An Adaptive Marking Strategy for Discontinuous Galerkin Methods

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Introduction

- Finite Element Approximation of solutions to Elliptic PDEs.
- *Adaptive Refinement* is required for efficient and accurate numerical approximation.
- Use of aposteriori estimators aids in the decision making process regarding *where* to refine.
- Modification of *SER* (Solve - Estimate - Refine) iterative procedure.(Dörfler, 1996)
- Joint work with Ohannes Karakashian.
- Use of the `triangle` and `showme` programs (Shewchuk, 1996) for generating initial mesh and pictures.

Why DG?

Discontinuous Galerkin (DG) methods sacrifice additional degrees of freedom (unknowns) for the following benefits:

- *Regular* refinement of triangular elements by connecting midpoints of edges.
- No continuity requirements of solutions along edges between elements allows for easy construction of trial and test spaces.
- Preserves minimum angle condition.
- Allows use of highly nonuniform and unstructured meshes.

DG Implementation

We have chosen the Symmetric Interior Penalty formulation of DG. There are several advantages to making this choice:

- Stiffness matrix is symmetric, positive definite.
- Allows for use of symmetric linear solvers such as Conjugate Gradient and Preconditioned Conjugate Gradient.

Model Problem

Let $\Omega \subset \mathbb{R}^d, d = 1, 2, 3$ be a bounded open polyhedral domain with Lipshitz continuous boundary.

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g_D & \text{on } \Gamma_D \\ \nabla u \cdot n = g_N & \text{on } \Gamma_N \end{cases} \qquad \text{(MP)}$$

where $\partial\Omega := \Gamma = \Gamma_D \cup \Gamma_N$ and n is the unit normal vector exterior to Ω. We also assume that $\mu_{d-1}(\Gamma_D) > 0$, $f \in L^2(\Omega)$, $g_N \in L^2(\Gamma_N)$. Notation follows that used in Karakashian and Pascal (2004)

DG Formulation

With the *Energy Spaces* defined by

$$E_h = \prod_{K \in \mathcal{T}_h} H^2(K)$$

we introduce the bilinear form $a_h^\gamma : E_h \times E_h \to \mathbb{R}$ as

$$a_h^\gamma(u,v) = \sum_{K \in \mathcal{T}_h} (\nabla u, \nabla v)_K - \sum_{e \in \mathcal{E}^I \cup \mathcal{E}_D^B} \Big(\langle \{\partial_n u\}, [v] \rangle_e + \langle \{\partial_n v\}, [u] \rangle_e - \gamma h_e^{-1} \langle [u], [v] \rangle_e \Big)$$

DG Formulation, contd.

- $h_e = \text{diam}(e)$
- $(u, v)_K = \int_K u \cdot v \, dx$
- $\langle u, v \rangle_\Gamma = \int_\Gamma uv \, ds, \quad |v|_\Gamma = \langle v, v \rangle_\Gamma^{1/2}$
- $[v]|_e = v^+|_e - v^-|_e, \quad v^+ = v|_{K^+}, \quad v^- = v|_{K^-}, e \in \mathcal{E}^I$
- We can use either the Arnold (Arnold, 1982) or Baker (Baker, 1977) formulation for $\{\partial_n v\}|_e, \quad e \in \mathcal{E}^I$

$$\{\partial_n v\}|_e = \frac{1}{2}\left(\frac{\partial v^+}{\partial n^+} + \frac{\partial v^-}{\partial n^+}\right)\Bigg|_e, \quad e \in \mathcal{E}^I \qquad \text{(Arnold)}$$

$$\{\partial_n v\}|_e = \frac{\partial v^+}{\partial n^+}\Bigg|_e, \quad e \in \mathcal{E}^I \qquad \text{(Baker)}$$

DG Formulation, contd.

a_h^γ is continuous and coercive

1. $|a_h^\gamma(u,v)| \leq (1+\gamma)\|u\|_{1,h}\|v\|_{1,h}, \quad \forall u,v \in E_h$
2. There exists positive constants γ_0 and c_a such that for $\gamma \geq \gamma_0$

$$a_h^\gamma(v,v) \geq c_a\|v\|_{1,h}^2, \quad \forall v \in V_h^r$$

where the *energy* norm on E_h is

$$\|v\|_{1,h} = \Big(\sum_{K\in\mathcal{T}_h} \|\nabla v\|_K^2 + \sum_{e\in\mathcal{E}^I\cup\mathcal{E}_D^B} \big(h_e|\{\partial_n v\}|_e^2 + h_e^{-1}|[v]|_e^2\big)\Big)^{1/2}$$

DG Formulation, contd.

Introducing the functional $F : E_h \to \mathbb{R}$

$$F(v) = (f, v) - \sum_{e \in \mathcal{E}_D^B} \langle g_D, \partial_n v - \gamma h_e^{-1} v \rangle_e + \sum_{e \in \mathcal{E}_N^B} \langle g_N, v \rangle_e, \quad \forall v \in E_h$$

implies the DG Finite Element approximation problem is to find $u_h^\gamma \in V_h^r$ such that

$$a_h^\gamma(u_h^\gamma, v) = F(v), \quad \forall v \in V_h^r \qquad \text{(DG)}$$

Remarks

- a_h^γ is symmetric, positive definite.
- The choice of γ varies as r^2 and is made to force coercivity of a_h^γ.
- Described in Karakashian and Pascal (2003, 2004).

