Twisted Commuting Squares

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Abstract

We consider generalizations of commuting squares, called twisted commuting squares, obtained by having the commuting square orthogonality condition hold with respect to the inner product given by a faithful state on a finite dimensional $\ast$-algebra. We present various examples of twisted commuting squares, most of which are computationally easy to work with, and we prove an isolation result. We also show how parametric families of (not necessarily) twisted commuting squares yield associative deformations of the matrix multiplication.

1 Introduction

Commuting squares were introduced by S. Popa in [12] (see also [5]), as invariants and construction data for subfactors. A commuting square is a square of inclusions:

$$
\begin{pmatrix}
P & \subset & R \\
\cup & \cup & , \tau \\
S & \subset & Q
\end{pmatrix}
$$

where $R$, $P$, $Q$, $S$ are finite dimensional $\ast$-algebras (i.e., of the form $\bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C})$) with a fixed trace $\tau$ on $R$, such that $P \ominus S \perp Q \ominus S$. We denoted by $P \ominus S$, $Q \ominus S$ the orthogonal complements of $S$ in $P$, $Q$. The orthogonality is with respect to the inner product $\langle x, y \rangle_{\tau} = \tau(y^*x)$ defined by $\tau$ on $R$.

Commuting squares arise naturally in the lattice of inclusions forming the standard invariant of a subfactor. Conversely, one can construct from a commuting square a finite index hyperfinite subfactor, by iterating Jones’ basic construction ([4]). Many of the known explicit examples of subfactors were obtained using this construction. However, computing the standard invariant of the subfactor associated to a commuting square (by using Ocneanu’s compactness argument, see [5]) turns out to be quite hard, and has been done so far only for a handful of examples (see for instance [5], [1], [9]). One of the problems is that even the easiest examples, based on complex Hadamard matrices, can be technically difficult to work with (see for instance [9]).

In this paper we weaken the commuting square condition, by replacing the trace $\tau$ with a faithful state $\varphi$. We call the resulting structure a twisted commuting square. Our main
motivation is the possibility to construct finite index subfactors from twisted commuting squares, which will be discussed in a future paper.

The paper is organized as follows: In Section 2 we introduce the main definitions and discuss possible normalizations for twisted commuting squares.

In Section 3 we prove an isolation result for twisted commuting squares, which is a generalization of the first author’s Span Condition from [8]. This allows us to determine, for many of the examples we consider, if they are isolated or part of parametric families.

Section 4 is dedicated to examples of twisted commuting squares similar to the commuting squares based on (complex) Hadamard matrices. They are obtained for $R = M_n(\mathbb{C})$, $P = D$ the diagonals, $Q = ud^*$ for some unitary $u$, and $\varphi(\cdot) = \tau(\cdot a)$ for some $a \in M_n(\mathbb{C})$ positive invertible. If the twisted commuting square condition is satisfied, we call $u$ an $a$-Hadamard matrix. When $a = I_n$, this is the same as a Hadamard matrix. While there aren’t many Hadamard matrices of small dimensions (see [3], [14]), looking at $a$-Hadamards provides a richer class of examples - see for instance Proposition 4.3.

Finally, in Section 5 we use parametric families of (twisted or not) Hadamard commuting squares to obtain associative deformations $m_\lambda$ of the matrix multiplication. Our results can also be extended to multiplications on finite dimensional $*$-algebras. It turns out that certain families of (twisted) commuting squares yield associative multiplications of the form

$$m_\lambda(x, y) = m(x, y) + (\lambda - 1)m'(x, y) + (\bar{\lambda} - 1)m''(x, y), \ |\lambda| = 1$$

where $m, m'$ and $m''$ are all associative multiplications.

Deformations of the multiplication are of interest in Quantum Algebra with applications to High Energy Physics. For instance, the approach to the theory of integrable systems via the Lenard-Magri scheme ([6]) uses compatible Poisson structures, which could be obtained from linear associative deformations of the multiplication (see for instance [10]).

2 Preliminaries

Definition 2.1. A twisted commuting square of matrix algebras is a square of inclusions:

$$\left( \begin{array}{c}
P \subset R \\
\cup \\
S \subset Q
\end{array} \right), \varphi$$

where $R$, $P$, $Q$, $S$ are finite dimensional $*$-algebras (i.e., of the form $\oplus_{i=1}^k M_{n_i}(\mathbb{C})$) and $\varphi$ is a faithful state on $R$, satisfying:

$$\text{proj}_{\varphi, P} \text{proj}_{\varphi, Q} = \text{proj}_{\varphi, Q} \text{proj}_{\varphi, P} \overset{\varphi}{=} \text{proj}_{\varphi, S}$$

where $\text{proj}_{\varphi, V}$ denotes the orthogonal projection from $R$ to the subspace $V$, with respect to the inner product defined by $\varphi$ on $R$: $\langle x, y \rangle = \varphi(y^* x)$.

Equivalently, the commuting projections condition above can be written as:
where the orthogonal complements and the orthogonality are considered with respect to the inner product defined by \( \varphi \).

One of the simplest (but rich in examples) classes of twisted commuting squares is obtained by letting \( P = D, Q = uDu^* \) be two copies of the diagonal matrices \( D \subset R = M_n(\mathbb{C}) \), where \( u \) is a unitary matrix. If we let \( \tau \) denote the normalized trace on \( M_n(\mathbb{C}) \), then the faithful state \( \varphi \) is of the form \( \varphi(x) = \tau(xa) \), for some positive invertible matrix \( a \in M_n(\mathbb{C}) \). The twisted commuting square condition becomes a relation between \( u \) and \( a \), generalizing the notion of complex Hadamard matrix.

