Twisted Commuting Squares

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Abstract

We consider generalizations of commuting squares, called twisted commuting squares, obtained by having the commuting square orthogonality condition hold with respect to the inner product given by a faithful state on a finite dimensional $*$-algebra. We present various examples of twisted commuting squares, most of which are computationally easy to work with, and we prove an isolation result. We also show how parametric families of (not necessarily) twisted commuting squares yield associative deformations of the matrix multiplication.

1 Introduction

Commuting squares were introduced by S. Popa in [12] (see also [5]), as invariants and construction data for subfactors. A *commuting square* is a square of inclusions:

$$\begin{pmatrix} P & \subset & R & \\ \cup & & \cup & ,\tau \\ S & \subset & Q & \end{pmatrix}$$

where R, P, Q, S are finite dimensional $*-$algebras (i.e., of the form $\oplus_{i=1}^{k}\mathrm{M_{n_i}}(\mathbb{C})$) with a fixed trace τ on R, such that $P \ominus S \perp Q \ominus S$. We denoted by$P \ominus S$, $Q \ominus S$ the orthogonal complements of S in P, Q. The orthogonality is with respect to the inner product $< x, y >_\tau = \tau(y^*x)$ defined by τ on R.

Commuting squares arise naturally in the lattice of inclusions forming the standard invariant of a subfactor. Conversely, one can construct from a commuting square a finite index hyperfinite subfactor, by iterating Jones' basic construction ([4]). Many of the known explicit examples of subfactors were obtained using this construction. However, computing the standard invariant of the subfactor associated to a commuting square (by using Ocneanu's compactness argument, see [5]) turns out to be quite hard, and has been done so far only for a handful of examples (see for instance [5], [1], [9]). One of the problems is that even the easiest examples, based on complex Hadamard matrices, can be technically difficult to work with (see for instance [9]).

In this paper we weaken the commuting square condition, by replacing the trace τ with a faithful state φ. We call the resulting structure a twisted commuting square. Our main motivation is the possibility to construct finite index subfactors from twisted commuting squares, which will be discussed in a future paper.

The paper is organized as follows: In Section 2 we introduce the main definitions and discuss possible normalizations for twisted commuting squares.

In Section 3 we prove an isolation result for twisted commuting squares, which is a generalization of the first author's Span Condition from [8]. This allows us to determine, for many of the examples we consider, if they are isolated or part of parametric families.

Section 4 is dedicated to examples of twisted commuting squares similar to the commuting squares based on (complex) Hadamard matrices. They are obtained for $R = \mathrm{M_n}(\mathbb{C})$, $P = \mathrm{D}$ the diagonals, $Q = u\mathrm{D}u^*$ for some unitary u, and $\varphi(\cdot) = \tau(\cdot a)$ for some $a \in \mathrm{M_n}(\mathbb{C})$ positive invertible. If the twisted commuting square condition is satisfied, we call u an $a-$Hadamard matrix. When $a = I_n$, this is the same as a Hadamard matrix. While there aren't many Hadamard matrices of small dimensions (see [3], [14]), looking at $a-$Hadamards provides a richer class of examples - see for instance Proposition 4.3.

Finally, in Section 5 we use parametric families of (twisted or not) Hadamard commuting squares to obtain associative deformations m_λ of the matrix multiplication. Our results can also be extended to multiplications on finite dimensional $*$-algebras. It turns out that certain families of (twisted) commuting squares yield associative multiplications of the form

$$m_\lambda(x, y) = m(x, y) + (\lambda - 1)m'(x, y) + (\bar{\lambda} - 1)m''(x, y),\ |\lambda| = 1$$

where m, m' and m'' are all associative multiplications.

Deformations of the multiplication are of interest in Quantum Algebra with applications to High Energy Physics. For instance, the approach to the theory of integrable systems via the Lenard-Magri scheme ([6]) uses compatible Poisson structures, which could be obtained from linear associative deformations of the multiplication (see for instance [10]).

2 Preliminaries

Definition 2.1. *A* twisted commuting square *of matrix algebras is a square of inclusions:*

$$\begin{pmatrix} P & \subset & R & \\ \cup & & \cup & ,\varphi \\ S & \subset & Q & \end{pmatrix}$$

where R, P, Q, S are finite dimensional $*-$algebras (i.e., of the form $\oplus_{i=1}^k \mathrm{M_{n_i}}(\mathbb{C})$) and φ is a faithful state on R, satisfying:

$$\mathrm{proj}_{\varphi,P}\mathrm{proj}_{\varphi,Q} = \mathrm{proj}_{\varphi,Q}\mathrm{proj}_{\varphi,P} = \mathrm{proj}_{\varphi,S}$$

where $\mathrm{proj}_{\varphi,V}$ denotes the orthogonal projection from R to the subspace V, with respect to the inner product defined by φ on R: $\langle x, y\rangle = \varphi(y^*x)$.

Equivalently, the commuting projections condition above can be written as:

$$P \ominus_{\varphi} S \perp_{\varphi} Q \ominus_{\varphi} S$$

where the orthogonal complements and the orthogonality are considered with respect to the inner product defined by φ.

One of the simplest (but rich in examples) classes of twisted commuting squares is obtained by letting $P = D, Q = uDu^*$ be two copies of the diagonal matrices $D \subset R = \mathrm{M_n}(\mathbb{C})$, where u is a unitary matrix. If we let τ denote the normalized trace on $\mathrm{M_n}(\mathbb{C})$, then the faithful state φ is of the form $\varphi(x) = \tau(xa)$, for some positive invertible matrix $a \in \mathrm{M_n}(\mathbb{C})$. The twisted commuting square condition becomes a relation between u and a, generalizing the notion of complex Hadamard matrix.

