International Journal of Bifurcation and Chaos, Vol. 11, No. 1 (2001) 73–89

OBSERVED ROTATION NUMBERS IN FAMILIES OF CIRCLE MAPS*

MICHAEL A. SAUM†
Department of Mathematics, University of Tennessee, Knoxville, TN 37996-1300, USA

TODD R. YOUNG‡
Department of Mathematics, Ohio U., Athens, OH 45701-2979, and IPST CSS Bldg, University of Maryland, 20742-2431, USA

Received December 3, 1999; Revised February 7, 2000

Noninvertible circle maps may have a rotation interval instead of a unique rotation number. One may ask which of the numbers or sets of numbers within this rotation interval may be observed with positive probability in term of Lebesgue measure on the circle. We study this question numerically for families of circle maps. Both the interval and "observed" rotation numbers are computed for large numbers of initial conditions. The numerical evidence suggests that within the rotation interval only a very narrow band or even a unique rotation number is observed. These observed rotation numbers appear to be either locally constant or vary wildly as the parameter is changed. Closer examination reveals that intervals with wild variation contain many subintervals where the observed rotation numbers are locally constant. We discuss the formation of these intervals. We prove that such intervals occur whenever one of the endpoints of the rotation interval changes. We also examine the effects of various types of saddle-node bifurcations on the observed rotation numbers.

1. Introduction

The rotation interval (see definition and references below) has been the chief object of study when one considers the topological properties of noninvertible circle maps. Results about the interval show that the interval predicts the existence of a multitude of orbits of various types (see below). Given the wealth of topological information carried by the rotation interval, it is natural to study the measure properties of this theory. For instance, although all the rotation numbers within the interval have a representative orbit in existence, one would not expect all numbers within the interval to be equally likely in terms of Lebesgue measure on the circle.

In [Young, 1998] a framework was presented in which one may pose the measure theoretic questions concerning Birkhoff averages of general dynamical systems in terms of *noninvariant measures*. In the next section we will present the rotation interval within this framework. Specifically, in the simplest case we define distributions of rotation numbers as one defines the distribution of a random variable in probability. However, as will be seen, complications may arise which require a more involved approach. The support of the distribution of rotation numbers we call the *likely rotation set*. When this set happens to be a unique number, or finite set of numbers, we call them the *likely rotation numbers*. Of course

*This work was partially supported by the Ohio University Research Council.
†E-mail: msaum@math.utk.edu
‡E-mail: young@math.ohiou.edu

rotation numbers are only defined asymptotically. To study them numerically the best one can do is to calculate a finite time rotation number for a specific orbit. As will be discussed, one should do this not for a single orbit, but for an ensemble of orbits whose initial conditions are chosen randomly. The resulting ensemble of finite time rotation numbers we call *observed rotation numbers*. The observed rotation numbers converge in an appropriate sense to the likely rotation set (see Propositions 1 and 2 below).

The body of this paper is dedicated to exploring rotation numbers numerically within the context of a family of circle maps. The families we consider can be represented in the form $x \mapsto f(x) + a \mod 1$, in particular we study numerically the standard family (see Sec. 3 below). Our main observation after many numerical studies is that the observed rotation numbers for a given map comprise a very narrow band, or even unique number within the rotation interval. This suggests that for most maps the likely rotation set consists of a single, or a few likely rotation numbers. We also find that the observed rotation numbers vary wildly as the parameter a is varied. Within intervals of wild variation, we find many "locking intervals", intervals where an observed rotation number is constant and often unique. We prove that locking intervals occur whenever the endpoints of the rotation interval changes. We also discuss the effects of different types of saddle-node bifurcations on the observed rotation numbers.

