elementary row operation to the $m \times m$ identity matrix

Ex. Add 4 times row 2 to row 4. ⇐ elem. row operation

$$A = \begin{bmatrix} 1 & 2 & 5 \\ -3 & 9 & 7 \\ 8 & 0 & 2 \\ 1 & 9 & -3 \end{bmatrix}$$

We take $m=4$. In general m is the number of rows of the matrix A to which the elementary row operation is to be applied.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[4 \times r_2 + r_4]{\text{Elem. row op}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \end{bmatrix} = E.$$

More generally

Elementary row operation	Corresponding $m \times m$ elementary matrix
Type 1 Multiply row i by $c \neq 0$	$\begin{bmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & & c & & \\ & & & & 1 & \\ 0 & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} = E = M_i^c$
Type 2 Interchange rows i and j	$\begin{array}{c} \\ \\ i \\ \\ \\ j \\ \\ \\ \end{array}\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0 & \cdots & 1 & \\ & & & \vdots & 1 & \vdots & \\ & & & \vdots & \ddots & \vdots & \\ & & & \vdots & & 1 \vdots & \\ & & & 1 & \cdots & 0 & \\ & & & & & & 1 \\ & & & & & & \ddots \\ & & & & & & & 1 \end{bmatrix} = E = P_{i,j}$
Type 3 Add μ times row i to row j, $j \neq i$	$\begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array}\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & \mu & & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix} = E = E_{i,j}^{\mu}$

Note: the notation M_i^c, $P_{i,j}$, $E_{i,j}^{\mu}$ is my own and is introduced for convenience.

As you see, elementary matrices are very easy to construct. They are also very useful in proving many important facts. Moreover, they can be used to show why/how some important algorithms work. e.g. finding the inverse of a matrix.

In Matlab, an easy way to generate elementary matrices can be done as follows

Type 1 $\quad E = \text{eye}(m)$ creates $m \times m$ identity matrix
$E(i,i) = c$ puts c in the (i,i) location

Type 2 $\quad E = \text{eye}(m)$

$E(i,i) = E(j,j) = 0$
$E(i,j) = E(j,i) = 1$

(There may be an easier way!)

Type 3 $\quad E = \text{eye}(m)$
$E(i,j) = \mu$

It turns out that there is a mathematically powerful way to represent elementary matrices using the identity matrix together with so-called rank-one matrices.

Let v be an $m \times 1$ matrix and w a $1 \times n$ matrix

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}, \quad w = [w_1, w_2, \dots, w_n].$$

The product vw makes sense and is an $m \times n$ matrix. It is sometimes called the exterior product of v and w. Note the

product wv makes sense only if $m = n$ and is the usual dot product of w and v. Now by definition

$$vw = \begin{bmatrix} v_1 w_1 & v_1 w_2 & \cdots & v_1 w_n \\ v_2 w_1 & v_2 w_2 & \cdots & v_2 w_n \\ \vdots & & & \\ v_m w_1 & v_m w_2 & \cdots & v_m w_n \end{bmatrix}$$

Exercise show that if $v \neq 0$ and $w \neq 0$, then vw

To construct the Elementary matrices E,

Type 1

Take $\underset{m \times 1}{v} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i$ $\quad w = \begin{array}{c} i \\ \downarrow \\ [0 - 0\; 1\; 0 - 0] \end{array}$ ~~$1 \times m$~~

and

$$\boxed{M_i^c = I + (c-1)vw}$$

$$vw = \begin{array}{c c} & \begin{array}{ccc} & i & \end{array} \\ i & \begin{bmatrix} 0 & \vdots & 0 \\ \cdots & 1 & \cdots \\ 0 & \vdots & 0 \end{bmatrix} \end{array}$$