Definition 2.2. *Let $a \in \mathrm{M_n}(\mathbb{C})$ be positive and invertible, with $\tau(a) = 1$. We say that a unitary $u \in \mathrm{M_n}(\mathbb{C})$ is $a-$*Hadamard *if*

$$\begin{pmatrix} \mathrm{D} & \subset & \mathrm{M_n}(\mathbb{C}) & \\ \cup & & \cup & ,\varphi \\ \mathbb{C}I_n & \subset & u\mathrm{D}\mathrm{u}^* & \end{pmatrix}$$

is a twisted commuting square, where $\varphi(x) = \tau(xa)$ for $x \in \mathrm{M_n}(\mathbb{C})$.

Remark 2.3. *Since $x - \varphi(x)1$ is orthogonal to $\mathbb{C}$, with respect to the inner product defined by φ, the commuting square condition can be written as $\varphi((y - \varphi(y)1)(x - \varphi(x)1)) = 0$, for all $x \in \mathrm{D}$ and $y \in u\mathrm{Du}^*$. After taking adjoints it also follows $\varphi((x - \varphi(x)1)(y - \varphi(y)1)) = 0$, for all $x \in \mathrm{D}$ and $y \in u\mathrm{Du}^*$. Equivalently:*

$$\varphi(xy) = \varphi(x)\varphi(y), \text{ for all } x \in \mathrm{D}, \mathrm{y} \in \mathrm{uDu}^*.$$

Note that this is also equivalent to $\varphi(yx) = \varphi(x)\varphi(y)$ for all $x \in \mathrm{D}, \mathrm{y} \in \mathrm{uDu}^*$.

Let $(e_{i,j})_{1 \leq i,j \leq n}$ denote the matrix units of $\mathrm{M_n}(\mathbb{C})$. It follows that u must satisfy $\tau(ue_{i\,i}u^*e_{j\,j}a) = \tau(ue_{i\,i}u^*a)\tau(e_{j\,j}a)$ for all i, j, or equivalently

$$\sum_{1 \leq k \leq n} a_{j\,k} u_{k\,i} \overline{u_{j\,i}} = \frac{a_{j\,j}}{n} \sum_{1 \leq l \leq n} \left(\sum_{1 \leq k \leq n} \overline{u_{k\,i}} a_{k\,l} \right) u_{l\,i}, \text{ for all } i, j. \tag{1}$$

Remark 2.4. *Note that for $a = I_n$, u being $a-$Hadamard is equivalent to u being a complex Hadamard matrix, i.e. u is unitary and all its entries are of the same absolute value, $|u_{k\,l}| = \frac{1}{\sqrt{n}}$.*

Definition 2.5. *We say that the twisted commuting square*

$$\mathfrak{C} = \begin{pmatrix} P & \subset & R & \\ \cup & & \cup & ,\varphi \\ S & \subset & Q & \end{pmatrix}$$

is isomorphic *to the twisted commuting square*

$$\widetilde{\mathfrak{C}} = \begin{pmatrix} \widetilde{P} & \subset & \widetilde{R} & \\ \cup & & \cup & ,\widetilde{\varphi} \\ \widetilde{S} & \subset & \widetilde{Q} & \end{pmatrix}$$

if there exists a $*-$algebra isomorphism $\psi : R \to \widetilde{R}$ such that $\psi(P) = \widetilde{P}$, $\psi(S) = \widetilde{S}$, $\psi(Q) = \widetilde{Q}$ and $\widetilde{\varphi}(\psi(x)) = \varphi(x)$ for all $x \in R$.

We now present the canonical way to normalize twisted commuting squares and a-Hadamard matrices. For algebras $B \subset A$, we recall the notation $B' \cap A = \{a \in A \text{ such that } ab = ba, \forall b \in B\}$.

Lemma 2.6. *Let R, P, Q, S be finite dimensional $*-$algebras with a fixed trace τ on R, let $a \in R$ be positive and invertible and let $\varphi(x) = \tau(ax)$, $x \in R$. For each unitary element,u, of R, let*

$$\mathfrak{C}(u) = \begin{pmatrix} P & \subset & R & \\ \cup & & \cup & ,\varphi \\ S & \subset & uQu^* & \end{pmatrix}.$$

Let $q \in Q$, $q' \in Q' \cap P$, $p \in a' \cap S' \cap P$, $p' \in a' \cap P' \cap R$ be unitary elements. Assume that $\mathfrak{C}(u)$ is a twisted commuting square. Then $\mathfrak{C}(pp'uqq')$ is a twisted commuting square isomorphic to $\mathfrak{C}(u)$.

Proof. Modifying u to the right by q, q' does not change the algebra uQu^* and thus does not change the twisted commuting square: $\mathfrak{C}(pp'u) = \mathfrak{C}(pp'uqq')$. By applying $\mathrm{Ad}((pp')^*)$ to $\mathfrak{C}(pp'u)$ (which leaves R, P, S invariant), we see that $\mathfrak{C}(pp'u)$ is isomorphic to $\mathfrak{C}(u)$. Indeed, we have $\varphi(\mathrm{Ad}(pp')(x)) = \varphi(x)$, since p and p' commute with a.