**Definition 2.2.** Let \( a \in M_n(\mathbb{C}) \) be positive and invertible, with \( \tau(a) = 1 \). We say that a unitary \( u \in M_n(\mathbb{C}) \) is a\(^{-}\)-Hadamard if

\[
\begin{pmatrix}
D \subset M_n(\mathbb{C}) \\
\cup \cup \\
\mathbb{C}I_n \subset uDu^* \\
\end{pmatrix}, \varphi
\]

is a twisted commuting square, where \( \varphi(x) = \tau(xa) \) for \( x \in M_n(\mathbb{C}) \).

**Remark 2.3.** Since \( x - \varphi(x)1 \) is orthogonal to \( \mathbb{C} \), with respect to the inner product defined by \( \varphi \), the commuting square condition can be written as \( \varphi((y - \varphi(y)1)(x - \varphi(x)1)) = 0 \), for all \( x \in D \) and \( y \in uDu^* \). After taking adjoints it also follows \( \varphi((x - \varphi(x)1)(y - \varphi(y)1)) = 0 \), for all \( x \in D \) and \( y \in uDu^* \). Equivalently:

\[
\varphi(xy) = \varphi(x)\varphi(y), \text{ for all } x \in D, y \in uDu^*.
\]

Note that this is also equivalent to \( \varphi(yx) = \varphi(x)\varphi(y) \) for all \( x \in D, y \in uDu^* \).

Let \( (e_{ij})_{1 \leq i, j \leq n} \) denote the matrix units of \( M_n(\mathbb{C}) \). It follows that \( u \) must satisfy \( \tau(u e_{ij} u^* e_{jk} a) = \tau(u e_{ij} u^* a) \tau(e_{jk} a) \) for all \( i, j \), or equivalently

\[
\sum_{1 \leq k \leq n} a_{jk} u_{ki} u_{ij} \frac{a_{ij}}{n} \sum_{1 \leq l \leq n} \left( \sum_{1 \leq k \leq n} u_{ki} a_{kl} \right) u_{li}, \text{ for all } i, j. \tag{1}
\]

**Remark 2.4.** Note that for \( a = I_n \), \( u \) being a\(^{-}\)-Hadamard is equivalent to \( u \) being a complex Hadamard matrix, i.e. \( u \) is unitary and all its entries are of the same absolute value, \( |u_{kl}| = \frac{1}{\sqrt{n}} \).

**Definition 2.5.** We say that the twisted commuting square

\[
\mathcal{C} = \begin{pmatrix}
P \subset R \\
\cup \cup \\
S \subset Q \\
\end{pmatrix}, \varphi
\]

is a twisted commuting square, where \( \varphi(x) = \tau(xa) \) for \( x \in M_n(\mathbb{C}) \).
is isomorphic to the twisted commuting square

\[ \tilde{C} = \left( \tilde{P} \subset \tilde{R}, \tilde{S} \subset \tilde{Q}, \tilde{\phi} \right) \]

if there exists a \(*\)-algebra isomorphism \( \psi : R \rightarrow \tilde{R} \) such that \( \psi(P) = \tilde{P}, \psi(S) = \tilde{S}, \psi(Q) = \tilde{Q} \) and \( \tilde{\phi}(\psi(x)) = \varphi(x) \) for all \( x \in R \).

We now present the canonical way to normalize twisted commuting squares and \( a \)-Hadamard matrices. For algebras \( B \subset A \), we recall the notation \( B' \cap A = \{ a \in A \text{ such that } ab = ba, \forall b \in B \} \).

**Lemma 2.6.** Let \( R, P, Q, S \) be finite dimensional \(*\)-algebras with a fixed trace \( \tau \) on \( R \), let \( a \in R \) be positive and invertible and let \( \varphi(x) = \tau(ax), x \in R \). For each unitary element \( u \) of \( R \), let

\[ C(u) = \left( P \subset R, S \subset uQu^*, \varphi \right). \]

Let \( q \in Q, q' \in Q' \cap P, p \in a' \cap S' \cap P, p' \in a' \cap P' \cap R \) be unitary elements. Assume that \( C(u) \) is a twisted commuting square. Then \( C(pp'uqq') \) is a twisted commuting square isomorphic to \( C(u) \).

**Proof.** Modifying \( u \) to the right by \( q, q' \) does not change the algebra \( uQu^* \) and thus does not change the twisted commuting square: \( C(pp'u) = C(pp'uqq') \). By applying \( \text{Ad}((pp')^*) \) to \( C(pp'u) \) (which leaves \( R, P, S \) invariant), we see that \( C(pp'u) \) is isomorphic to \( C(u) \). Indeed, we have \( \varphi(\text{Ad}(pp')(x)) = \varphi(x) \), since \( p \) and \( p' \) commute with \( a \). \( \blacksquare \)

**Definition 2.7.** Let \( a \in M_n(\mathbb{C}) \) be a positive, invertible matrix. We say that two \( a \)-Hadamard matrices \( u_1, u_2 \) are equivalent, written \( u_1 \sim u_2 \), if there exist unitary diagonal matrices \( d_1, d_2 \) and permutation matrices \( p_1, p_2 \) with \( p_1d_1 \) commuting with \( a \), such that \( u_2 = p_1d_1u_1d_2p_2 \).

**Remark 2.8.** Lemma 2.6 shows that equivalent \( a \)-Hadamard matrices yield isomorphic twisted commuting squares. The converse is also true, since the normalizer of \( D \) in \( M_n(\mathbb{C}) \) is the set of elements of the form \( pd \), with \( d \) diagonal unitary and \( p \) a permutation matrix. The fact that \( p_1d_1 \) commutes with \( a \) follows from the preservation of \( \varphi \) in the definition of the isomorphism of twisted commuting squares: \( \varphi(\text{Ad}(p_1d_1)) = \varphi \).

