Chapter 1: Systems of linear Equations and matrices

§1.1 Systems of linear equations

Ex. $x + 3y = 7$
$\frac{1}{2}x - y + 3z = -1$ 2 equations, 3 unknowns

Ex. nonlinear systems

$$x + 3y^2 = 4 \qquad 3x + 2y - xy = 5$$
$$\sin x + y = 0 \qquad \sqrt{x_1} + 2x_2 + x_3 = 1$$

General linear system: m equations, n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

One of the most fundamental issues in Linear Algebra is whether a given linear system of equations has a solution or not.

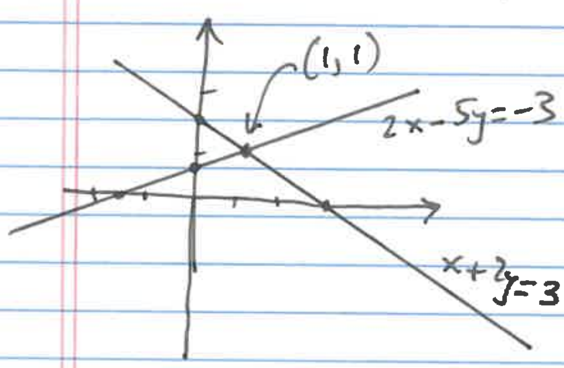
Defn. A linear system that has a solution is called consistent. Otherwise inconsistent.

Obviously a linear system can be either consistent or inconsistent.

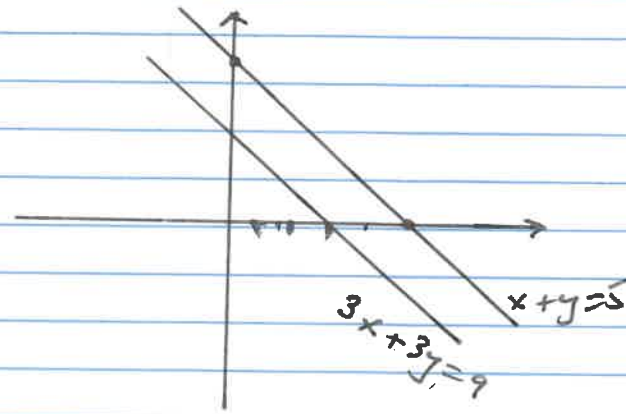
Ex. $x + 2y = 3$
$2x - 5y = -3$ is consistent since by inspection we see that $x = 1, y = 1$ is a solution

Ex. $x + y = 5$
$3x + 3y = 9$ is inconsistent.

Geometric interpretation



solution is intersection of the two lines representing the equations in the system



The two lines are parallel, => no intersection, i.e. no solution.

Remark For a general system of m equations in n unknowns it is almost impossible to use geometry to ascertain whether system is consistent or not.

Instead, we shall develop algebraic theory to do this.

Theorem (proof later) A linear system can have
(i) No solution, system is inconsistent
(ii) unique solution
(iii) infinitely many solutions

$\left.\begin{array}{l}\text{(ii)}\\ \text{(iii)}\end{array}\right\}$ system is consistent.

Ex. A system with infinitely many solutions

$$\begin{aligned} x - y + 2z &= 5 \\ 2x - 2y + 4z &= 10 \\ 3x - 3y + 8z &= 15 \end{aligned}$$

we see that the 2nd equ. is redundant with the first. So we can eliminate it

$$\Rightarrow \begin{aligned} x - y + 2z &= 5 \\ 3x - 3y + 8z &= 15 \end{aligned}$$

normal vector $(1,-1,2)$
" " $(3,-3,8)$

Any triple of the form $(t, t-5, 0)$ is a solution.

§1.2 Gaussian Elimination

Is a general ad effective method for solving linear systems.

First step: write system in augmented matrix form
The letters we use for the unknowns are not important. we can get rid of them. This step is not necessary. It only reduces clutter and displays the essential information contained in the system

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] \longleftrightarrow [A \,\vdots\, b]$$

Elimination step: Reduce augmented matrix into upper triangular form.

Solution step: Solve system using back substitution

Elimnation step is carried out using a sequence of elementary row operations. These come in 3 flavors

1) Multiply a row by a nonzero constant
2) Interchang two rows
3) Add a constant multiple of one row to another

> **Theorem** The solution set of a linear system remains the same upon application of any sequence of elementary row operations.

proof. (later!)