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Definition 2.7. *Let $a \in \mathrm{M_n}(\mathbb{C})$ be a positive, invertible matrix. We say that two a-Hadamard matrices u_1, u_2 are* equivalent*, written $u_1 \sim u_2$, if there exist unitary diagonal matrices d_1, d_2 and permutation matrices p_1, p_2 with $p_1 d_1$ commuting with a, such that $u_2 = p_1 d_1 u_1 d_2 p_2$.*

Remark 2.8. *Lemma 2.6 shows that equivalent a-Hadamard matrices yield isomorphic twisted commuting squares. The converse is also true, since the normalizer of* D *in* $\mathrm{M_n}(\mathbb{C})$ *is the set of elements of the form pd, with d diagonal unitary and p a permutation matrix. The fact that $p_1 d_1$ commutes with a follows from the preservation of φ in the definition of the isomorphism of twisted commuting squares: $\varphi(Ad(p_1 d_1)) = \varphi$.*

Remark 2.9. *It is easy to see that if $u \in \mathrm{M_n}(\mathbb{C})$ is a-Hadamard and if $d \in \mathrm{D}$ is unitary, then dud^* is dad^*-Hadamard. Thus, we may reduce ourselves to the case when a has the entries $a_{11}, a_{12}, ..., a_{1n}$ real.*

While complex Hadamard matrices can be normalized (up to equivalence) to have the first row and column made of 1's, twisted Hadamard matrices do not allow for such nice normalizations. The main problem is that we can modify from the left only by elements which commute with a. The only normalization that we will use is the following.

Remark 2.10. *Since we can modify $u \to ud$ by diagonal unitaries d, we may assume (up to equivalence) that the entries $u_{11}, u_{12}, ..., u_{1n}$ are all real.*

3 An Isolation Result for Twisted Hadamard Matrices

In this section we generalize the isolation result obtained by the first author in [8] to twisted commuting squares: we prove that if a certain Span Condition holds, then the twisted commuting square is isolated. For simplicity, we will only do this for twisted commuting squares arising from a-Hadamard matrices. All examples to which we apply the Span Condition in the next section are of this form.

Definition 3.1. *Let $a \in \mathrm{M_n}(\mathbb{C})$ be positive and invertible, and let u be an a-twisted Hadamard matrix. We say that u is* isolated *if there exists $\delta > 0$ such that if v is any a-twisted Hadamard with $||v - u|| < \delta$, then v is equivalent to u.*

Let u be $a-$Hadamard and let $\mathfrak{C}$ be the twisted a-Hadamard commuting square associated to u

$$\mathfrak{C} = \begin{pmatrix} P & \subset & \mathrm{M_n}(\mathbb{C}) & \\ \cup & & \cup & , \varphi \\ \mathbb{C} & \subset & Q & \end{pmatrix}$$

where $P = \mathrm{D}, \mathrm{Q} = \mathrm{uDu^*}$ and $\varphi(x) = \tau(xa)$.

For two subspaces V, W of a $*-$algebra R we will use the notation

$$[V, W] = \mathrm{span}\{vw - wv : v \in V, w \in W\}$$

Definition 3.2. *We say that $\mathfrak{C}$ satisfies* the Span Condition *if*

$$[aP_0, Q] + [P_0 a, Q] + (a' \cap P) + Q = \mathrm{M_n}(\mathbb{C})$$

where $P_0 = P \ominus_\varphi \mathbb{C}I_n$

Remark 3.3. *When $a = I_n$, the span condition becomes $[P, Q] + P + Q = \mathrm{M_n}(\mathbb{C})$, which is the span condition for Hadamard matrices introduced by the first author in [8].*

Remark 3.4. *For most a's that we will apply this result to, we have $a' \cap P = \mathbb{C}$, so the span condition becomes $[aP_0, Q] + [P_0 a, Q] + Q = \mathrm{M_n}(\mathbb{C})$.*

Theorem 3.5. *Let u be an a-Hadamard matrix. If the associated twisted commuting square $\mathfrak{C}$ satisfies the Span Condition, then u is isolated.*

Proof. Assume that the span condition is satisfied, but u is not isolated. Then there exists unitaries u_n, $u_n \to I$, such that $u_n u$ are all $a-$Hadamard matrices, not equivalent to u. So

$$\mathfrak{C}(\mathfrak{u}_\mathfrak{n}) = \begin{pmatrix} P & \subset & \mathrm{M_n}(\mathbb{C}) & \\ \cup & & \cup & ,\varphi \\ \mathbb{C} & \subset & u_n Q u_n^* & \end{pmatrix}$$

are all twisted commuting squares. By a compactness argument, we may assume after eventually passing to a subsequence that $h = \lim_{n\to\infty} \frac{u_n - I}{i||u_n - I||}$ exists. Note that $||h|| = 1$. Since u_n are unitaries, it follows that h is self-adjoint. By the same argument as in Lemma 1.8 from [8], after eventually modifying $u_n \to p_n u_n q_n$ with $p_n \in a' \cap P$ and $q_n \in Q$ unitaries convering to I, we may assume that h is orthogonal to the vector space $(a' \cap P) + Q$. By orthogonal we mean with respect to the canonical inner product on $\mathrm{M_n}(\mathbb{C})$, given by the trace.

Writing the twisted commuting square condition for each u_n gives $\varphi(pu_n qu_n^*) = \varphi(p)\varphi(u_n qu_n^*)$. For $\varphi(p) = 0$, we can rewrite $\varphi(pu_n qu_n^*) = 0$ as:

$$\varphi(p(u_n - I)qu_n^*) + \varphi(pq(u_n - I)^*) = 0$$

After dividing by $i||u_n - I||$ and taking the limit as $n \to \infty$, we obtain:

$$\varphi(phq) - \varphi(pqh) = 0$$

Equivalently: $\tau(phqa) - \tau(pqha) = 0$. Which by using the properties of the trace is equivalent to

$$\tau(h[ap, q]) = 0$$

This shows that h is orthogonal to $[aP_0, Q]$, where $P_0 = P \ominus_\varphi \mathbb{C}I_n$. Since h is self-adjoint and P_0, Q are closed to $*$, it follows that h will also be orthogonal to $[aP_0, Q]^* = [P_0 a, Q]$. Thus, h is orthogonal to $[aP_0, Q] + [P_0 a, Q] + (a' \cap P) + Q = \mathrm{M_n}(\mathbb{C})$, which implies $h = 0$, contradicting $||h|| = 1$. ■

4 Examples

In this section we present various examples of twisted commuting squares, arising from unitaries u which are a-Hadamard, for some a positive and invertible. We will avoid the examples that yield classical commuting squares (without a twist), i.e. based on complex Hadamard matrices. A catalogue of the known complex Hadamard matrices can be found at [14]. We start by observing that diagonal matrices a will only give Hadamard examples.