**Remark 2.9.** It is easy to see that if \( u \in M_n(\mathbb{C}) \) is an \( a \)-Hadamard and if \( d \in D \) is unitary, then \( dud^* \) is \( dad^* \)-Hadamard. Thus, we may reduce ourselves to the case when \( a \) has the entries \( a_{11}, a_{12}, \ldots, a_{1n} \) real.
While complex Hadamard matrices can be normalized (up to equivalence) to have the first row and column made of 1’s, twisted Hadamard matrices do not allow for such nice normalizations. The main problem is that we can modify from the left only by elements which commute with \(a\). The only normalization that we will use is the following.

Remark 2.10. Since we can modify \(u \rightarrow ud\) by diagonal unitaries \(d\), we may assume (up to equivalence) that the entries \(u_{11}, u_{12}, \ldots, u_{1n}\) are all real.

3 An Isolation Result for Twisted Hadamard Matrices

In this section we generalize the isolation result obtained by the first author in [8] to twisted commuting squares: we prove that if a certain Span Condition holds, then the twisted commuting square is isolated. For simplicity, we will only do this for twisted commuting squares arising from \(a\)-Hadamard matrices. All examples to which we apply the Span Condition in the next section are of this form.

Definition 3.1. Let \(a \in M_n(\mathbb{C})\) be positive and invertible, and let \(u\) be an \(a\)-twisted Hadamard matrix. We say that \(u\) is isolated if there exists \(\delta > 0\) such that if \(v\) is any \(a\)-twisted Hadamard with \(||v - u|| < \delta\), then \(v\) is equivalent to \(u\).

Let \(u\) be \(a\)-Hadamard and let \(\mathcal{C}\) be the twisted \(a\)-Hadamard commuting square associated to \(u\)

\[
\mathcal{C} = \left( \begin{array}{c}
P \subset M_n(\mathbb{C}) \cup \bigcup_{\mathcal{C} \subset Q} \\
C \subset Q
\end{array} \right), \varphi
\]

where \(P = D, Q = uDu^*\) and \(\varphi(x) = \tau(xa)\).

For two subspaces \(V, W\) of a \(*\)-algebra \(R\) we will use the notation

\[
[V, W] = \text{span}\{vw - wv : v \in V, w \in W\}
\]

Definition 3.2. We say that \(\mathcal{C}\) satisfies the Span Condition if

\[
[aP_0, Q] + [P_0a, Q] + (a' \cap P) + Q = M_n(\mathbb{C})
\]

where \(P_0 = P \oplus \varphi CI_n\)

Remark 3.3. When \(a = I_n\), the span condition becomes \([P, Q] + P + Q = M_n(\mathbb{C})\), which is the span condition for Hadamard matrices introduced by the first author in [8].

Remark 3.4. For most \(a\)'s that we will apply this result to, we have \(a' \cap P = C\), so the span condition becomes \([aP_0, Q] + [P_0a, Q] + Q = M_n(\mathbb{C})\).

Theorem 3.5. Let \(u\) be an \(a\)-Hadamard matrix. If the associated twisted commuting square \(\mathcal{C}\) satisfies the Span Condition, then \(u\) is isolated.
Proof. Assume that the span condition is satisfied, but $u$ is not isolated. Then there exists unitaries $u_n, u_n \to I$, such that $u_n u$ are all $a$–Hadamard matrices, not equivalent to $u$. So

$$C(u_n) = \left( \begin{array}{c}
\mathbb{C} \subset M_n(\mathbb{C}) \\
\cup \\
\cup \\
\mathbb{C} \subset u_nQu_n^* , \varphi
\end{array} \right)$$

are all twisted commuting squares. By a compactness argument, we may assume after eventually passing to a subsequence that $h = \lim_{n \to \infty} \frac{u_n - I}{||u_n - I||}$ exists. Note that $||h|| = 1$. Since $u_n$ are unitaries, it follows that $h$ is self-adjoint. By the same argument as in Lemma 1.8 from [8], after eventually modifying $u_n \to p_n u_q q_n$ with $p_n \in a' \cap P$ and $q_n \in Q$ unitaries converging to $I$, we may assume that $h$ is orthogonal to the vector space $(a' \cap P) + Q$. By orthogonal we mean with respect to the canonical inner product on $M_n(\mathbb{C})$, given by the trace.

Writing the twisted commuting square condition for each $u_n$ gives $\varphi(pu_nqu_n^*) = \varphi(p)\varphi(u_nqu_n^*)$. For $\varphi(p) = 0$, we can rewrite $\varphi(pu_nqu_n^*) = 0$ as:

$$\varphi(p(u_n - I)qu_n^*) + \varphi(pq(u_n - I)^*) = 0$$

After dividing by $i||u_n - I||$ and taking the limit as $n \to \infty$, we obtain:

$$\varphi(phq) - \varphi(pqh) = 0$$

Equivalently: $\tau(phqa) - \tau(pqha) = 0$. Which by using the properties of the trace is equivalent to

$$\tau(h[ap, q]) = 0$$

This shows that $h$ is orthogonal to $[aP_0, Q]$, where $P_0 = P \ominus \varphi CI_n$. Since $h$ is self-adjoint and $P_0, Q$ are closed to $\ast$, it follows that $h$ will also be orthogonal to $[aP_0, Q]^* = [P_0a, Q]$. Thus, $h$ is orthogonal to $[aP_0, Q] + [P_0a, Q] + (a' \cap P) + Q = M_n(\mathbb{C})$, which implies $h = 0$, contradicting $||h|| = 1$. 

