Math 251 Spring 2019. Exam I

Name: ______________________________

1	2	3	4	5	6	7	Total

Show all calculations to receive full credit

1. (20 pts) Let $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & -4 \\ -1 & 3 & 2 \end{bmatrix}$.

Which of the expressions $AB, (A-B)^T, AB^T - (BA)^T$ are defined? Compute the ones that are.

AB is not defined: 3×3 times 2×3 don't match

$A-B$ is not defined: 3×3 and 2×3 => $(A-B)^T$ is not defined

$$AB^T = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 10 & -5 \\ -8 & 10 \\ -10 & 11 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 0 & -4 \\ -1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -2 & -8 & -16 \\ 7 & 10 & 17 \end{bmatrix}$$

$$AB^T - (BA)^T = \begin{bmatrix} 10 & -5 \\ -8 & 10 \\ -10 & 11 \end{bmatrix} - \begin{bmatrix} -2 & 7 \\ -8 & 10 \\ -16 & 17 \end{bmatrix} = \begin{bmatrix} 12 & -12 \\ 0 & 0 \\ 6 & -6 \end{bmatrix}$$

2. (20 pts.) Consider the system

$$\begin{aligned} x - 2y + z + 2w &= 1 \\ x + y - z + w &= 2\ . \\ x + 7y - 5z &= 3 \end{aligned}$$

Is the system consistent? If yes, give the set of all possible solutions.

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 1 & 1 & -1 & 1 & 2 \\ 1 & 7 & -5 & 0 & 3 \end{array}\right] \xrightarrow{-1\times r_1 + r_2} \left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 0 & 3 & -2 & -1 & 1 \\ 0 & 9 & -6 & -2 & 2 \end{array}\right] \xrightarrow{-3r_2+r_3} \left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 0 & 3 & -2 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$-1 \times r_1 \times r_3$

z is free variable since there is no leading nonzero in column 3.

$\boxed{w=-1}$. From 3rd Equation

From 2nd Equ. $3y - 2z - w = 1 \Rightarrow y = \frac{2z + w + 1}{3} = \boxed{\frac{2}{3}z}$ ($w = -1$)

1st. Eqn. $\Rightarrow x - 2y + z + 2w = 1 \Rightarrow x = 2y - z - 2w + 1 =$

$= 2(\frac{2}{3}z) - z + 2 + 1 = \frac{z}{3} + 3$

solution set is $(\frac{z}{3}+3, \frac{2}{3}z, z, -1) = \boxed{(3, 0, 0, -1) + z(\frac{1}{3}, \frac{2}{3}, 1, 0) \text{ for all } z}$

3. (15 pts.) Let A and B be two matrices such that BA is defined and B is invertible. Show that $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{0}$ if and only if $BA\mathbf{x} = \mathbf{0}$.

Suppose $Ax = 0$. want to show $BAx = 0$.

To do this, simply multiply both sides of $Ax=0$ by B

$\Rightarrow \quad BAx = B0 = 0$ ✓

Conversely, suppose $BAx = 0$. want to show $Ax = 0$.

For this part, since B is invertible, multiply both sides of $BAx = 0$ by B^{-1}

$$\Rightarrow \underbrace{B^{-1}(BAx)}_{Ax} = B^{-1}0$$

$$Ax = 0$$ ✓

4. (25 pts.) Use elementary row operations to compute the inverse of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$. If you kept track of the operations, then you should be able to compute $\det(A)$ very easily.

$$\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&1&0&1&0\\1&2&0&0&0&1\end{array}\right] \xrightarrow{-r_1+r_3} \left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&1&0&1&0\\0&2&0&-1&0&1\end{array}\right] \xrightarrow{-2r_2+r_3} \left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&1&0&1&0\\0&0&-2&-1&-2&1\end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}\times r_3} \left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&1&0&1&0\\0&0&1&\frac{1}{2}&1&-\frac{1}{2}\end{array}\right] \xrightarrow{-r_3+r_2} \left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&-\frac{1}{2}&0&\frac{1}{2}\\0&0&1&\frac{1}{2}&1&-\frac{1}{2}\end{array}\right] \Leftarrow$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}$$

what was done is: $A \to \to \to \to I$. we used 3 operations of type 3. These don't change the determinant. In step 3 we used a type 1 operation which multiplies $\det(A)$ by $-\frac{1}{2}$. Thus

$$\left(-\frac{1}{2}\right)\det(A) = \det(I) = 1 \Rightarrow \boxed{\det(A) = -2}$$

5. (15 pts.) Suppose A is a square matrix such that the second row is 4 times the third row. Show that $\det(A) = 0$.

Let B be the matrix obtained from A by adding -4 times row 3 to row 2.

As a consequence, row 2 of B is zero $\Rightarrow \boxed{\det(B) = 0}$

since the operation ~~of~~ used is type 3, $\boxed{\det(A) = \det(B)}$

Hence $\det(A) = 0$.

Another approach $\quad B = M_3^4 A \Rightarrow \det(B) = 4\det(A)$

B has rows 2 and 3 identical $\Rightarrow \det(B) = 0 = 4 \cdot \det(A)$

$\Rightarrow \det(A) = 0$.

6. (25 pts.) Compute the adjoint of A where $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$. Use it to compute $\det(A)$ and A^{-1}.

$C_{11} = +\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 \qquad C_{12} = -\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = 2 \qquad C_{13} = +\begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} = 1$

$C_{21} = -\begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} = 2 \qquad C_{22} = +\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 \qquad C_{23} = -\begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} = 2$

$C_{31} = +\begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix} = 1 \qquad C_{32} = -\begin{vmatrix} 2 & 0 \\ -1 & -1 \end{vmatrix} = 2 \qquad C_{33} = +\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$

$$\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$A \operatorname{adj}(A) = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4 \cdot I$$

$$\Rightarrow \boxed{\det(A) = 4}, \quad A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

7. (15 pts.) Supppose the matrix A is invertible and also symmetric, i.e. $A^T = A$. Show that A^{-1} is also symmetric.

want to show $(A^{-1})^T = A^{-1}$.

<u>Method I</u> By a Theorem, $(A^{-1})^T = (A^T)^{-1} = A^{-1}$ since $A^T = A$

<u>Method II</u> $AA^{-1} = I \Rightarrow (AA^{-1})^T = I^T = I$ ✓

$$\begin{aligned} (AA^{-1})^T &= I \\ (A^{-1})^T A^T &= I \qquad A^T = A \\ (A^{-1})^T A &= I \qquad \text{Mult. by } A^{-1} \\ (A^{-1})^T &= I \cdot A^{-1} \\ &= A^{-1} \end{aligned}$$ ✓

<u>Method III</u> $A = A^T \Rightarrow \operatorname{adj}(A)$ is symmetric $\Rightarrow A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$ is symm.

✓