2. Distributions of Rotation Numbers

2.1. *Notation, definitions and background*

Let f be a noninvertible map of the circle $\mathbf{S}^1$ into itself. We will assume throughout that f is C^r, $r \geq 0$, and degree one, i.e. homotopic to the identity. Let π be the usual projection map $\pi : \mathbb{R} \to \mathbf{S}^1 : x \mapsto z = \exp 2\pi i x$. Let F be a lift of f, i.e. a one-periodic map from $\mathbb{R}$ into itself such that $\pi \circ F = f \circ \pi$. Let

$$\rho(z) \equiv F(x) - x\,,$$

where $x \in \pi^{-1}(z)$. An interpretation of ρ is as a measure of the angle of precession.

Given $z \in \mathbf{S}^1$, let $\{z_i\}_{i=0}^{\infty}$ be the forward orbit of z under f, ($z_0 = z$ and $z_{i+1} = f(z_i)$) and consider the sequence:

$$\rho_i(F,\, z) \equiv \frac{1}{i}\sum_{j=0}^{i-1} \rho(z_j)\,. \tag{1}$$

The rotation set of the orbit of z, denoted as $I(F,\, z)$, may be defined as the set of limit points of the sequence (1). It is well known that $I(F,\, z)$ is either a point or a closed interval.

The rotation interval, denoted as $I(F)$, was introduced in [Newhouse *et al.*, 1983] and may be defined in several ways (see [Bamon *et al.*, 1984; Block *et al.*, 1980; Boyland, 1986; Chenciner *et al.*, 1984]). One way to define it is as the union of all the rotation sets, i.e.

$$I(F) = \cup_{z \in \mathbf{S}^1} I(F,\, z)\,. \tag{2}$$

If we let

$$\rho^+(z,\, F) = \limsup_{i \to \infty} \rho_i(z,\, F)$$

$$\text{and} \qquad \rho^-(z,\, F) = \liminf_{i \to \infty} \rho_i(z,\, F)\,, \tag{3}$$

then it is equivalent to define $I(F)$ as the closure of the range of ρ^+ or ρ^-.

Perhaps the most important fact about the rotation interval is that given any $\alpha \in I(F)$ there exists $z \in \mathbf{S}^1$ such that $I(F,\, z) = \{\alpha\}$. In fact, given any subinterval $[\alpha,\, \beta] \subset I(F)$, there exists $z \in \mathbf{S}^1$ such that $I(F,\, z) = [\alpha,\, \beta]$ [Bamon *et al.*, 1984]. Given these remarkable existence results, we now move on to the question of measure theory. Specifically, one is led to ask: Given some $\alpha \in I(F)$ or some Borel subset S of $I(F)$ what is the measure of the subset of $\mathbf{S}^1$ for which $I(z,\, F)$ is α or is contained in S?

That is, if we choose a point $z \in \mathbf{S}^1$ randomly (using a uniform distribution on the circle) what is the probability that $I(z,\, F)$ is α or in S?

The real valued function on the state space of a dynamical system, such as $\rho : \mathbf{S}^1 \to \mathbb{R}$ is often called an *observable*. The limit (if it exists) of a sum of the form (1) is commonly called a Birkhoff average of the observable. In [Young, 1998] the distribution of a general Birkhoff average was studied and a *probability distribution* was defined, the meaning of which in the present context is roughly what is called for in the question above. We now describe the distribution in the present context.

Let m denote the projected Lebesgue measure on $\mathbf{S}^1$, $(m(\mathbf{S}^1) = 1)$. *We do not assume m is invariant under f.* Actually, we could let m be any Borel probability measure, but Lebesgue measure is the only one we find interesting. A natural partition of

$\mathbf{S}^1$ is as follows:

$$A(f) = \left\{ z \in \mathbf{S}^1 : I(F, z) = \left\{ \frac{p}{q} \right\} \in \mathbb{Q} \right\}$$
$$B(f) = \{ z \in \mathbf{S}^1 : I(F, z) = \{\alpha\} \in \mathbb{R} \backslash \mathbb{Q} \} \quad (4)$$
$$C(f) = \{ z \in \mathbf{S}^1 : I(F, z) \text{ is not a point} \} .$$

If $I(F)$ is not a point, then $A(f)$, $B(f)$, and $C(f)$ are all nonempty [Bamon *et al.*, 1984]. One might ask the following: "*Are* $A(f)$, $B(f)$ *and* $C(f)$ m*-measurable? If so, what are their measures?*"

An affirmative answer to the first question is easily obtained (see [Young, 1998]).