Proposition 4.1. *Let $a \in \mathrm{D}$ be positive and invertible, $\tau(a) = 1$. Then, u is $a-$Hadamard if and only if u is a complex Hadamard matrix.*

Proof. Let e_{ij} denote the matrix units of $\mathrm{M_n}(\mathbb{C})$. Suppose u is a Hadamard matrix. Since $ae_{jj} \in \mathrm{D}$, we have $\tau(ue_{ii}u^*e_{jj}a) = \tau(ue_{ii}u^*)\tau(e_{jj}a) = \tau(ue_{ii}u^*a)\tau(e_{jj}a)$.

Conversely, suppose u is $a-$Hadamard and $a = \sum_{1\leq k\leq n} \lambda_k e_{k\,k}$. Then

$$\lambda_j |u_{j\,i}|^2 = \lambda_j \tau(ue_{i\,i}u^*e_{j\,j}) = \tau(ue_{i\,i}u^*e_{j\,j}a) = \tau(ue_{i\,i}u^*a)\tau(e_{j\,j}a) = \frac{\lambda_j}{n}\tau(ue_{i\,i}u^*a).$$

Therefore, we have $|u_{j\,i}| = |u_{k\,i}|$ for any fixed i and $1 \leq j, k \leq n$. Because u is unitary, it follows $|u_{j\,i}| = \frac{1}{\sqrt{n}}$ for all j, i. Hence, u is a Hadamard matrix.

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Remark 4.2. *It is possible for a Hadamard matrix to be a-Hadamard, even if a is not diagonal. For instance, if a is the matrix having* 1 *on the diagonal and all the other entries equal to some* $t \in (0,1)$*, then it is easy to see that any Hadamard matrix having the first column made of* $1'$*s (so in particular any Hadamard matrix in normalized form) is also* $a-$*Hadamard.*

The next proposition shows that for each $n > 4$ there exists at least one pair (a, u) with a positive invertible, and u an a-Hadamard matrix which is not a complex Hadamard matrix. Moreover, the matrices a, u are circulant matrices. In particular, they commute.

Proposition 4.3. *Let* $n > 4$ *and let* e_{ij} *denote the matrix units of* $\mathrm{M_n}(\mathbb{C})$*. Let* $B = \sum_{i,j=1}^{n} e_{ij}$ *and let* $X = \sum_{i=1}^{n} e_{ii+1}$*, where the index* $n+1$ *is identified with* 1*. Let* $u = \frac{2}{n}B - X$ *and let* $a = I_n + \frac{n-4}{2n-4}(B - I_n)$*. Then* u *is* a*-Hadamard.*

Proof. Using $B^2 = nB$, $XB = BX = B$ and $X^*B = BX^* = B$, it is immediate to check that $uu^* = (\frac{2}{n}B - X)(\frac{2}{n}B - X^*) = \frac{4}{n^2}nB + I_n - \frac{4}{n}B = I_n$. Thus u is a unitary. It is easy to see that a is positive and invertible.

We now check that the a-Hadamard condition $\tau(ae_{kk}ue_{ll}u^*) = \tau(ae_{kk})\tau(aue_{ll}u^*)$ holds for all k, l. This is equivalent to

$$(u^*a)_{lk} \cdot u_{kl} = \frac{1}{n} a_{kk} \cdot (u^*au)_{ll}.$$

Since $ua = au$, the right part of the equality above is $\frac{1}{n}$. For the left part, notice that $u^*a = (\frac{2}{n}B - X^*)(I_n + \frac{n-4}{2n-4}(B - I_n)) = \frac{1}{2}B - \frac{n}{2n-4}X^*$. This matrix has all entries equal to $\frac{1}{2}$, except those on positions $k+1, k$, which equal $\frac{1}{2-n}$. Since u has the entries equal to $\frac{2}{n}$, except the entries $u_{k,k+1} = \frac{2-n}{n}$, it follows that $(u^*a)_{lk} \cdot u_{kl} = \frac{1}{n}$.

■

We now classify the unitaries u that are $a-$Hadamard, for an arbitrary 2×2 positive invertible matrix a, $\tau(a) = 1$. By remarks 2.9 and 2.10, we may assume $a = \begin{pmatrix} \alpha & \lambda \\ \lambda & 2-\alpha \end{pmatrix} \in \mathrm{M}_2(\mathbb{R})$ where $0 < \alpha < 2$ and $|\lambda| < \sqrt{2\alpha - \alpha^2}$, and $u_{1\,1}$, $u_{1\,2} \in \mathbb{R}$. Then (1) yields the following equations:

$$\alpha u_{1\,k}^2 + \lambda u_{1\,k}u_{2\,k} = \frac{\alpha}{2}\left(\alpha u_{1\,k}^2 + (2-\alpha)\,|u_{2\,k}|^2 + 2\lambda u_{1\,k}\Re u_{2\,k}\right) \tag{2}$$

$$\lambda u_{1\,k}\overline{u_{2\,k}} + (2-\alpha)\,|u_{2\,k}|^2 = \frac{2-\alpha}{2}\left(\alpha u_{1\,k}^2 + (2-\alpha)\,|u_{2\,k}|^2 + 2\lambda u_{1\,k}\Re u_{2\,k}\right) \tag{3}$$