4 Examples

In this section we present various examples of twisted commuting squares, arising from unitaries $u$ which are $a$-Hadamard, for some $a$ positive and invertible. We will avoid the examples that yield classical commuting squares (without a twist), i.e. based on complex Hadamard matrices. A catalogue of the known complex Hadamard matrices can be found at [14]. We start by observing that diagonal matrices $a$ will only give Hadamard examples.

Proposition 4.1. Let $a \in \mathbb{D}$ be positive and invertible, $\tau(a) = 1$. Then, $u$ is $a$–Hadamard if and only if $u$ is a complex Hadamard matrix.

Proof. Let $e_{ij}$ denote the matrix units of $M_n(\mathbb{C})$. Suppose $u$ is a Hadamard matrix. Since $ae_{jj} \in \mathbb{D}$, we have $\tau(u e_{ij} u^* e_{jj} a) = \tau(u e_{ij} u^*)\tau(e_{jj} a) = \tau(u e_{ij} u^* a)\tau(e_{jj} a).$
Conversely, suppose $u$ is $a$–Hadamard and $a = \sum_{1 \leq k \leq n} \lambda_k e_{kk}$. Then

$$\lambda_j |u_{jj}|^2 = \lambda_j \tau(ue_{ii}u^*e_{jj}) = \tau(ue_{ii}u^*e_{jj}) = \tau(ue_{ii}u^*a)\tau(e_{jj}a) = \frac{\lambda_j}{n} \tau(ue_{ii}u^*a).$$

Therefore, we have $|u_{jj}| = |u_{kk}|$ for any fixed $i$ and $1 \leq j,k \leq n$. Because $u$ is unitary, it follows $|u_{jj}| = \frac{1}{\sqrt{n}}$ for all $j,i$. Hence, $u$ is a Hadamard matrix. 

\[\square\]

**Remark 4.2.** It is possible for a Hadamard matrix to be $a$–Hadamard, even if $a$ is not diagonal. For instance, if $a$ is the matrix having $1$ on the diagonal and all the other entries equal to some $t \in (0,1)$, then it is easy to see that any Hadamard matrix having the first column made of $1$’s (so in particular any Hadamard matrix in normalized form) is also $a$–Hadamard.

The next proposition shows that for each $n > 4$ there exists at least one pair $(a,u)$ with $a$ positive invertible, and $u$ an $a$–Hadamard matrix which is not a complex Hadamard matrix. Moreover, the matrices $a,u$ are circulant matrices. In particular, they commute.

**Proposition 4.3.** Let $n > 4$ and let $e_{ij}$ denote the matrix units of $M_n(\mathbb{C})$. Let $B = \sum_{i,j=1}^n e_{ij}$ and let $X = \sum_{i=1}^n e_{i+1}$, where the index $n+1$ is identified with $1$. Let $u = \frac{2}{n} B - X$ and let $a = I_n + \frac{n-4}{2n-4}(B - I_n)$. Then $u$ is an $a$–Hadamard.

**Proof.** Using $B^2 = nB$, $XB = BX = B$ and $X^*B = BX^* = B$, it is immediate to check that $uu^* = (\frac{2}{n} B - X)(\frac{2}{n} B - X^*) = \frac{4}{n^2} B + I_n - \frac{4}{n} B = I_n$. Thus $u$ is a unitary. It is easy to see that $a$ is positive and invertible.

We now check that the $a$–Hadamard condition $\tau(ae_{kk}ue_{ll}u^*) = \tau(ue_{kk})\tau(au_{ll}u^*)$ holds for all $k,l$. This is equivalent to

$$(u^*a)_{lk} \cdot u_{kl} = \frac{1}{n} a_{kk} \cdot (u^*au)_{ll}. \quad (u^*a)_{lk} \cdot u_{kl} = \frac{1}{n}.$$

Since $ua = au$, the right part of the equality above is $\frac{1}{n}$. For the left part, notice that $u^*a = (\frac{2}{n} B - X^*)(I_n + \frac{n-4}{2n-4}(B - I_n)) = 1\frac{1}{2} B - \frac{n}{2n-4} X^*$. This matrix has all entries equal to $\frac{1}{2}$, except those on positions $k+1,k$, which equal $\frac{1}{2n-4}$. Since $u$ has the entries equal to $\frac{2}{n}$, except the entries $u_{k,k+1} = 2\frac{n}{n}$, it follows that $(u^*a)_{lk} \cdot u_{kl} = \frac{1}{n}$. 

\[\square\]

We now classify the unitaries $u$ that are $a$–Hadamard, for an arbitrary $2 \times 2$ positive invertible matrix $a$, $\tau(a) = 1$. By remarks 2.9 and 2.10, we may assume $a = \begin{pmatrix} \alpha & \lambda \\ \lambda & 2 - \alpha \end{pmatrix} \in M_2(\mathbb{R})$ where $0 < \alpha < 2$ and $|\lambda| < \sqrt{2\alpha - \alpha^2}$, and $u_{11}, u_{12} \in \mathbb{R}$. Then (1) yields the following equations:

$$\alpha u_{1k}^2 + \lambda u_{1k}u_{2k} = \frac{\alpha}{2}(u_{1k}^2 + (2 - \alpha)|u_{2k}|^2 + 2\lambda u_{1k}Re_{2k}) \quad (2)$$

$$\lambda u_{1k} Re_{2k} + (2 - \alpha)|u_{2k}|^2 = \frac{2 - \alpha}{2}(u_{1k}^2 + (2 - \alpha)|u_{2k}|^2 + 2\lambda u_{1k} Re_{2k}) \quad (3)$$
Since the right hand sides of equations 2 and 3 are all real, \( u_{2k} \in \mathbb{R} \). Using this and the previous equations, we obtain

\[
\alpha(2 - \alpha) \left(|u_{1k}|^2 - |u_{2k}|^2\right) = 2(\alpha - 1)\lambda u_{1k}u_{2k}. \tag{4}
\]