Type 2

$$\underset{m \times 1}{v} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \leftarrow j \\ \\ \\ \\ \end{array}, \quad \underset{1 \times m}{w} = \begin{array}{c} i \qquad\quad j \\ \downarrow \qquad\quad \downarrow \\ [0 - 0\; 1\; 0 - 0 \; -1\; 0 - 0] \end{array}$$

$$\boxed{P_{ij} = I - vw}$$

$$vw = \begin{array}{c} \begin{array}{cc} i & j \\ \downarrow & \downarrow \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow i \\ \leftarrow j \end{array}$$

Type 3

$$\underset{m \times 1}{v} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ \leftarrow j \\ \\ \\ \\ \end{array}, \quad \underset{1 \times m}{w} = \begin{array}{c} [0 - 0\; 1\; 0 - 0] \\ \uparrow \\ i \end{array}$$

$$\boxed{E_{ij}^{\mu} = I + \mu vw}$$

$$vw = \begin{array}{c c} & \\ j \rightarrow & \begin{bmatrix} \vdots & & \\ \cdots 1 & \cdots & \cdots \\ \vdots & & \end{bmatrix} \\ & \begin{array}{c} i \end{array} \end{array} \leftarrow j$$

The relationship between an elementary row operation and the corresponding elementary matrix is stated in the following result

Theorem 1.5.1 Let E denote the $m\times m$ elementary matrix corresponding a given elementary row operation. If A is an $m\times n$ matrix, then EM is the (same) matrix that would be obtained by applying the elementary row operation to A.

Ex. $m=3$

Elem. row operation: Add $-2 \times$ row 3 to row 1

$$\Updownarrow$$

Elementary 3×3 matrix $E = E^{-2}_{\underset{i}{3},\underset{j}{1}} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 5 & -3 & 4 \\ 2 & 0 & -5 & 7 \\ 4 & 8 & -3 & 9 \end{bmatrix}$

Now

$$A \xrightarrow{\text{row op}} \begin{bmatrix} -7 & -11 & 3 & -14 \\ 2 & 0 & -5 & 7 \\ 4 & 8 & -3 & 9 \end{bmatrix}.$$

On the other hand

$$E = I + (-2)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow EA = A + (-2)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 5 & -3 & 4 \\ 2 & 0 & -5 & 7 \\ 4 & 8 & -3 & 9 \end{bmatrix}$$

$$= A + (-2)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 4 & 8 & -3 & 9 \end{bmatrix} = A + \begin{bmatrix} -8 & -16 & 6 & -18 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & -11 & 3 & -14 \\ 2 & 0 & -5 & 7 \\ 4 & 8 & -3 & 9 \end{bmatrix} \checkmark$$

proof. just as in the example above, the proof consists in a simple verification of the statement for each type.

Remark It is easy to see that every elementary row operation has an inverse which is also an elementary row operation of the same type. By inverse we mean an operation that cancels the effect of the operation.

	Elem row operation	inverse elementary row operation	
$M_i^c \longleftrightarrow$	Multiply row i by $c \neq 0$	Multiply row i by $1/c$	$\longleftrightarrow M_i^{1/c}$
$P_{ij} \longleftrightarrow$	interchange rows i and j	interchange rows i and j	$\longleftrightarrow P_{ij}$
$E_{i,j}^{\mu} \longleftrightarrow$	Add $\mu \times$ row i to row j	Add $-\mu \times$ row i to row j	$\longleftrightarrow E_{ij}^{-\mu}$

Lemma Elementary matrices are invertible. Also, the inverse of an elementary matrix, is the elementary matrix corresponding to the inverse elementary row operation. In other words

$$(M_i^c)^{-1} = M_i^{1/c}$$

$$(P_{ij})^{-1} = P_{ij}$$

$$(E_{ij}^{\mu})^{-1} = E_{ij}^{-\mu}$$

proof $M_i^c M_i^{1/c} = (I + (c-1)vw)(I + (\frac{1}{c} - 1)vw)$

$$= I + (c - 1 + \tfrac{1}{c} - 1)vw + (c-1)(\tfrac{1}{c} - 1)vwvw.$$

Note that

$$vw = \begin{array}{c} \\ i \rightarrow \\ \\ \end{array}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{array}{c} \ i\ \\ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \end{array}$$

and

$$vwvw = v(wv)w = \begin{array}{c} \\ i \rightarrow \\ \\ \end{array}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \underbrace{\left(\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)}_{=1} \begin{array}{c} \ i\ \\ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \end{array} = vw$$

Hence
$$M_i^c M_i^{1/c} = I + \underbrace{\left(c-1+\frac{1}{c}-1+(c-1)\left(\frac{1}{c}-1\right)\right)}_{=0} vw = I.$$

Hence by uniqueness of the inverse, $M_i^{1/c} = (M_i^c)^{-1}$.