Error Estimator

We utilize a residual type aposteriori error estimator based on the RHS of the following:

Theorem (Karakashian and Pascal (2004)). *Let* $e = u - u_h^\gamma$. *Then*

$$\sum_{K\in\mathcal{T}_h} \|\nabla e\|_K^2 \le c\Big(\sum_{K\in\mathcal{T}_h} h_K^2 \|f + \Delta u_h^\gamma\|_K^2 + \sum_{e\in\mathcal{E}^I} h_e |[\partial_n u_h^\gamma]|_e^2 + \sum_{e\in\mathcal{E}_N^B} h_e |g_N - \partial_n u_h^\gamma|_e^2 + \gamma^2 \sum_{e\in\mathcal{E}^I} h_e^{-1} |[u_h^\gamma]|_e^2 + \gamma^2 \sum_{e\in\mathcal{E}_D^B} h_e^{-1} |g_D - u_h^\gamma|_e^2 \Big)$$

Marking vs Refinement

- It is important to distinguish between the marking of elements to be refined based on the aposteriori error indicator and the actual process of refinement.
- We impose conditions on the refinement process such that a *1-irregular* mesh is obtained (i.e., only one hanging node per edge can exist in the mesh). Note that this implies that no single triangle can have more than two triangles as neighbors.

Marking vs Refinement, contd.

- Note that no continuity of solution constraints are imposed on the midpoint nodes of the edges in the DG case whereas with Standard Galerkin these nodes are not *free*.
- We will focus here on the *Marking Strategy* for refinement only, noting that a similar *coarsening* marking strategy can be implemented for evolution PDEs.

Marking Strategy

Assume that on $\mathcal{T}_h$ that the solution u_h^γ has been calculated, where η is the estimator being used. (Dörfler, 1996)

1. $\forall K \in \mathcal{T}_h$
   (a) Calculate η_K
   (b) Calculate $\eta_{\max}$
   (c) Calculate $\eta^2_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \eta^2_K$
2. $s = 0, \quad \tau = 1.$
3. While $s < \theta^2 \eta^2_{\mathcal{T}_h}$
   (a) $\tau = \tau - \nu$
   (b) $\forall K \in \mathcal{T}_h$
      i. If K is not marked **AND** If $\eta_K > \tau \eta_{\max}$
         A. Mark K for refi nement
         B. $s = s + \eta^2_K$

Marking Strategy, contd.

- This algorithm is guaranteed to stop because as $\tau \to 0$ eventually s will exceed the threshold $\theta^2 \eta^2_{\mathcal{T}_h}$.
- Choices for θ, ν greatly affect how many elements are marked each sweep through $\mathcal{T}_h$.
- Once this algorithm finishes, the set of marked triangles have the largest estimated error and are then refined, after which the solver is called to produce new error estimates and the process is repeated until the total estimated error is less than some tolerance ϵ, i.e., $\eta^2_{\mathcal{T}_h} < \epsilon^2$.

Marking Strategy, contd.

Conceptually this process can be thought of in the following manner:

- Consider a point singularity in Ω, the typical error distribution will be high for triangles near the singularity and low for triangles away from the singularity.
- Multiple sweeps through $\mathcal{T}_h$ are made with the largest *error* triangles marked first.
- After a sweep through $\mathcal{T}_h$ is made, if the accumulated (marked) error is still less than the threshold θ of the total error, then triangles with less error are considered for marking, controlled by τ and ν.

Parameter Choices

Question. How does the choice of θ, ν, and ϵ affect the resultant meshes (and thus the approximate solutions)?

To investigate this question better, we will use Preconditioned Conjugate Gradient (with Full Multigrid as the preconditioner) to obtain approximations to the following problems:

- P1 Smooth function
- P2 Point singularity in the interior of Ω
- P3 Mixed boundary condition induced gradient mismatch at a number of points on the boundary.

Parameter Choices, contd.

We will analyze how many triangles are refined and reduction in total estimated error for various choices of θ, ν, and ϵ. The finite element implementation uses quadratic ($r = 3$) functions to approximate the solution on each triangle in $\mathcal{T}_h$.

P1

$$\begin{cases} -\Delta u = 2\pi^2 \sin(\pi x)\cos(\pi x) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \end{cases} \qquad \text{(P1)}$$