We note that \( \alpha = 1 \) or \( \lambda = 0 \) yields \( |u_{11}| = |u_{21}| \) and \( |u_{12}| = |u_{22}| \) and \( u \) unitary would then imply \( |u_{11}| = |u_{21}| = |u_{12}| = |u_{22}| = \frac{1}{\sqrt{2}} \). Thus, \( u \) is equivalent to the unitary

\[
u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & -1 \end{pmatrix}.
\]

For \( \lambda \neq 0 \) and \( \alpha \neq 1 \), let \( \omega = \frac{2(\alpha - 1)}{\alpha(2 - \alpha)} \lambda \). Note that each of \( |u_{ij}| \neq 0 \) for \( 1 \leq i, j \leq 2 \) (else, a column of \( u \) would be 0) and \( \omega \neq 0 \). Let \( r = \frac{u_{1k}}{u_{2k}} \) and note that \( r, \omega \in \mathbb{R} \). Dividing equation 4 by \( u_{2k}^2 \), we get \( r^2 - 1 = \omega r \). Letting \( r_1 = \frac{\omega + \sqrt{\omega^2 + 1}}{2} \), we see that \( u_{1k} = ru_{2k} \) with \( r \in \{r_1, -\frac{1}{r_1}\} \). Letting \( u_{11} = ru_{21} \) and \( u_{12} = r' u_{22} \) with \( r, r' \in \{r_1, -\frac{1}{r_1}\} \), the unitary conditions are \( 1 = r^2 u_{21}^2 + (r')^2 u_{22}^2 = u_{21}^2 + u_{22}^2 \) and \( 0 = ru_{21}^2 + r'u_{22}^2 \) which imply \( r \neq r' \). Hence, \( u \) is equivalent to a unitary of the form:

\[
\begin{pmatrix} rr_1 & \frac{-1}{r}r_2 \\ r_1 & r_2 \end{pmatrix}
\]

where \( r_1 = \frac{1}{\sqrt{1 + r^2}} \), \( r_2 = \frac{r}{\sqrt{1 + r^2}} \), \( \omega = \frac{2(\alpha - 1)}{\alpha(2 - \alpha)} \lambda \), and \( r = \frac{\omega + \sqrt{\omega^2 + 1}}{2} \).

**Example 4.4.** For \( a = \left(\frac{1 + \sqrt{5}}{2}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3} - \sqrt{5}}{3}\right) \), we obtain \( u = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{-1}{\sqrt{2}}\right) \). A calculation via mathematica shows the span condition is satisfied, so \( u \) is isolated among the \( a \)-Hadamard matrices.

We now give a parametric family of matrices of order 3 with a corresponding family of twisted Hadamard matrices.

**Example 4.5.** Let \( a = \left(\frac{1}{\sqrt{2 + t^2}}, \frac{2t^2 - 1}{t}, \frac{t}{t^2 + 1}\right) \) where \( 0 < t < 1 \). Let \( x = -1 + t^4 - t\sqrt{2 - 3t^2 + t^6}, y = -1 + t^4 + t\sqrt{2 - 3t^2 + t^6}, w = -t + t^3 - \sqrt{2 - 3t^2 + t^6}, \) and \( z = -t + t^3 + \sqrt{2 - 3t^2 + t^6} \). Then

\[
u = \begin{pmatrix} \frac{1}{\sqrt{2 + t^2}} & \sqrt{\frac{t}{2(t^2 + 2)(t^2 - 1)}} & \sqrt{\frac{y}{2(t^2 + 2)(t^2 - 1)}} \\ \frac{1}{\sqrt{2 + t^2}} & -\sqrt{\frac{t^2 - 1}{2(t^2 + 2)}} & \sqrt{\frac{2y(t^2 - 1)}{2(t^2 + 2)}} \\ \frac{t}{\sqrt{2 + t^2}} & \frac{1}{w} \sqrt{\frac{2x(t^2 - 1)}{(t^2 + 2)}} & \frac{1}{z} \sqrt{\frac{2y(t^2 - 1)}{(t^2 + 2)}} \end{pmatrix}
\]

is a-Hadamard.

While the example above looks hard to work with in general, certain values of \( t \) give unitaries with rational entries.
Example 4.6. For \( t = \frac{1}{2} \), we obtain 
\[
a = \begin{pmatrix}
1 & -\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} & 1 & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & 1
\end{pmatrix}
\]
and 
\[
u = \begin{pmatrix}
\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\
\frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\
-\frac{2}{3} & \frac{2}{3} & \frac{1}{3}
\end{pmatrix}.
\]

A calculation via Mathematica shows the span condition is satisfied, so \( u \) is an isolated \( a-\)Hadamard matrix.

We now show how for any \( n \) non-prime, \( n = k \cdot m \), parametric families of \( a \)-Hadamards (a fixed) can be constructed from fixed \( a \)-Hadamards of smaller orders. This construction is a natural generalization of the Dita-Haagerup type Hadamard matrices (see [3],[14]).