Ex. Solve the linear system

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right]$$

$\downarrow$

Add 1 x row 1 to row 2
Add 3 x row 1 to row 3

(why did we do this?)

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right]$$

$\downarrow$

Add (-10) x row 2 to row 3

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & 0 & -52 & -104 \end{array}\right]$$

At this point, we are done with Gauss Elimination in the sense that we have reduced the augment matrix i.e. the original system, to <u>upper triangular form</u>.

The "new" augmented matrix represents the equivalent system

$$\begin{aligned} x_1 + x_2 + 2x_2 &= 8 \\ -x_2 + 5x_3 &= 9 \\ -52x_3 &= -104 \end{aligned}$$

we now start back subst.
From 3rd Equation

$$\Rightarrow x_3 = \frac{-104}{-52} = 2 = x_3$$

using 2nd equation: $-x_2 = 9 - 5x_3 = 9 - 5(2) = -1$

$$\Rightarrow x_2 = 1$$

Finally, using the first equation:

$$x_1 = 8 - x_2 - 2x_2 = 8 - (1) - 2(2) = 3 = x_1$$

we call this last process back substitution since we are solving for the unknowns in <u>reverse order</u>

In this example we had a case of a <u>unique solution</u>

<u>Ex.</u> <u>An inconsistent system</u>

$$\begin{aligned} x + 3y - z &= 2 \\ x + y + 2z &= 4 \\ 2y - 3z &= 5 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 1 & 1 & 2 & 4 \\ 0 & 2 & -3 & 5 \end{array}\right]$$

add (-1) x row 1 to row 2

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 0 & -2 & 3 & 2 \\ 0 & 2 & -3 & 5 \end{array}\right]$$

add 1 x row 2 to row 3

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 0 & -2 & 3 & 2 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

The new system is equivalent to the original system and is <u>inconsistent</u>. <u>why?</u>

<u>reason</u> The last equation of the new system says

$$0 \cdot x + 0 \cdot y + 0 \cdot z = 7 \quad \text{i.e. } 0 = 7. \text{ impossible.}$$

<u>Ex.</u> <u>A system with infinitely many solutions</u>

$$\begin{aligned} x + 3y - z &= 2 \\ x + y + 2z &= 4 \\ 2y - 3z &= -2 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 1 & 1 & 2 & 4 \\ 0 & 2 & -3 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 0 & -2 & 3 & 2 \\ 0 & 2 & -3 & -2 \end{array}\right]$$

add 1 x row 2 to row 3

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 0 & -2 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

we will designate z as a <u>free</u> variable (we will explain this later) in the sense that z can assume any value and then x and y can be solved in terms of z. Indeed, from the 2nd equation of the transformed system

$$-2y + 3z = 2 \Rightarrow \boxed{y = \frac{2 - 3z}{-2}}$$

Then, from the first equation $x + 3y - z = 2$

$$x = 2 - 3y + z = 2 - 3\left(\frac{2-3z}{-2}\right) + z$$

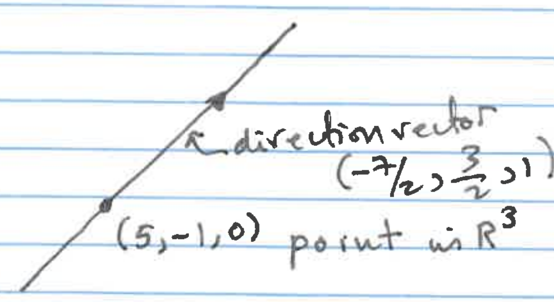
$$= 2 + 3 - \frac{9}{2} z + z = 5 - \frac{7}{2} z$$

$\Rightarrow$ <u>General solution</u> : $\left(5 - \frac{7}{2} z, -1 + \frac{3}{2} z, z\right) \quad z \in \mathbb{R}$

we say that we have a one parameter (z) infinite family of solutions.

can also be written as $(5, -1, 0) + z\left(-\frac{7}{2}, \frac{3}{2}, 1\right)$

Every point on this line is a solution of the system.



<u>Defn.</u> A matrix A is said to be in <u>reduced row echelon form</u> (rref) if

(1) If there are any rows that consist entirely of zeros, then they are placed/grouped at the bottom of the matrix

(2) If a row has a nonzero, then the first nonzero in that row must equal 1. It is called a <u>leading 1</u>

(3) In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row is to the right of the leading 1 of the previous row.