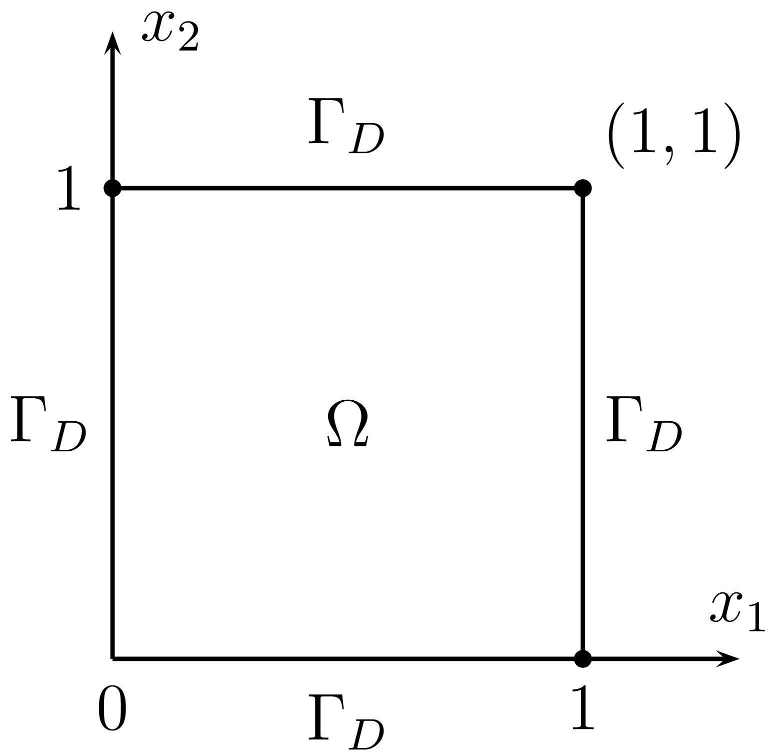


Figure 1: P1 Geometry

P1 Solution

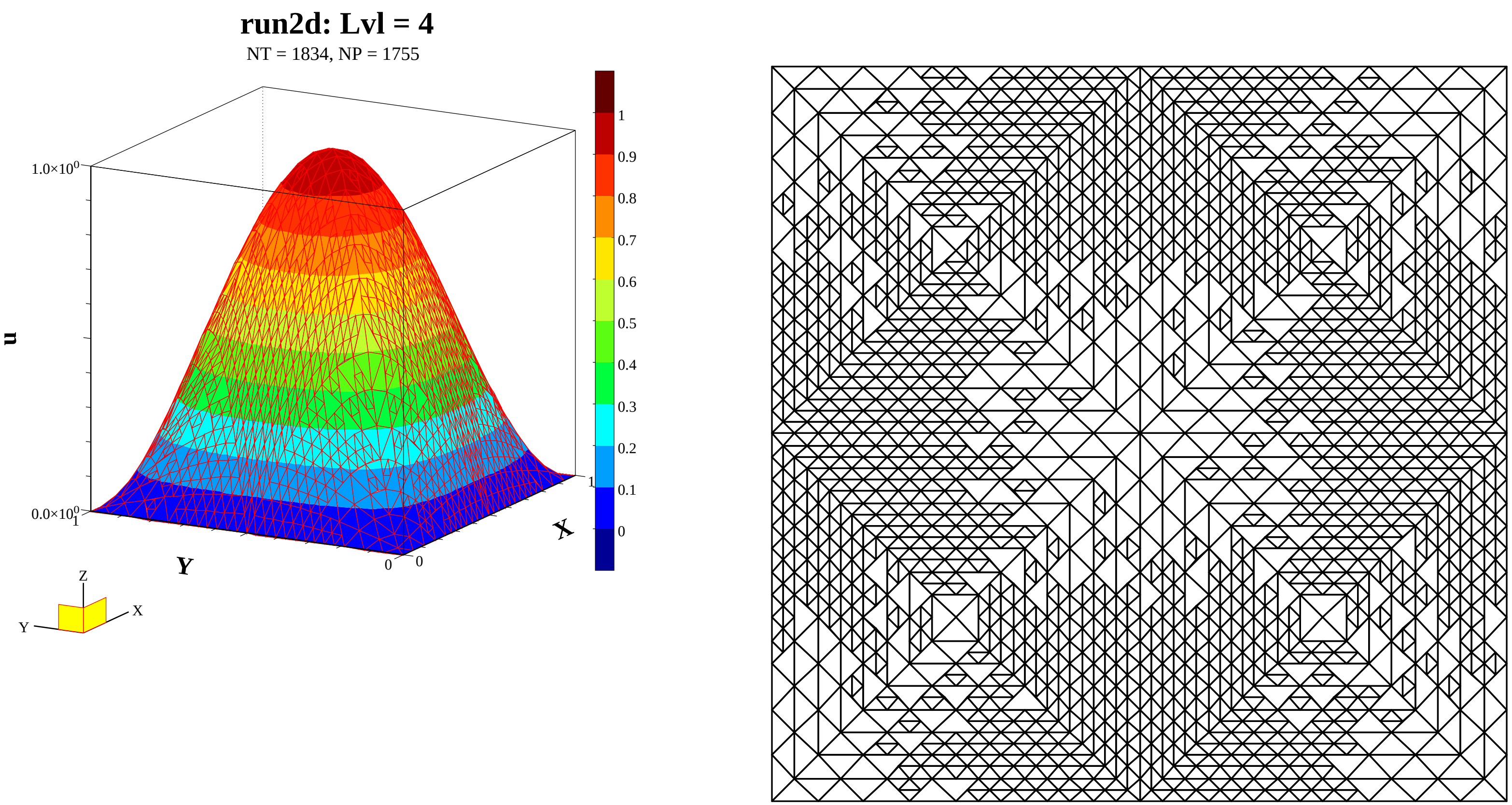
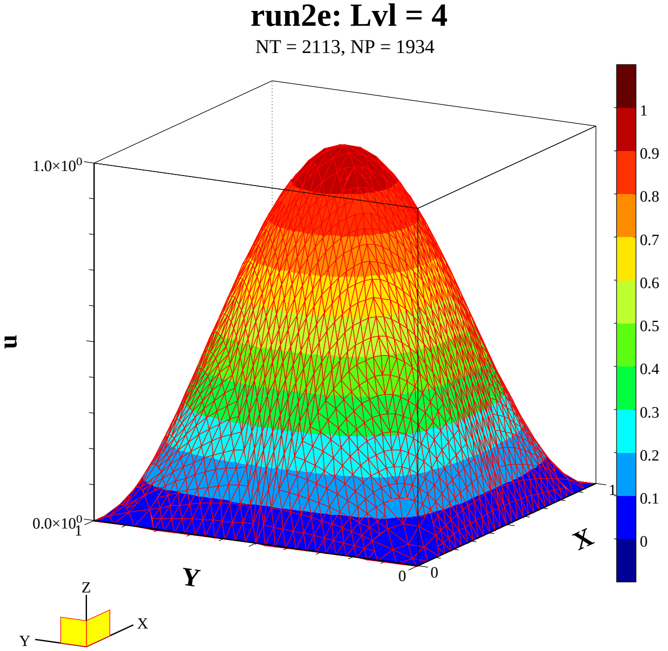