**Proposition 4.7.** Let \( a_1 \in M_m(\mathbb{C}) \) and \( a_2 \in M_k(\mathbb{C}) \) be positive, invertible matrices. Suppose that \( b \) is \( a_2 \)-Hadamard and \( c_1, c_2, \ldots, c_k \) are \( a_1 \)-Hadamard. Then

\[
u = \begin{bmatrix}
\begin{bmatrix}
 b_{11}c_1 & b_{12}c_2 & \cdots & b_{1k}c_k \\
b_{21}c_1 & b_{22}c_2 & \cdots & b_{2k}c_k \\
\vdots & \vdots & \ddots & \vdots \\
b_{k1}c_1 & b_{k2}c_2 & \cdots & b_{kk}c_k
\end{bmatrix}
\end{bmatrix}
\]
is \( a_1 \otimes a_2 \)-Hadamard

**Proof.** Let \((e_{ij})_{1 \leq i, j \leq m}\) and \((f_{i'j'})_{1 \leq i', j' \leq k}\) be the matrix units for \( M_m(\mathbb{C}) \) and \( M_k(\mathbb{C}) \) respectively. Making an appropriate identification of \( M_{km}(\mathbb{C}) \) with \( M_m(\mathbb{C}) \otimes M_k(\mathbb{C}) \), we have 

\[
u = \sum_{p,q,r,s} b_{pq}c_q \otimes f_{pq} a_1 \otimes u^* = \sum_{r,s} b_{rs} c_r^* \otimes f_{sr}. \]

We need to verify that for each \( i, i', j, j' \)

\[
\tau(u(e_{ii} \otimes f_{i'i'})u^*(e_{jj} \otimes f_{j'j'})a) = \tau(u(e_{ii} \otimes f_{i'i'})u^*a) \tau((e_{jj} \otimes f_{j'j'})a). \tag{6}
\]

Using the linearity of \( \tau \), we have the left side of equation 6 is:

\[
\sum_{p,q,r,s} \tau(b_{pq}c_q \otimes e_{ij} \otimes f_{pq} f_{sr} f_{j'j'} a_2).
\]

Note that \( f_{pq} f_{i'i'} f_{sr} f_{j'j'} \neq 0 \iff q = i' = s, r = j' \). Thus, the left side of (6) reduces to

\[
\sum_{p} \tau(c_{i'i} e_{ii} e_{jj} a_1) \tau(b_{i'i} f_{i'i'} f_{j'j'} a_2). \tag{7}
\]

Using similar arguments as above, the right side reduces to:

\[
\sum_{p,r} \tau(c_{i'i} e_{ii} c_{i'i} \tau(b_{i'i} b_{jr} f_{jr} a_2) \tau(e_{jj} a_1) \tau(f_{j'j'} a_2). \tag{8}
\]

Since \( b = \sum_{p,q} b_{pq} f_{pq} \) is \( a_2 \)-Hadamard and for a fixed \( i', c_{i'i} \) is \( a_1 \)-Hadamard, we have:

\[
\sum_{p} \tau(b_{i'i} f_{i'i'} f_{j'j'} a_2) = \sum_{p,r} \tau(b_{i'i} b_{jr} f_{jr} a_2) \tau(f_{j'j'} a_2) \tag{9}
\]

\[
\tau(c_{i'i} e_{ii} c_{i'i} e_{jj} a_1) = \tau(c_{i'i} e_{ii} c_{i'i} a_1) \tau(e_{jj} a_1). \tag{10}
\]

Finally, we plug (10) and (9) into (7) to get (8).
Using the previous proposition, we now give a parametric family of $4 \times 4$ of $a$–Hadamard matrices, with $a$ fixed.

**Example 4.8.** Let $a = \begin{pmatrix} \alpha & \lambda \\ \lambda & 2 - \alpha \end{pmatrix}$ where $0 < \alpha < 2$ and $0 < \lambda < \sqrt{2\alpha - \alpha^2}$. Let
\[
\omega = \frac{2\lambda(a-1)}{a(2-a)} \quad \text{and} \quad r = \frac{\omega + \sqrt{\omega^2 + 4}}{2}.
\]
Let $r_1 = \frac{1}{\sqrt{1+r^2}}$ and $r_2 = \frac{r}{\sqrt{1+r^2}}$. Then equation (5) gives
\[
c_1 = \begin{pmatrix} rr_1 & -r_2 \\ r_1 & r_2 \end{pmatrix} \quad \text{and} \quad c_2(t) = \begin{pmatrix} rr_1 & -r_2 \exp(it) \\ r_1 & r_2 \exp(it) \end{pmatrix}
\]
are $a$–Hadamard for any real number $t$. Furthermore,
\[
u(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} c_1 & c_1 \\ c_2(t) & -c_2(t) \end{pmatrix}
\]
is a one parameter family of $a \otimes I_2$-Hadamard matrices.

The simplest parametric family that we were able to obtain is the following:

**Example 4.9.** Let $a = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} & \frac{3-\sqrt{5}}{2} \end{pmatrix}$ and $\lambda \in \mathbb{T}$. Then the following is a parametric family of $a$-Hadamards:
\[
u_{\lambda} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \lambda & -\frac{\sqrt{3}}{2} & \frac{1}{2} \lambda \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \lambda & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \lambda \end{pmatrix}.
\]
The span condition can not hold for $\nu_{\lambda}$, as it is not isolated. Indeed, a Mathematica calculation shows that the dimension of the space from the span condition is 13 for $\lambda = 1$.

## 5 Twisted Hadamard Matrices and Associative Deformations of the Matrix Product

Starting from a parametric family of twisted Hadamard commuting squares, we obtain a continuous family $m_t$ ($t \in \mathbb{R}$) of associative multiplications on $M_n(\mathbb{C})$, where $m_0$ is the canonical matrix multiplication. Moreover, $m_t$ will coincide with $m_0$ when restricted to any of the two MASAs which are the corners of the initial twisted commuting square.