(4) Each column that contains a leading 1 has zeros everywhere <u>else</u> in that column.

<u>Defn.</u> A matrix that satisfies conditions ①, ②, ③ is said to be in <u>row echelon form</u>.

<u>Defn.</u> A matrix that satisfies ① and ③ (replace leading 1 by first nonzero) is said to be in <u>upper triangular form</u>.

$$\boxed{\text{reduced row echelon form} \Rightarrow \text{row echelon form} \Rightarrow \text{upper tri. form}}$$

<u>Ex.</u>
$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

are all in rref form.

<u>Ex.</u>
$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

are in row echelon form but <u>not</u> in rref

<u>Ex.</u>
$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & -2 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \left[\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 0 & -2 & 3 & 2 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

are in upper triangular form but not in row echelon or reduced row echelon form!

<u>Theorem</u> Any matrix can be transformed into reduced row echelon form by a finite sequence of elementary row operations. Furthermore, the rref of a matrix is unique in the sense that it does not depend on any specific sequence used.

The process of going from the matrix to its rref is called <u>Gauss-Jordan Elimination</u>

<u>Defn.</u> The rank of a matrix is the number of nonzero rows in its rref.

<u>Theorem</u> A linear system $Ax=b$ if and only if

$$\operatorname{rank}(A) = \operatorname{rank}(A \mid b).$$

<u>Note</u> It is easy to see that if a matrix B is obtained by adding a number of columns to a matrix A, then

$$\operatorname{rank}(A) \leq \operatorname{rank}(B)$$

So the above Theorem says that while

$$\operatorname{rank}(A) \leq \operatorname{rank}(A \mid b),$$

the system $Ax=b$ is consistent iff adding b as a last column to A does not increase the rank of A.

Free variables, pivot variables

Suppose the augmented matrix $[A \mid b]$ of a system has been reduced to rref

Ex.

$$\left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This system has a solution (why?)

we have 6 variables, say $x_1, x_2, \dots, x_6$.

x_1, x_3, x_6 are called pivot variables in the sense that they correspond to columns that contain a leading 1.

The remaining variables, x_2, x_4, x_5 are called free variables.

Fact: we can assign arbitrary values to the free variables and then uniquely solve for the pivot variables.

3rd Eqn. $\Rightarrow$ $\boxed{x_6 = 3}$

2nd Eqn. $\Rightarrow$ $\boxed{x_3 = 2 - 2x_4}$ $\quad \forall x_4$

1st Eqn $\Rightarrow$ $\boxed{x_1 = -3x_2 - 4x_4 - 2x_5}$

$\Rightarrow$ General solution set

$$(-3x_2 - 4x_4 - 2x_5,\ \underset{\substack{\uparrow \\ \text{free}}}{x_2},\ 2 - 2x_4,\ \underset{\substack{\uparrow \\ \text{free}}}{x_4},\ \underset{\substack{\uparrow \\ \text{free}}}{x_5},\ 3)$$

The system $Ax=b$ is called <u>homogeneous</u> if $b=0$.

<u>Ex.</u>
$$x_1 + x_2 - 4x_3 = 0$$
$$x_1 - 5x_2 + 7x_3 = 0$$

<u>Thm.</u> A homogeneous system is consistent: $x=0$ is always a solution. $x=0$ is called the <u>trivial solution</u>

<u>proof.</u> obvious!

$$a_{11}\cdot 0 + a_{12}\cdot 0 + \cdots + a_{1n}\cdot 0 = 0$$
$$a_{21}\cdot 0 + a_{22}\cdot 0 + \cdots + a_{2n}\cdot 0 = 0$$
$$\vdots$$
$$a_{m1}\cdot 0 + a_{m2}\cdot 0 + \cdots + a_{mn}\cdot 0 = 0 \quad \checkmark$$

<u>Remark</u> Since a homogeneous system always has the trivial solution, the only alternatives left are the possibility of either <u>one</u> (unique) solution or infinitely many solutions.

<u>Lemma</u> A homogeneous system $Ax=0$ with $m<n$, i.e. more variables than equations has infinitely many solutions.