Figure 2: $\theta = 0.8$, $\nu = 0.0005$, $\epsilon = 0.02$

P1 Solution



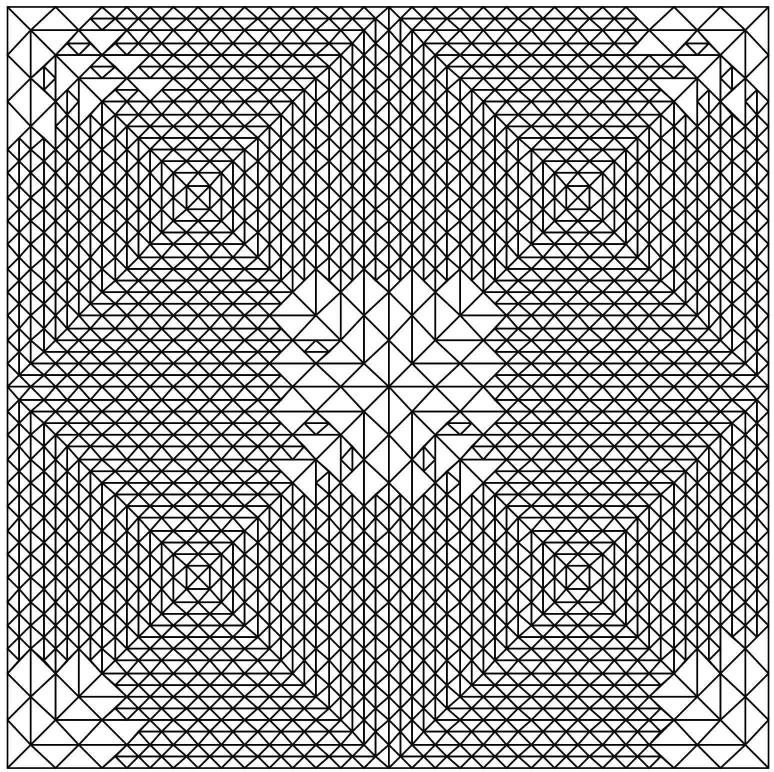


Figure 3: $\theta = 0.9$, $\nu = 0.0005$, $\epsilon = 0.02$

P2

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = r^{2/3}\sin(2/3\theta) & \text{on } \Gamma_D \end{cases} \qquad \text{(P2)}$$