Since the right hand sides of equations 2 and, 3 are all real, $u_{2\,k} \in \mathbb{R}$. Using this and the previous equations, we obtain

$$\alpha(2-\alpha)\left(|u_{1\,k}|^2 - |u_{2\,k}|^2\right) = 2(\alpha-1)\lambda u_{1\,k}u_{2\,k}. \tag{4}$$

We note that $\alpha = 1$ or $\lambda = 0$ yields $|u_{1\,1}| = |u_{2\,1}|$ and $|u_{1\,2}| = |u_{2\,2}|$ and u unitary would then imply $|u_{1\,1}| = |u_{2\,1}| = |u_{1\,2}| = |u_{2\,2}| = \frac{1}{\sqrt{2}}$. Thus, u is equivalent to the unitary $u = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

For $\lambda \neq 0$ and $\alpha \neq 1$, let $\omega = \frac{2(\alpha-1)}{\alpha(2-\alpha)}\lambda$. Note that each of $|u_{i\,j}| \neq 0$ for $1 \leq i,\, j \leq 2$ (else, a column of u would be 0) and $\omega \neq 0$. Let $r = \frac{u_{1\,k}}{u_{2\,k}}$ and note that $r,\, \omega \in \mathbb{R}$. Dividing equation 4 by $u_{2\,k}^2$, we get $r^2 - 1 = \omega r$. Letting $r_1 = \frac{\omega+\sqrt{\omega^2+4}}{2}$, we see that $u_{1\,k} = ru_{2\,k}$ with $r \in \{r_1, \frac{-1}{r_1}\}$. Letting $u_{1\,1} = ru_{2\,1}$ and $u_{1\,2} = r'u_{2\,2}$ with $r,\, r' \in \{r_1, \frac{-1}{r_1}\}$, the unitary conditions are $1 = r^2u_{2\,1}^2 + (r')^2u_{2\,2}^2 = u_{2\,1}^2 + u_{2\,2}^2$ and $0 = ru_{2\,1}^2 + r'u_{2\,2}^2$ which imply $r \neq r'$. Hence, u is equivalent to a unitary of the form:

$$\begin{pmatrix} rr_1 & \frac{-1}{r}r_2 \\ r_1 & r_2 \end{pmatrix} \tag{5}$$

where $r_1 = \frac{1}{\sqrt{1+r^2}}$, $r_2 = \frac{r}{\sqrt{1+r^2}}$, $\omega = \frac{2(\alpha-1)}{\alpha(2-\alpha)}\lambda$, and $r = \frac{\omega+\sqrt{\omega^2+4}}{2}$.

Example 4.4. *For* $a = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} & \frac{3-\sqrt{5}}{2} \end{pmatrix}$, *we obtain* $u = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$. *A calculation via mathematica shows the span condition is satisfied, so u is isolated among the a-Hadamard matrices.*

We now give a parametric family of matrices of order 3 with a corresponding family of twisted Hadamard matrices.

Example 4.5. *Let* $a = \begin{pmatrix} 1 & 2t^2-1 & t \\ 2t^2-1 & 1 & t \\ t & t & 1 \end{pmatrix}$ *where* $0 < t < 1$. *Let* $x = -1 + t^4 - t\sqrt{2-3t^2+t^6}$, $y = -1 + t^4 + t\sqrt{2-3t^2+t^6}$, $w = -t + t^3 - \sqrt{2-3t^2+t^6}$, *and* $z = -t + t^3 + \sqrt{2-3t^2+t^6}$. *Then*

$$u = \begin{pmatrix} \frac{1}{\sqrt{2+t^2}} & \sqrt{\frac{x}{2(t^2+2)(t^2-1)}} & \sqrt{\frac{y}{2(t^2+2)(t^2-1)}} \\ \frac{1}{\sqrt{2+t^2}} & -\sqrt{\frac{t^2-1}{2x(t^2+2)}} & -\sqrt{\frac{t^2-1}{2(t^2+2)y}} \\ \frac{t}{\sqrt{2+t^2}} & \frac{1}{w}\sqrt{\frac{2x(t^2-1)}{(t^2+2)}} & \frac{1}{z}\sqrt{\frac{2y(t^2-1)}{(t^2+2)}} \end{pmatrix}$$

is $a-$*Hadamard.*

While the example above looks hard to work with in general, certain values of t give unitaries with rational entries.

Example 4.6. *For $t = \frac{1}{2}$, we obtain* $a = \begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$ *and* $u = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix}$.
A calculation via Mathematica shows the span condition is satisfied, so u is an isolated $a-$Hadamard matrix.

We now show how for any n non-prime, $n = k \cdot m$, parametric families of a-Hadamards (a fixed) can be constructed from fixed a-Hadamards of smaller orders. This construction is a natural generalization of the Dita-Haagerup type Hadamard matrices (see [3],[14])

Proposition 4.7. *Let $a_1 \in \mathrm{M_m}(\mathbb{C})$ and $a_2 \in \mathrm{M_k}(\mathbb{C})$ be positive, invertible matrices. Suppose that b is a_2-Hadamard and $c_1, c_2, \ldots, c_k$ are a_1-Hadamard. Then*

$$u = \begin{bmatrix} b_{1\,1}c_1 & b_{1\,2}c_2 & \cdots & b_{1\,k}c_k \\ b_{2\,1}c_1 & b_{2\,2}c_2 & \cdots & b_{2\,k}c_k \\ \vdots & \vdots & & \vdots \\ b_{k\,1}c_1 & b_{k\,2}c_2 & \cdots & b_{k\,k}c_k \end{bmatrix} \quad \textit{is } a = a_1 \otimes a_2\textit{-Hadamard}$$