<u>proof</u>
$$[A \mid 0] \longrightarrow \underbrace{[\underbrace{\mathrm{rref}(A)}_{n} \mid \underbrace{0}_{1}]}_{n+1} \updownarrow m$$

$m<n \Rightarrow$ There is at least one free variable which can be assigned arbitrary values $\Rightarrow \infty$-many sols. ∎

§1.3 Matrices and Matrix Operations.

Defn. An $m \times n$ matrix A is a rectangular array of elements

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad a_{ij} \text{ or } (A)_{ij}, \quad \begin{matrix} i = 1, \ldots, m \\ j = 1, \ldots, n \end{matrix}$$

we say it has m rows and n columns

Defn. A matrix is square if $m = n$

Defn. Two matrices A and B are equal if

$$\underbrace{m(A) = m(B), \; n(A) = n(B)}_{\text{we say } A \text{ and } B \text{ have the same size}} \text{ and } a_{ij} = b_{ij} \quad \begin{matrix} i = 1, \ldots, m \\ j = 1, \ldots, n. \end{matrix}$$

Defn. Addition of two (or more matrices)

Suppose A and B are two matrices of the same size, say m by n.
Then, the sum matrix, denoted $A+B$ is the m by n matrix such that $(A+B)_{ij} = a_{ij} + b_{ij}$ $\quad i = 1, \ldots, m$, $j = 1, \ldots, n$

Ex.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 9 \end{bmatrix} + \begin{bmatrix} -7 & 4 & 2 \\ 4 & 9 & 5 \end{bmatrix} = \begin{bmatrix} -6 & 6 & 5 \\ 3 & 11 & 14 \end{bmatrix}$$

Ex. Cannot "add" $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 3 & 0 & 5 \\ 4 & 1 & 9 \end{bmatrix}$

Defn. <u>Scalar multiplication</u>

Let A be an $m \times n$ matrix and c a <u>scalar</u> (a real number). Then cA is the m by n matrix with $(cA)_{ij} = c a_{ij}$ $\quad i=1,\dots,m;\ j=1,\dots,n$

Ex.

$$A = \begin{bmatrix} 2 & 5 & -7 \\ 4 & 2 & 1/2 \end{bmatrix}. \text{ then}$$

$$\frac{1}{2}A = \begin{bmatrix} 1 & 5/2 & -7/2 \\ 2 & 1 & 1/4 \end{bmatrix}$$

Defn. <u>Matrix-Matrix Multiplication</u>

Let A be an $m \times n$ matrix
Let B be an $n \times p$ matrix

i.e. the number of cols. of A must be equal to the number of rows of B. The we define the <u>product</u> AB as the $m \times p$ matrix

$$(AB)_{ij} = (\text{row } i \text{ of } A) \cdot (\text{col } j \text{ of } B) = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$(a_{i1}\ a_{i2} \cdots a_{in}) \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix}$$

$i = 1,\dots,m$
$j = 1,\dots,p$

Ex.

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 5 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -1 & 5 \\ 4 & 2 & 1 \\ 5 & 9 & 3 \end{bmatrix}$$

$m=2,\ n=3$ $\qquad$ $n=3,\ p=3$

AB is 2×3

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & 5 & 7 \end{bmatrix} \begin{bmatrix} 7 & -1 & 5 \\ 4 & 2 & 1 \\ 5 & 9 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (2)(7)+(3)(4)+(-1)(5)=21 & (2)(-1)+(3)(2)+(-1)(9)=-5 & (2)(5)+(3)(1)+(-1)(3)=10 \\ (4)(7)+(5)(4)+(7)(5)=83 & (4)(-1)+(5)(2)+(7)(9)=69 & (4)(5)+(5)(1)+(7)(3)=46 \end{bmatrix}$$

In particular, if A is an $m \times n$ matrix and x is an $n \times 1$ matrix (i.e. vector) then

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

we call this a <u>linear combination</u> of the columns of A

<u>Remark</u> $AB \neq BA$ in general.

<u>Thm.</u> Let $C = AB$. Then

j-th column of C is $A \cdot j$-th column of B

i-th row of C is ith row of $A \cdot B$.

Defn: If A is an $m \times n$ matrix, its transpose A^T is the $n \times m$ matrix such that

$$(A^T)_{ij} = (A)_{ji}, \quad \begin{matrix} i = 1, \ldots, n \\ j = 1, \ldots, m \end{matrix}$$

Ex. $A = \begin{bmatrix} 2 & 7 & -1 \\ 4 & 10 & 3 \end{bmatrix}$

$$A^T = \begin{bmatrix} 2 & 4 \\ 7 & 10 \\ -1 & 3 \end{bmatrix}$$

"rows of A are columns of A^T" in same order.
" cols. of A are rows of A^T"

Defn. A square matrix is symmetric if $A^T = A$.