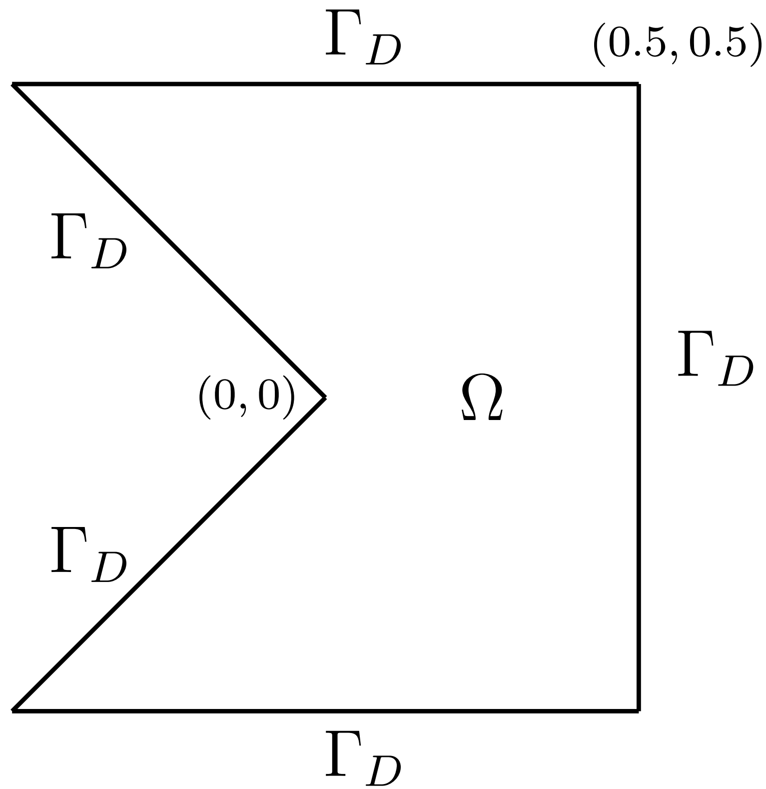


Figure 4: P2 Geometry

P2 Solution

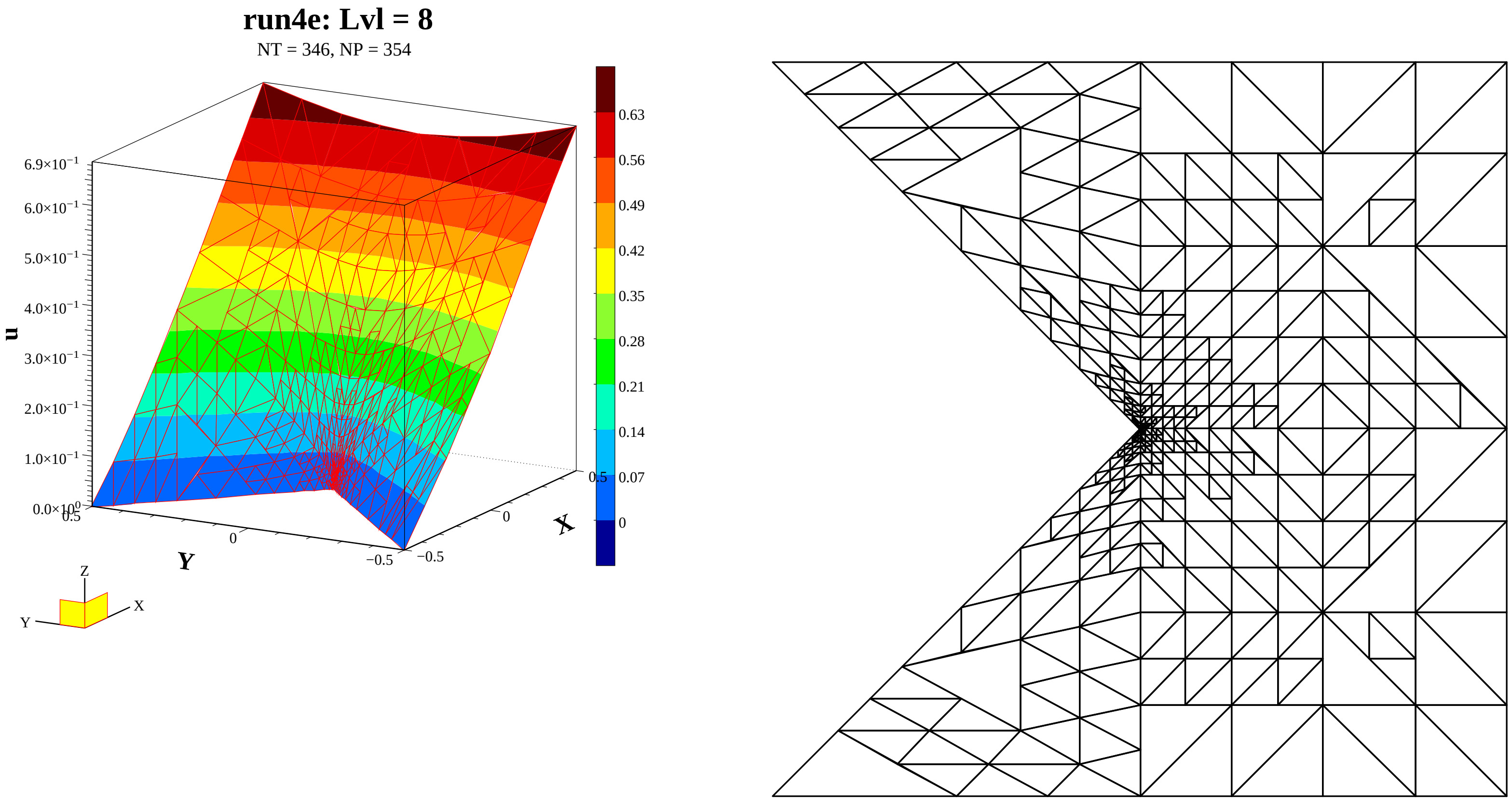


Figure 5: $\theta = 0.85$, $\nu = 0.0005$, $\epsilon = 0.02$

P2 Solution

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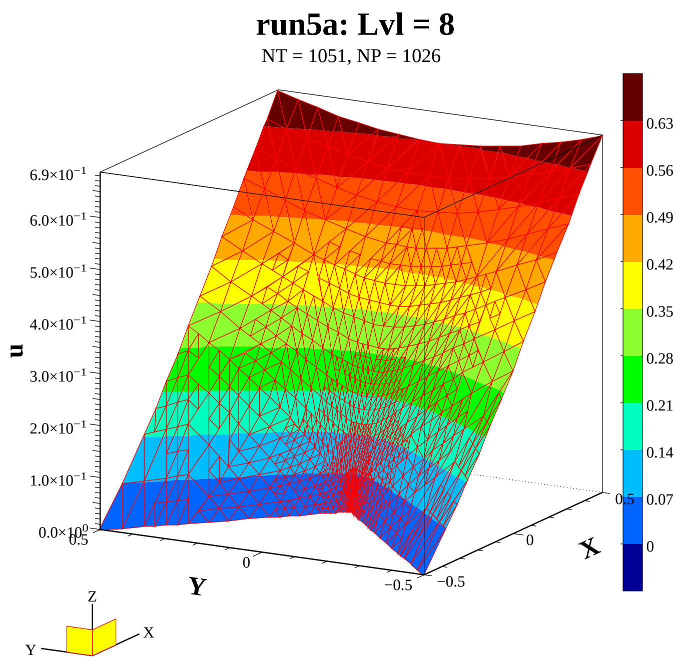