Proof. Let $(e_{i\,j})_{1 \le i,j \le m}$ and $(f_{i'\,j'})_{1 \le i',j' \le k}$ be the matrix units for $\mathrm{M_m}(\mathbb{C})$ and $\mathrm{M_k}(\mathbb{C})$ respectively. Making an appropriate identification of $\mathrm{M_{km}}(\mathbb{C})$ with $\mathrm{M_m}(\mathbb{C}) \otimes \mathrm{M_k}(\mathbb{C})$, we have $u = \sum_{p,q} b_{p\,q} c_q \otimes f_{p\,q}$ and $u^* = \sum_{r,s} \overline{b_{r\,s}} c_s^* \otimes f_{s\,r}$. We need to verify that for each i, i', j, j'

$$\tau\left(u(e_{i\,i} \otimes f_{i'\,i'})u^*(e_{j\,j} \otimes f_{j'\,j'})a\right) = \tau\left(u(e_{i\,i} \otimes f_{i'\,i'})u^* a\right) \tau\left((e_{j\,j} \otimes f_{j'\,j'})a\right). \tag{6}$$

Using the linearity of τ, we have the left side of equation 6 is:

$$\sum_{p,q,r,s} \tau\left(b_{p\,q} c_q e_{i\,i} \overline{b_{r\,s}} c_s^* e_{j\,j} a_1 \otimes f_{p\,q} f_{i'\,i'} f_{s\,r} f_{j'\,j'} a_2\right).$$

Note that $f_{p\,q} f_{i'\,i'} f_{s\,r} f_{j'\,j'} \neq 0 \Leftrightarrow q = i' = s$, $r = j'$. Thus, the left side of (6) reduces to

$$\sum_{p} \tau\left(c_{i'} e_{i\,i} c_{i'}^* e_{j\,j} a_1\right) \tau\left(b_{p\,i'} \overline{b_{j'\,i'}} f_{p\,j'} a_2\right). \tag{7}$$

Using similar arguments as above, the right side reduces to:

$$\sum_{p,r} \tau\left(c_{i'} e_{i\,i} c_{i'}^* a_1\right) \tau\left(b_{i'\,q} \overline{b_{r\,i'}} f_{p\,r} a_2\right) \tau\left(e_{j\,j} a_1\right) \tau\left(f_{j'\,j'} a_2\right). \tag{8}$$

Since $b = \sum_{p\,q} b_{p\,q} f_{p\,q}$ is a_2-Hadamard and for a fixed i', $c_{i'}$ is a_1-Hadamard, we have:

$$\begin{aligned} \sum_{p} \tau\left(b_{p\,i'} \overline{b_{j'\,i'}} f_{p\,j'} a_2\right) &= \sum_{p\,r} \tau\left(b_{p\,i'} \overline{b_{r\,i'}} f_{p\,r}\right) \tau\left(f_{j'\,j'} a_2\right) &(9) \\ \tau\left(c_{i'} e_{i\,i} c_{i'}^* e_{j\,j} a_1\right) &= \tau\left(c_{i'} e_{i\,i} c_{i'}^* a_1\right) \tau\left(e_{j\,j} a_1\right). &(10) \end{aligned}$$

Finally, we plug (10) and (9) into (7) to get (8).

■

Using the previous proposition, we now give a parametric family of 4×4 of $a-$Hadamard matrices, with a fixed.

Example 4.8. *Let* $a = \begin{pmatrix} \alpha & \lambda \\ \lambda & 2-\alpha \end{pmatrix}$ *where* $0 < \alpha < 2$ *and* $0 < \lambda < \sqrt{2\alpha - \alpha^2}$. *Let* $\omega = \frac{2\lambda(\alpha-1)}{\alpha(2-\alpha)}$ *and* $r = \frac{\omega + \sqrt{\omega^2+4}}{2}$. *Let* $r_1 = \frac{1}{\sqrt{1+r^2}}$, *and* $r_2 = \frac{r}{\sqrt{1+r^2}}$. *Then equation (5) gives*

$$c_1 = \begin{pmatrix} rr_1 & -\frac{r_2}{r} \\ r_1 & r_2 \end{pmatrix} \text{ and } c_2(t) = \begin{pmatrix} rr_1 & \frac{-r_2}{r}\exp(\mathrm{i}t) \\ r_1 & r_2\exp(\mathrm{i}t) \end{pmatrix}$$

are $a-$*Hadamard for any real number* t. *Furthermore,*

$$u(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} c_1 & c_1 \\ c_2(t) & -c_2(t) \end{pmatrix}$$

is a one parameter family of $a \otimes I_2$*-Hadamard matrices.*

The simplest parametric family that we were able to obtain is the following:

Example 4.9. *Let* $a = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} & \frac{3-\sqrt{5}}{2} \end{pmatrix}$ *and* $\lambda \in \mathbb{T}$. *Then the following is a parametric family of a-Hadamards:*

$$u_\lambda = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} & \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2}\lambda & \frac{-\sqrt{3}}{2} & \frac{1}{2}\lambda \\ \frac{1}{2} & \frac{\sqrt{3}}{2}\lambda & \frac{-1}{2} & \frac{-\sqrt{3}}{2}\lambda \end{pmatrix}.$$

The span condition can not hold for u_λ, *as it is not isolated. Indeed, a Mathematica calculation shows that the dimension of the space from the span condition is 13 for* $\lambda = 1$.

5 Twisted Hadamard Matrices and Associative Deformations of the Matrix Product

Starting from a parametric family of twisted Hadamard commuting squares, we obtain a continuous family m_t ($t \in \mathbb{R}$) of associative multiplications on $\mathrm{M_n}(\mathbb{C})$, where m_0 is the canonical matrix multiplication. Moreover, m_t will coincide with m_0 when restricted to any of the two MASAs which are the corners of the initial twisted commuting square.