Ex.
$$A = \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}, \quad A^T = \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix} = A.$$

Defn. Let A be a square $m \times m$ matrix. Its trace, denoted $tr(A)$ is the sum of the diagonal elements of A, i.e.

$$\boxed{tr(A) = \sum_{i=1}^{m} a_{ii}}.$$

Lemma $tr(A+B) = tr(A) + tr(B)$.

Thm. (i) $(A+B)^T = A^T + B^T$

(ii) $(AB)^T = B^T A^T$.

(iii) $(cA)^T = c A^T$

proof.

(i) $((A+B)^T)_{ij} = (A+B)_{ji} = (A)_{ji} + (B)_{ji} = (A^T)_{ij} + (B^T)_{ij}$ ✓

(ii) $((AB)^T)_{ij} = (AB)_{ji} = \sum_{k=1}^{p} A_{jk} B_{ki}$

$$= \sum_{k=1}^{p} (A^T)_{kj} (B^T)_{ik} = \sum_{k=1}^{p} (B^T)_{ik} (A^T)_{kj}$$

$$= (B^T A^T)_{ij}$$ ✓

(iii) $((cA)^T)_{ij} = (cA)_{ji} = c(A)_{ji} = c(A^T)_{ij}$ ✓

$\boxed{\text{Summary of sections 1.1, 1.2}}$

Linear system, augmented matrix $[A \,\vdots\, b]$
consistent linear system: Has at least one solution $< \begin{matrix} 1 \\ \infty \end{matrix}$
inconsistent linear system: No (or zero) solution.

To solve a linear system, use Gaussian Elimination
Aims at transforming A into upper triangular form.

Elementary row operations: 3 types
1) Multiply a row by a nonzero constant
2) Interchange two rows
3) Add a multiple of a row to another row (designed to introduce a zero in a specific place)

Thm. $[A \,;\, b] \xrightarrow[\text{Elimination}]{\text{Gauss}} [\tilde{A}, \tilde{b}]$

Both systems have exactly same solution set.

1) upper triangular form: satisfies ① and ③ on page 11
2) row echelon form: satisfies ①, ② and ③
3) reduced row echelon form: satisfies ①, ②, ③, ④

$\boxed{\text{reduced row echelon} \Rightarrow \text{row echelon} \Rightarrow \text{upper triangular}}$

Defn. rank of a matrix is number of nonzero rows after reduction to upper tri. form or row echelon form or rref

Thm. Adding a row or a column to a matrix cannot decrease the rank of the matrix.

Thm. Reduce $[A|b]$ into $[\tilde{A}|\tilde{b}]$ upper tri form or row echelon or reduced row ech. form
Then
$[A|b]$ has a solution iff # of nonzero rows in $\tilde{A}$ equals # of nonzero rows in $[\tilde{A}|\tilde{b}]$

pivot and free variables

Reduce $[A|b]$ into $[\tilde{A}|\tilde{b}]$ upper tri. form.

There are as many pivot variables as there are non zero rows in $\tilde{A}$. Note If system is consistent Then # of nonzero rows in $\tilde{A}$ = # of nonzero rows in $[\tilde{A}|\tilde{b}]$

A variable among $x_1, x_2, \ldots, x_n$ is a pivot variable if its corresponding column in $\tilde{A}$ has a leading nonzero (or one if you're looking at the row echelon form)
Non-pivot variables are called free variables.

Thm. If a system $[A|b]$ is consistent, Then The free variables can be assigned any value. The pivot variables can Then be calculated.

Note. Suppose system is consistent. Then
1) No free variables => unique solution.
2) At least one free variable => infinitely many solns.

Defn. A system $[A|b]$ is Homogeneous if $b=0$ i.e. $b_1 = b_2 = \cdots = b_m = 0$. Otherwise, it is called Inhomogeneous or Non homogeneous (at least one b_i is nonzero

Thm. A homogeneous system $[A\,|\,0]$ always has the zero or trivial solution $x_1 = x_2 = \cdots = x_n = 0$