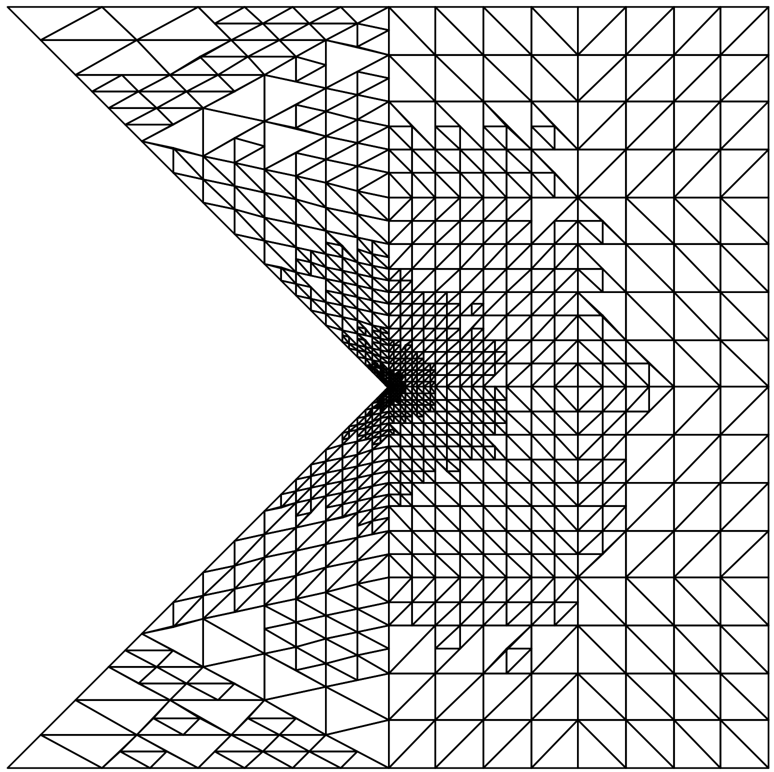


Figure 6: $\theta = 0.95$, $\nu = 0.0005$, $\epsilon = 0.001$

P2 Meshes

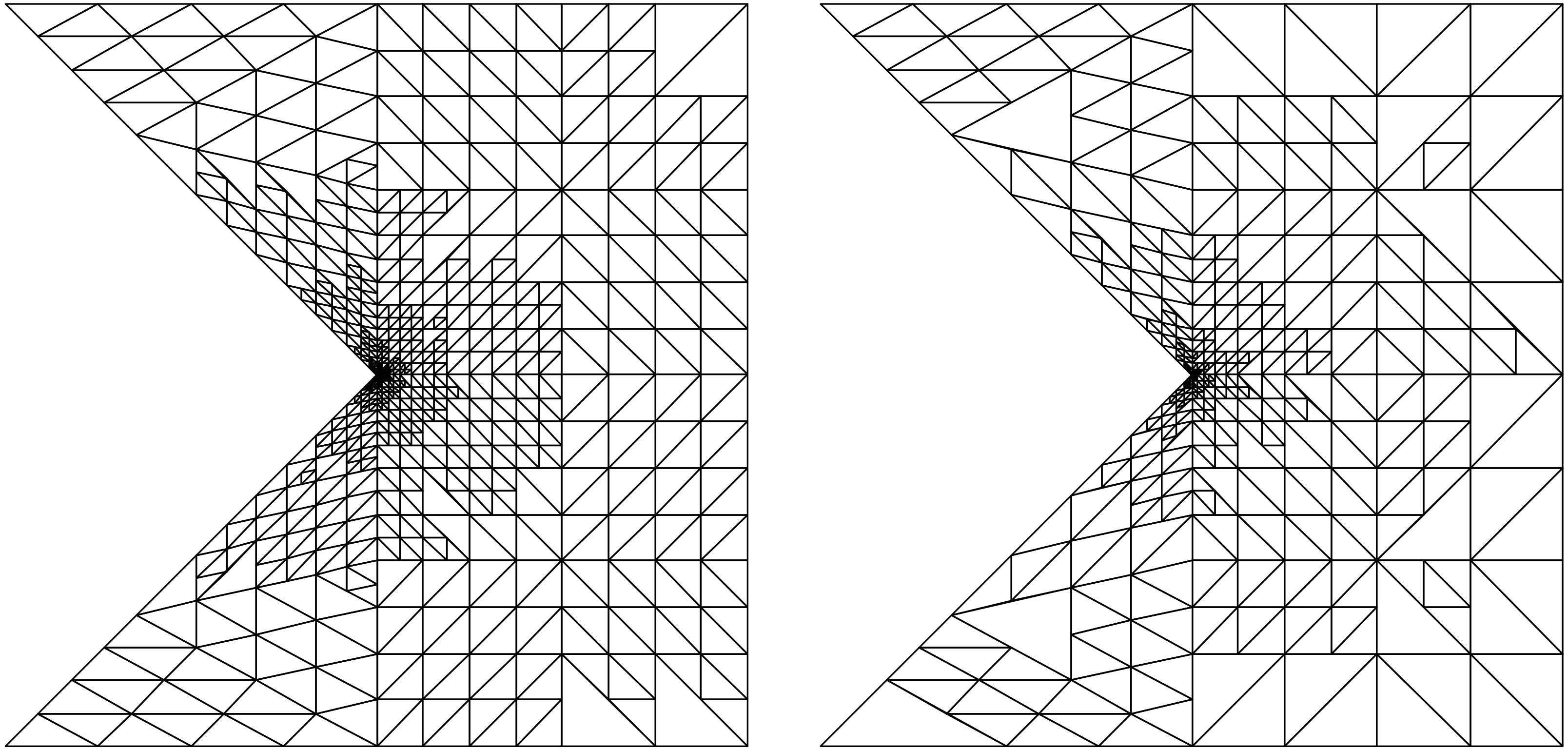


Figure 7: $\theta = (0.9, 0.85)$, $\nu = 0.0005$, $\epsilon = 0.001$

P3

$$
\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \frac{\partial u}{\partial n} = -1 & \text{on } \Gamma_N \end{cases} \qquad \text{(P3)}
$$