Associative deformations of the multiplication are a subject of interest in Quantum Algebra with applications to High Energy Physics. For instance, the approach to the theory of integrable systems via the Lenard-Magri scheme ([6]) uses compatible Poisson structures, which could be obtained from linear associative deformations of the multiplication (see for instance [10]).

Due to the rigidity of semi-simple associative algebras, for t small, the multiplications m_t must be of the form $m_t(x,y) = \varphi_t^{-1}(\varphi_t(x)\varphi_t(y))$, where $\varphi_t : \mathrm{M_n}(\mathbb{C}) \to \mathrm{M_n}(\mathbb{C})$ is a linear isomorphism. We will construct, from twisted commuting squares, such families φ_t for which the structural constants of the multiplications m_t are easy to compute. Moreover, we will see that certain families of commuting squares, such as those arising from Petrescu's Hadamard matrices, yield associative multiplications of the form

$$m_\lambda(x,y) = m(x,y) + (\lambda - 1)m'(x,y) + (\bar{\lambda} - 1)m''(x,y),\ |\lambda| = 1$$

where m, m' and m'' are all associative multiplications, with $m = m_0$. Equivalently, this will give a parametric family of multiplications m_t of degree two in t.

Let $a \in M_n(\mathbb{C})$ be positive invertible, let $\varphi(x) = \tau(ax)$ and let u be an a-Hadamard unitary with no non-zero entries. Denote $P = \mathrm{D}$, $Q = uDu^*$. Let p_i and $q_i = up_iu^*$, $1 \leq i \leq n$, denote the minimal projections of P, respectively Q. We begin by finding some easy formulas for the structural constants of the canonical matrix multiplication m, in the bases given by p_i, q_i. Note that p_iq_j span $\mathrm{M_n}(\mathbb{C})$:

Proposition 5.1. *Let $u \in \mathrm{M_n}(\mathbb{C})$ be a unitary, $P = \mathrm{D}$ and $Q = u\mathrm{D}u^*$. Then*

$$\mathrm{span}\{\mathrm{pq} : \mathrm{p} \in \mathrm{P}, \mathrm{q} \in \mathrm{Q}\} = \mathrm{M_n}(\mathbb{C})$$

if and only if u has no zero entries.

Proof. $\mathrm{span}\{\mathrm{pq} : \mathrm{p} \in \mathrm{P}, \mathrm{q} \in \mathrm{Q}\} = \mathrm{M_n}(\mathbb{C})$ if and only if $v_{i,j} = p_iq_j, 1 \leq i,j \leq n$ form a basis for $\mathrm{M_n}(\mathbb{C})$. We have: $\tau(v_{k,l}v^*_{k',l'}) = \tau(p_kq_lq_{l'}p_{k'}) = \delta_k^{k'}\delta_l^{l'}\tau(p_kq_l) = \frac{1}{n}\delta_k^{k'}\delta_l^{l'}|u_{kl}|^2$. So the $n^2 \times n^2$ matrix with entries $\tau(v_{k\,l}v^*_{k'\,l'})$ is diagonal and it is invertible if and only if all the diagonal entries $|u_{kl}|^2$ are non-zero. ■

It follows that for all $1 \leq k, l, k', l' \leq n$, the structural coefficients $c^{ij}_{klk'l'}$ of the multiplication satisfy: $(p_kq_l)(p_{k'}q_{l'}) = \Sigma^n_{i,j=1}c^{ij}_{klk'l'}p_iq_j$. By multiplying by p_k to the left and $q_{l'}$ to the right, it follows $(p_kq_l)(p_{k'}q_{l'}) = c^{kl'}_{klk'l'}p_kq_{l'}$, so $c^{kl'}_{klk'l'} = \varphi(p_kq_lp_{k'}q_{l'})/\varphi(p_k)\varphi(q_{l'})$ and $c^{ij}_{klk'l'} = 0$ for $(i,j) \neq (k,l')$. Thus, we obtain:

$$(p_kq_l)(p_{k'}q_{l'}) = \gamma^{k'l'}_{kl}\,p_kq_{l'},\ \text{ where } \gamma^{k'l'}_{kl} = \frac{\varphi(p_kq_lp_{k'}q_{l'})}{\varphi(p_k)\varphi(q_{l'})}. \tag{11}$$

Let now u_t $(t \in \mathbb{R})$ be a continuous parametric family of $a - Hadamard$ unitaries, with $u_0 = u$. Denote $Q_t = u_t\mathrm{D}u_t^*$, so $Q_0 = Q$. Let $\varphi_t : \mathrm{M_n}(\mathbb{C}) \to \mathrm{M_n}(\mathbb{C})$ be given by

$$\varphi_t(p_kup_lu^*) = p_ku_tp_lu_t^*$$

for any $1 \leq k, l \leq n$. For t small, the entries of u_t will also be non-zero, so $p_ku_tp_lu_t^*$, $1 \leq k, l \leq n$ form a basis of $\mathrm{M_n}(\mathbb{C})$. It follows that φ_t extends to a linear isomorphism of $\mathrm{M_n}(\mathbb{C})$, and $\varphi_t(pq) = p\mathrm{Ad}(u_tu_0^*)(q)$ for all $p \in P, q \in Q$. Let $m_t(x,y) = \varphi_t^{-1}(\varphi_t(x)\varphi_t(y))$.