Associative deformations of the multiplication are a subject of interest in Quantum Algebra with applications to High Energy Physics. For instance, the approach to the theory of integrable systems via the Lenard-Magri scheme ([6]) uses compatible Poisson structures, which could be obtained from linear associative deformations of the multiplication (see for instance [10]).
Due to the rigidity of semi-simple associative algebras, for $t$ small, the multiplications $m_t$ must be of the form $m_t(x, y) = \varphi_t^{-1}(\varphi_t(x) \varphi_t(y))$, where $\varphi_t : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a linear isomorphism. We will construct, from twisted commuting squares, such families $\varphi_t$ for which the structural constants of the multiplications $m_t$ are easy to compute. Moreover, we will see that certain families of commuting squares, such as those arising from Petrescu’s Hadamard matrices, yield associative multiplications of the form

$$m_\lambda(x, y) = m(x, y) + (\lambda - 1)m'(x, y) + (\bar{\lambda} - 1)m''(x, y), \quad |\lambda| = 1$$

where $m, m'$ and $m''$ are all associative multiplications, with $m = m_0$. Equivalently, this will give a parametric family of multiplications $m_t$ of degree two in $t$.

Let $a \in M_n(\mathbb{C})$ be positive invertible, let $\varphi(x) = \tau(ax)$ and let $u$ be an $a$-Hadamard unitary with no non-zero entries. Denote $P = D$, $Q = uDu^*$. Let $p_i$ and $q_i = up_iu^*$, $1 \leq i \leq n$, denote the minimal projections of $P$, respectively $Q$. We begin by finding some easy formulas for the structural constants of the canonical matrix multiplication $m$, in the bases given by $p_i, q_i$. Note that $p_iq_j$ span $M_n(\mathbb{C})$:

**Proposition 5.1.** Let $u \in M_n(\mathbb{C})$ be a unitary, $P = D$ and $Q = uDu^*$. Then

$$\text{span}\{pq : p \in P, q \in Q\} = M_n(\mathbb{C})$$

if and only if $u$ has no zero entries.

**Proof.** $\text{span}\{pq : p \in P, q \in Q\} = M_n(\mathbb{C})$ if and only if $v_{i,j} = p_iq_j, 1 \leq i, j \leq n$ form a basis for $M_n(\mathbb{C})$. We have: $\tau(v_{i,j}v_{k,l}') = \tau(p_kv_qp_kq_l') = \delta^k_i\delta^l_j\tau(p_kv_q) = \frac{1}{n}\delta^k_i\delta^l_j|u_{kl}|^2$. So the $n^2 \times n^2$ matrix with entries $\tau(v_{k,l}v_{p,q}')$ is diagonal and it is invertible if and only if all the diagonal entries $|u_{kl}|^2$ are non-zero. \hfill \blacksquare

It follows that for all $1 \leq k, l, k', l' \leq n$, the structural coefficients $c^i_{jkl'}$ of the multiplication satisfy: $(p_kq_l)(p_{k'}q_{l'}) = \Sigma c^i_{jkl'}p_{q_j}q_j$. By multiplying by $p_k$ to the left and $q_l$ to the right, it follows $(p_kq_l)(p_{k'}q_{l'}) = c^i_{jkl'}p_{k'}q_{l'}$, so $c^i_{jkl'} = \varphi(p_kq_ip_{k'}q_{l'})/\varphi(p_k)\varphi(q_l)$ and $c^i_{jkl'} = 0$ for $(i, j) \neq (k, l')$. Thus, we obtain:

$$
(p_kq_l)(p_{k'}q_{l'}) = \gamma^i_{k'l'}p_{k'}q_{l'}, \quad \text{where} \quad \gamma^i_{k'l'} = \frac{\varphi(p_kq_ip_{k'}q_{l'})}{\varphi(p_k)\varphi(q_{l'})}.
$$

(11)

Let now $u_t (t \in \mathbb{R})$ be a continuous parametric family of $a$-Hadamard unitaries, with $u_0 = u$. Denote $Q_t = u_tD^*_t$, so $Q_0 = Q$. Let $\varphi_t : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be given by

$$\varphi_t(p_kq_ip_{l'}u_q^*) = p_ku_ip_{l'}u_q^*$$

for any $1 \leq k, l \leq n$. For $t$ small, the entries of $u_t$ will also be non-zero, so $p_ku_ip_{l'}u_q^*, 1 \leq k, l \leq n$ form a basis of $M_n(\mathbb{C})$. It follows that $\varphi_t$ extends to a linear isomorphism of $M_n(\mathbb{C})$, and $\varphi_t(pq) = p\text{Ad}(u_tu_0^*)(q)$ for all $p \in P, q \in Q$. Let $m_t(x, y) = \varphi_t^{-1}(\varphi_t(x)\varphi_t(y))$.

**Proposition 5.2.** The associative multiplication $m_t$ satisfies: $m_t(x, y) = m(x, y)$ for all $(x, y)$ with $x \in P$ or $y \in Q$.
Proof. We have \( m_t(pq, q') = \varphi^{-1}(p\text{Ad}(u_t^*)u_t^*)(q\text{Ad}(u_t^*)u_t^*)(q') = \varphi^{-1}(p\text{Ad}(u_t^*)u_t^*)(qq') = pq = m(pq, q') \), for all \( p \in P \) and \( q, q' \in Q \). This shows that \( m_t(x, y) = m(x, y) \) for all \( y \in Q \). The other equality follows by a similar argument.