Now
$$P_{ij} P_{ij} = (I - vw)(I - vw)$$
$$= I - 2vw + (vw)(vw)$$
$$v = \begin{array}{c} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow i \\ \leftarrow j \end{array}, \quad w = \begin{array}{ccc} & i & j \\ & \downarrow & \downarrow \\ [& 1 & -1 &] \end{array}$$

Now
$$(vw)(vw) = v(wv)w \text{ by associativity.}$$

Also, $wv = [1 \quad -1]\begin{bmatrix} 1 \\ -1 \end{bmatrix} = (1)(1) + (-1)(-1) = 2$ (This is a dot prod.)

Hence
$$(vw)(vw) = v(wv)w = v(2)(w) = 2vw.$$

Thus
$$P_{ij} P_{ij} = I - 2vw + 2vw = I. \Rightarrow (P_{ij})^{-1} = P_{ij}.$$

Finally,
$$E_{ij}^{\mu} E_{ij}^{-\mu} = (I + \mu vw)(I - \mu vw)$$
$$= I + \overbrace{\mu vw - \mu vw}^{0} - \underbrace{(\mu vw)(\mu vw)}_{\mu^2 vwvw}$$
$$= I - \mu^2 vwvw.$$

Note that $v = \begin{bmatrix} 1 \end{bmatrix} \leftarrow j$; $w = \begin{array}{c} i \\ \downarrow \\ [0 \; 1 \; 0] \end{array}$, $i \neq j$; hence $wv = 0$.

This shows that $E_{ij}^{\mu} E_{ij}^{-\mu} = I$, i.e. $(E_{ij}^{\mu})^{-1} = E_{ij}^{-\mu}$ ∎

Theorem 1.53 If A is an $n \times n$ matrix, then the following statements are equivalent.

(a) A is invertible

(b) $Ax = 0 \Rightarrow x = 0$, i.e. $Ax = 0$ has only the trivial solution

(c) $\text{rref}(A) = I$

(d) A is the product of elementary matrices.

proof $\boxed{(a) \Rightarrow (b)}$.

A invertible means $\exists !$ inverse of A, A^{-1} such that $A^{-1}A = I$.

Now multiply both sides of $Ax=0$ by A^{-1}

$$\Rightarrow \underbrace{A^{-1}A}_{I}x = \underbrace{A^{-1}0}_{0} \Rightarrow x=0$$

$\boxed{(b) \Rightarrow (c)}$

Consider the augmented matrix $[A|0]$ of the homogeneous system $Ax=0$. We bring $[A|0]$ into reduced row echelon form by a sequence of elementary row operations. We claim that there can be no zero rows in the reduced row echelon form which must be of the form $[\tilde{A}|0]$. Indeed, if the last row of $\tilde{A}$ consisted of zeros entirely, then that would mean that there are free variables, i.e. infinitely many solutions, contradicting the assumption that the only solution is $x=0$. Essentially, we have shown that $\tilde{A} = I_n$, which was the desired result.

$\boxed{(c) \Rightarrow (d)}$

(c) means $\text{rref}(A) = I_n$ whereas

$A \xrightarrow{op^1} \xrightarrow{op^2} \longrightarrow \cdots \xrightarrow{op^s} \text{rref}(A)$ can be expressed as

$E_s \cdots E_1 A = I$, where E_i is the elementary matrix corresponding to the i-th elem. row operation.