Proposition 5.2. *The associative multiplication m_t satisfies: $m_t(x,y) = m(x,y)$ for all (x,y) with $x \in P$ or $y \in Q$*

Proof. We have $m_t(pq,q') = \varphi_t^{-1}(p\mathrm{Ad}(u_tu_0^*)(q)\mathrm{Ad}(u_tu_0^*)(q')) = \varphi_t^{-1}(p\mathrm{Ad}(u_tu_0^*)(qq')) = pqq' = m(pq,q')$, for all $p \in P$ and $q,q' \in Q$. This shows that $m_t(x,y) = m(x,y)$ for all $y \in Q$. The other equality follows by a similar argument. ■

We now find the structural constants of the multiplication m_t. From the formula for m_t, we have: $m_t(p_kup_lu^*, p_{k'}up_{l'}u^*) = \varphi^{-1}(p_ku_tp_lu_t^*p_{k'}u_tp_{l'}u_t^*)$. By arguments similar to those that lead to (11), we have

$$p_ku_tp_lu_t^*p_{k'}u_tp_{l'}u_t^* = \gamma_{kl}^{k'l'}(t)p_ku_tp_{l'}u_t^*, \text{ with } \gamma_{kl}^{k'l'}(t) = \frac{\varphi(p_ku_tp_lu_t^*p_{k'}u_tp_{l'}u_t^*)}{\varphi(p_k)\varphi(u_tp_{l'}u_t^*)}.$$

So we obtain:

$$m_t(p_kup_lu^*, p_{k'}up_{l'}u^*) = \gamma_{kl}^{k'l'}(t)p_kup_{l'}u^*.$$

In the case when $a = \frac{1}{n}I_n$, so $u_t = (u_{kl}(t))_{1\leq k,l\leq n}$ are all Hadamard matrices, we have $\gamma_{kl}^{k'l'}(t) = n^2\tau(p_ku_tp_lu_t^*p_{k'}u_tp_{l'}u_t^*) = n\frac{u_{kl}u_{k'l'}}{u_{k'l}u_{kl'}}$. It follows:

Proposition 5.3. *Consider a continuous family of Hadamard matrices $u_t = (u_{kl}(t))_{1\leq k,l\leq n}$, t real. Then the following is a parametric family of associative multiplications of* $\mathrm{M_n}(\mathbb{C})$*:*

$$m_t(p_kup_lu^*, p_{k'}up_{l'}u^*) = \gamma_{kl}^{k'l'}(t)p_kup_{l'}u^*, \text{ where } \gamma_{kl}^{k'l'}(t) = n\frac{u_{kl}(t)u_{k'l'}(t)}{u_{k'l}(t)u_{kl'}(t)}$$

We now observe that certain families of Hadamard matrices, which depend linearly on a parameter $\lambda \in \mathbb{T}$, yield multiplications m_λ linear in $\lambda, \bar{\lambda}$. This is somewhat surprising, as the formula for the structural constants $\gamma_{kl}^{k'l'}(\lambda)$ of m_λ involves products of four entries of u.

We first recall the family of complex Hadamard matrices of order 4 found by Haagerup in [3]:

$$F_4(\lambda) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & \lambda & -\lambda \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -\lambda & \lambda \end{pmatrix}, \lambda \in \mathbb{T}$$

We also recall the following family of complex Hadamard matrices of order 7 found by Petrescu in [11]. Let $w = cos\frac{2\pi}{6} + isin\frac{2\pi}{6}$ and let

$$P_7(\lambda) = \frac{1}{\sqrt{7}}\begin{pmatrix} \lambda w & \lambda w^4 & w^5 & w^3 & w^3 & w & 1 \\ \lambda w^4 & \lambda w & w^3 & w^5 & w^3 & w & 1 \\ w^5 & w^3 & \bar{\lambda} w & \bar{\lambda} w^4 & w & w^3 & 1 \\ w^3 & w^5 & \bar{\lambda} w^4 & \bar{\lambda} w & w & w^3 & 1 \\ w^3 & w^3 & w & w & w^4 & w^5 & 1 \\ w & w & w^3 & w^3 & w^5 & w^4 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \lambda \in \mathbb{T} \tag{12}$$

Proposition 5.4. *If $u_\lambda = F_4(\lambda)$ or $u_\lambda = P_7(\lambda)$ ($\lambda \in \mathbb{T}$), then the parametric family of associative multiplications m_λ associated to u_λ, as in Proposition 5.3, is of the form*

$$m_\lambda = m + (\lambda - 1)m' + (\bar{\lambda} - 1)m'', \lambda \in \mathbb{T}$$

where m is the canonical matrix multiplication and m', m'' are associative matrix multiplications.

Proof. To show that m_λ can be expressed as $m + (\lambda - 1)m' + (\bar{\lambda} - 1)m''$, it is enough to show that none of the coefficients $\gamma_{k,l}^{k',l'}(\lambda)$ contain λ^2 or $\bar{\lambda}^2$. This is clear, since the families of Hadamard matrices above have the property: no single row or column contains multiples of both λ and $\bar{\lambda}$, and if $u_{kl}(\lambda)$ and $u_{k'l'}(\lambda)$ are both multiples of λ (respectively $\bar{\lambda}$), then so are $u_{kl'}(\lambda)$ and $u_{k'l}(\lambda)$.

The fact that m', m'' must be multiplications follows by writing the associativity of m_λ and identifying the coefficients of $\lambda^2, \bar{\lambda}^2$, which are linearly independent of $1, \lambda, \bar{\lambda}$ for $\lambda \in \mathbb{T}$. ■

Remark 5.5. *The same holds true for the deformation of multiplication arising from the twisted a-Hadamard matrices from Example 4.8. It is also true for several other known linear parametric families of Hadamard matrices.*

Remark 5.6. *If we use $\bar{\lambda} = 1/\lambda$, we can write λm_λ in the form $A + B\lambda + C\lambda^2$, with $\lambda \in \mathbb{T}$. It easily follows that $A + Bt + Ct^2$ is an associative multiplication for any parameter t. Note that A and C, but not B, are also associative multiplications (for the examples above).*

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