We now find the structural constants of the multiplication \( m_t \). From the formula for \( m_t \), we have: \( m_t(p_k u_p u_t u_r^*, p_k' u_p u_t u_r^*) = \varphi^{-1}(p_k u_p u_t u_r^* p_k' u_p u_t u_r^*) \). By arguments similar to those that lead to (11), we have

\[
p_k u_p u_t u_r^* p_k' u_p u_t u_r^* = \gamma_{kl}^{k'l'}(t)p_k u_t u_r^*, \quad \text{with} \quad \gamma_{kl}^{k'l'}(t) = \frac{\varphi(p_k u_p u_t u_r^* p_k' u_p u_t u_r^*)}{\varphi(p_k)\varphi(u_t u_r^*)}.
\]

So we obtain:

\[
m_t(p_k u_p u_t u_r^*, p_k' u_p u_t u_r^*) = \gamma_{kl}^{k'l'}(t)p_k u_p u_t u_r^*.
\]

In the case when \( a = \frac{1}{n}I_n \), so \( u_t = (u_{kl}(t))_{1 \leq k, l \leq n} \) are all Hadamard matrices, we have \( \gamma_{kl}^{k'l'}(t) = n^2\tau(p_k u_p u_t u_r^* p_k' u_p u_t u_r^*) = \frac{u_{k'l'}(t)}{u_{kl}(t)u_{kl'}(t)} \). It follows:

**Proposition 5.3.** Consider a continuous family of Hadamard matrices \( u_t = (u_{kl}(t))_{1 \leq k, l \leq n}, t \text{ real} \). Then the following is a parametric family of associative multiplications of \( M_n(\mathbb{C}) \):

\[
m_t(p_k u_p u_t u_r^*, p_k' u_p u_t u_r^*) = \gamma_{kl}^{k'l'}(t)p_k u_p u_t u_r^*, \quad \text{where} \quad \gamma_{kl}^{k'l'}(t) = \frac{u_{kl}(t)u_{k'l'}(t)}{u_{kl}(t)u_{kl'}(t)}
\]

We now observe that certain families of Hadamard matrices, which depend linearly on a parameter \( \lambda \in \mathbb{T} \), yield multiplications \( m_\lambda \) linear in \( \lambda, \bar{\lambda} \). This is somewhat surprising, as the formula for the structural constants \( \gamma_{kl}^{k'l'}(\lambda) \) of \( m_\lambda \) involves products of four entries of \( u \).

We first recall the family of complex Hadamard matrices of order 4 found by Haagerup in [3]:

\[
F_4(\lambda) = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & \lambda & -\lambda \\
1 & 1 & -1 & -1 \\
1 & -1 & -\lambda & \lambda \\
\end{pmatrix}, \lambda \in \mathbb{T}
\]

We also recall the following family of complex Hadamard matrices of order 7 found by Petrescu in [11]. Let \( w = \cos \frac{2\pi}{6} + isin \frac{2\pi}{6} \) and let

\[
P_7(\lambda) = \frac{1}{\sqrt{7}} \begin{pmatrix}
\lambda w & \lambda w^4 & w^5 & w^3 & w^3 & w^3 & 1 \\
\lambda w^4 & \lambda w & w^3 & w^5 & w^3 & w^3 & 1 \\
w^5 & w^3 & \lambda w & w^4 & w^3 & w^3 & 1 \\
w^3 & w^5 & \lambda w^4 & \lambda w & w^3 & w^3 & 1 \\
w^3 & w^3 & w & w^4 & w^5 & 1 \\
w^3 & w^3 & w^3 & w^5 & w^4 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}, \lambda \in \mathbb{T}
\]
Proposition 5.4. If $u_\lambda = F_4(\lambda)$ or $u_\lambda = P_7(\lambda)$ ($\lambda \in \mathbb{T}$), then the parametric family of associative multiplications $m_\lambda$ associated to $u_\lambda$, as in Proposition 5.3, is of the form

$$m_\lambda = m + (\lambda - 1)m' + (\bar{\lambda} - 1)m'',$$

where $m$ is the canonical matrix multiplication and $m', m''$ are associative matrix multiplications.

Proof. To show that $m_\lambda$ can be expressed as $m + (\lambda - 1)m' + (\bar{\lambda} - 1)m''$, it is enough to show that none of the coefficients $\gamma_{k,l}^{k',l'}(\lambda)$ contain $\lambda^2$ or $\bar{\lambda}^2$. This is clear, since the families of Hadamard matrices above have the property: no single row or column contains multiples of both $\lambda$ and $\bar{\lambda}$, and if $u_{kl}(\lambda)$ and $u_{k'l'}(\lambda)$ are both multiples of $\lambda$ (respectively $\bar{\lambda}$), then so are $u_{k'l'}(\lambda)$ and $u_{k'l}(\lambda)$.

The fact that $m', m''$ must be multiplications follows by writing the associativity of $m_\lambda$ and identifying the coefficients of $\lambda^2, \bar{\lambda}^2$, which are linearly independent of $1, \lambda, \bar{\lambda}$ for $\lambda \in \mathbb{T}$. □

Remark 5.5. The same holds true for the deformation of multiplication arising from the twisted $a$-Hadamard matrices from Example 4.8. It is also true for several other known linear parametric families of Hadamard matrices.

Remark 5.6. If we use $\bar{\lambda} = 1/\lambda$, we can write $\lambda m_\lambda$ in the form $A + B\lambda + C\lambda^2$, with $\lambda \in \mathbb{T}$. It easily follows that $A + Bt + Ct^2$ is an associative multiplication for any parameter $t$. Note that $A$ and $C$, but not $B$, are also associative multiplications (for the examples above).

References