We know that each E_i is invertible and E_i^{-1} is an elementary matrix (of the same type). We also know that the product of any finite number of invertible matrices is invertible and the inverse of the product is the product of inverses in reverse order. In other words

$$E_s \cdots E_1 A = I \Rightarrow A = E_1^{-1} \cdots E_s^{-1} I$$
$$= E_1^{-1} \cdots E_s^{-1}.$$

(d) => (a)

(d) means A is the product of elementary matrices. Since such matrices are invertible, so is the product, which is A. ✓ ∎

Remark One efficient way to calculate the inverse of an invertible matrix A is done as follows

(1) Apply a sequence of elem. row operations to reduce A to rref(A). Note A invertible $\iff$ rref$(A) = I$.

(2) Apply the same sequence of elem. row operations to I. At the end, the resulting matrix is A^{-1}.

$$\left[\begin{array}{c} A \\ \hline I \end{array}\right] \xrightarrow{op1} \xrightarrow{op2} \cdots \xrightarrow{ops} \left[\begin{array}{c} I \\ \hline A^{-1} \end{array}\right]$$

(Rigorous) Justification

$$E_s \cdots E_1 A = I$$

$$\Rightarrow \quad A = E_1^{-1} \cdots E_s^{-1} I = E_s^{-1} \cdots E_s^{-1}$$

$$\Rightarrow \quad A^{-1} = (E_s^{-1} \cdots E_s^{-1})^{-1} = (E_s^{-1})^{-1} \cdots (E_s^{-1})^{-1}$$

$$\Rightarrow \quad \boxed{A^{-1} = E_s \cdots E_1 = E_s \cdots E_1 I}$$

Ex.

$$A=\begin{bmatrix}1&2&3\\2&5&3\\1&0&8\end{bmatrix} \qquad I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\Big\downarrow \begin{matrix}-2r_1+r_2\\-r_1+r_3\end{matrix} \qquad\qquad \Big\downarrow \begin{matrix}-2r_1+r_2\\-r_1+r_3\end{matrix} \Big\rangle \text{ may interchange order}$$

$$\begin{bmatrix}1&2&3\\0&1&-3\\0&-2&5\end{bmatrix} \qquad \begin{bmatrix}1&0&0\\-2&1&0\\-1&0&1\end{bmatrix}$$

$$\Big\downarrow 2r_2+r_3 \qquad\qquad \Big\downarrow 2r_2+r_3$$

$$\begin{bmatrix}1&2&3\\0&1&-3\\0&0&-1\end{bmatrix} \qquad \begin{bmatrix}1&0&0\\-2&1&0\\-5&2&1\end{bmatrix}$$

$$\Big\downarrow -1\times r_3 \qquad\qquad \Big\downarrow -1\times r_3$$

$$\begin{bmatrix}1&2&3\\0&1&-3\\0&0&1\end{bmatrix} \qquad \begin{bmatrix}1&0&0\\-2&1&0\\5&-2&-1\end{bmatrix}$$

$$\Big\downarrow -2r_2+r_1 \qquad\qquad \Big\downarrow -2r_2+r_1$$

$$\begin{bmatrix}1&0&9\\0&1&-3\\0&0&1\end{bmatrix} \qquad \begin{bmatrix}5&-2&0\\-2&1&0\\5&-2&-1\end{bmatrix}$$

$$\Big\downarrow \begin{matrix}3r_3+r_2\\-9r_3+r_1\end{matrix} \qquad\qquad \Big\downarrow \begin{matrix}3r_3+r_2\\-9r_3+r_1\end{matrix} \Big\rangle \text{ may interchange order}$$

$$I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \qquad \begin{bmatrix}-40&16&9\\13&-5&-3\\5&-2&-1\end{bmatrix}=A^{-1}$$

The fact that rref$(A)=I$ shows that A is invertible.

Now, to find A^{-1} exactly the same operations that were used in the first column must be used on I. Also, the same order must be followed in the calculation of A^{-1}.

It is possible to interchange certain operations, e.g. as shown above. However it is safer not to take chances and stick to exactly the same order.