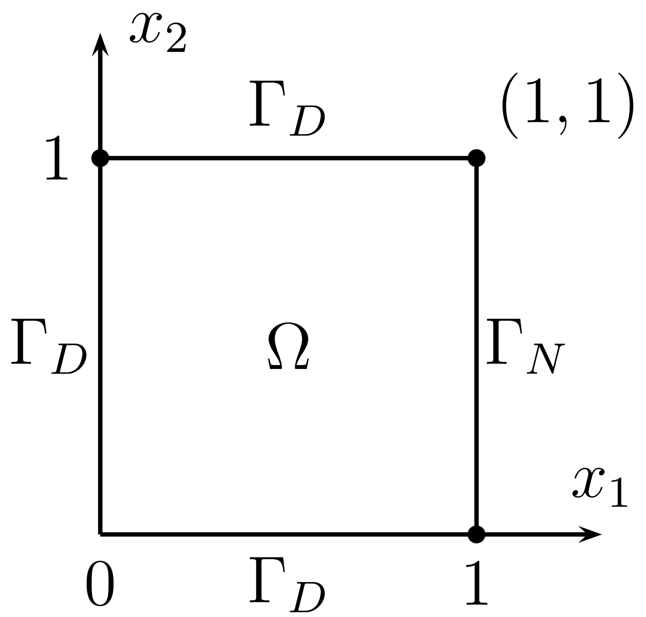
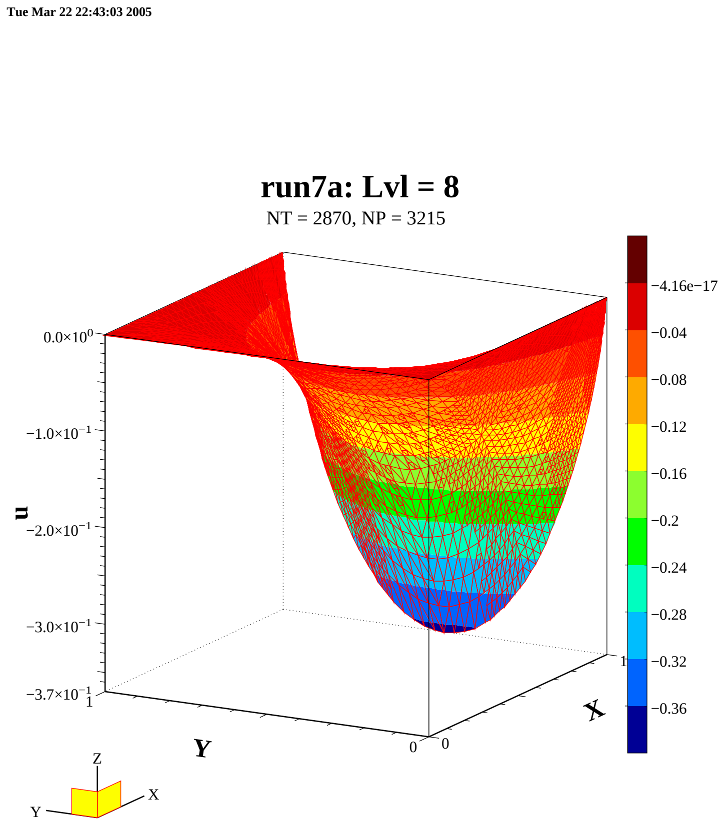


Figure 8: P3 Geometry

P3 Solution



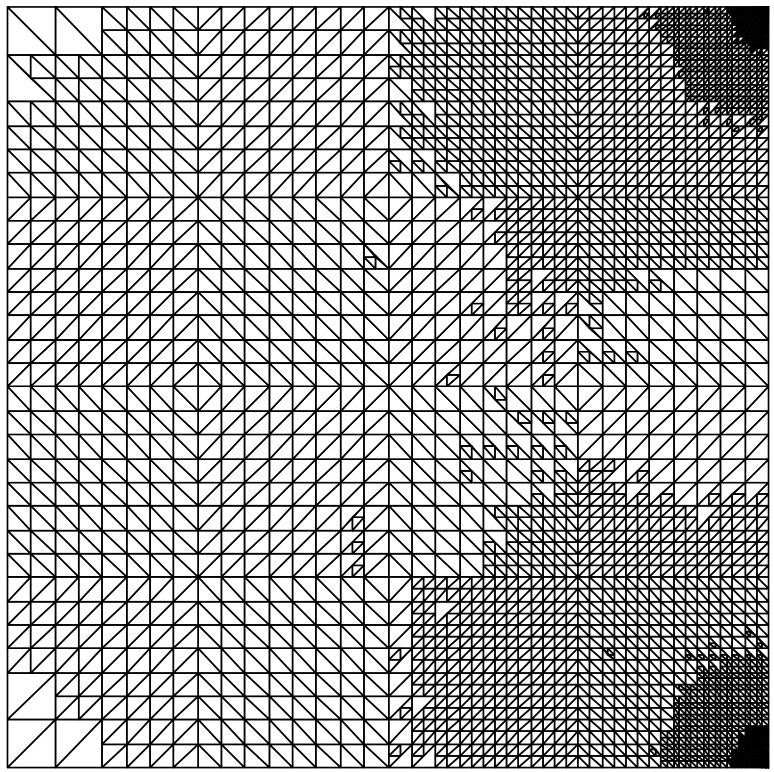


Figure 9: $\theta = 0.95$, $\nu = 0.0005$, $\epsilon = 0.001$

P3 Meshes

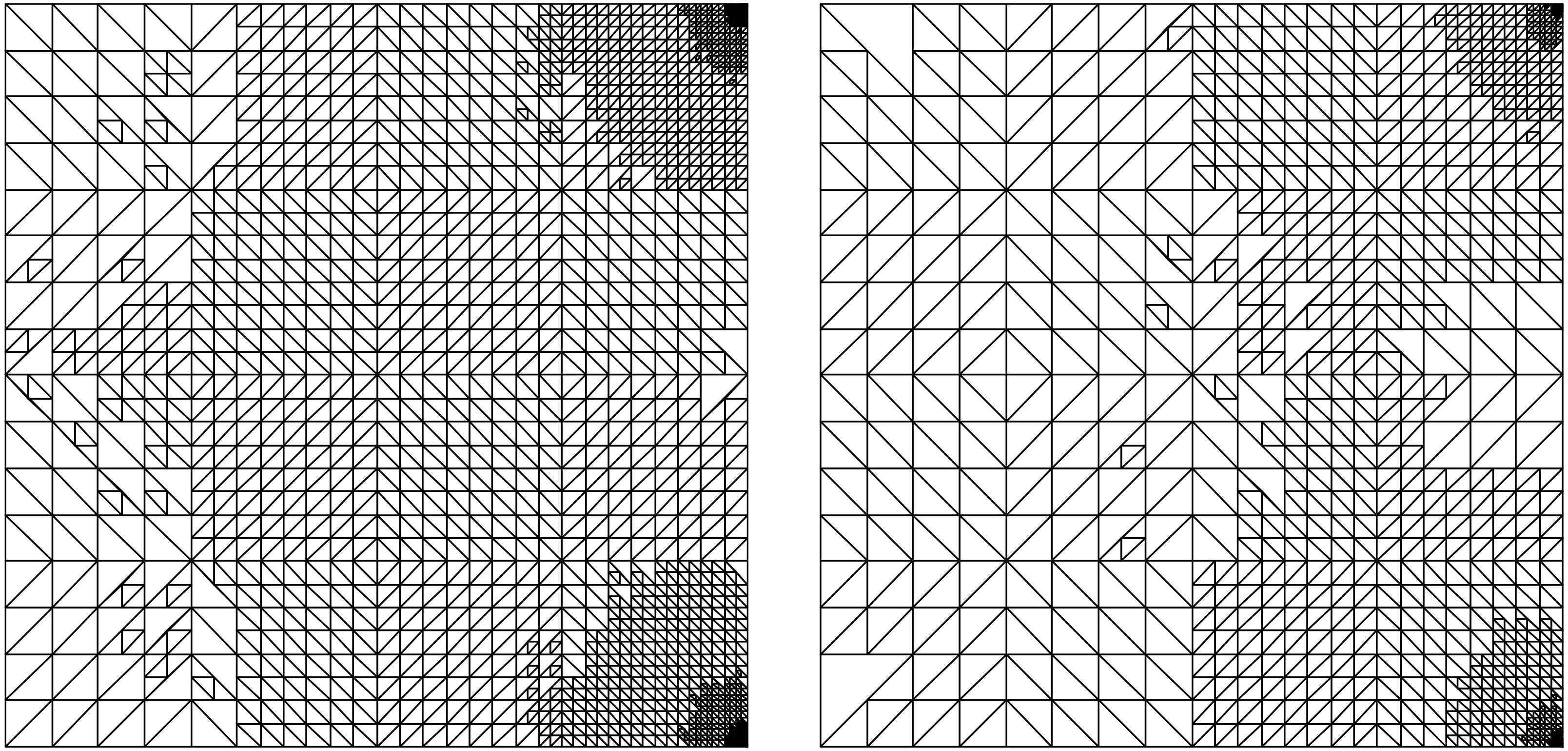


Figure 10: $\theta = (0.90, 0.85)$, $\nu = 0.0005$, $\epsilon = 0.001$

Conclusions

- Efficient procedure for selectively refining only the largest estimated errors under control of θ, ν, and ϵ.
- This procedure eliminates the need for sorting the triangles by estimated error.
- Flexible marking strategy can be modified to include marking triangles for coarsening.
- ν should be chosen small enough to select just enough triangles to mark for refinement without overrefining the mesh.

Conclusions, contd.

- Large θ coupled with small ν produces a good mesh for singularity problems.
- ϵ should be chosen small enough only once one has an understanding of the behavior of the estimator for the particular problem under study.
- The accuracy of the iterative solver might have to be increased for highly refined meshes.

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