In some cases a partial answer to the second question is known. For instance, if f has a hyperbolic attracting periodic point or a measurable periodic attractor [Milnor, 1985], then A will contain the basin of this attractor and so A will have positive Lebesgue measure. Or, if m is an invariant measure, then the Birkhoff–Khinchin Ergodic Theorem (see [Cornfeld *et al.*, 1982]) implies that $m(A \cup B) = 1$. If m is ergodic then there exists α such that $\rho(z) = \alpha$ for almost every $z \in \mathbf{S}^1$. However, in general, a complete answer to this question is out of reach (see [de Melo & van Strien, 1993, p. 328]).

2.2. *The distribution of rotation numbers*

In the case $m(C) = 0$ we may define a distribution in a standard way. First note that the statement $m(C) = 0$ is equivalent to the statement that $\{\rho_i(z)\}$ converges almost everywhere as $i \to \infty$. Define $\bar{\rho}(z)$ to be the limit of the sequence wherever it exists and assign it an arbitrary value elsewhere (i.e. on C). Since $\rho_i(z)$ is a sequence of measurable (in fact continuous) functions, $\bar{\rho}(z)$ is measurable.

Definition 1. For $m(C) = 0$, define μ to be a real valued function on the Borel sets of $\mathbb{R}$ given by

$$\mu(S) = \bar{\rho}_* m(S) = m(\{z \in \mathbf{S}^1 : \bar{\rho}(z) \in S\}) ,$$
$$S \in \mathfrak{B} ,$$

i.e. the pushforward of m along $\bar{\rho}$. We call the measure μ the *rotation distribution* of F.

In the terminology of probability theory, $\bar{\rho}$ is a *random variable* since it is a measurable real valued function and $\mu = m_\rho \equiv m \circ \rho^{-1}$ is the *distribution* of $\bar{\rho}$. (see e.g. [Shiryayev, 1979, p. 168].)

Definition 2. For $m(C) = 0$, we call the support of μ the *likely rotation set* of f.

Under the condition $m(C) = 0$, the measure μ carries the measure theoretic information about the rotation interval. For instance, if f has a periodic attractor which attracts m-almost every $z \in \mathbf{S}^1$, then μ will be the atomic probability measure supported on the rotation number of the attractor.

Before we proceed to the case $m(C) > 0$, we first recall a result which illustrates the potential usefulness of this concept and which is helpful in motivating the extension of the distribution.

Definition 3. For any $i \in \mathbb{N}$ and any Borel set S define

$$\mu_i(S) = \rho_{i*} m(S) = m(\{z \in \mathbf{S}^1 : \rho_i(z) \in S\})$$

We call μ_i the ith rotation distribution.

It was shown [Young, 1998] that μ_i converges to the rotation distribution μ as $i \to \infty$. However, each μ_i is absolutely continuous whereas μ may be singular (as in the case where m is ergodic). Thus we cannot expect convergence to occur in a strong sense, i.e. $\mu_i(S) \to \mu(S)$ as $i \to \infty$ for every Borel set S. Instead, we use the definition of weak convergence (sometimes called convergence in distribution).

Definition 4. Let $\{\mu_i\}_{i=1}^{\infty}$ be a sequence of Borel probability measures. We say that μ_i converges weakly to a Borel measure μ, denoted $\mu_i \rightharpoonup \mu$, if for every test function $\psi \in C_b(\mathbb{R})$ (bounded continuous) we have:

$$\int_{\mathbb{R}} \psi d\mu_i \to \int_{\mathbb{R}} \psi d\mu, \quad \text{as} \quad i \to \infty .$$

Proposition 1. [*Young, 1998*]. *If* $m(C) = 0$, *then* $\mu_i \rightharpoonup \mu$.

The significance of this proposition is that μ_i may be approximated numerically to any degree of accuracy and thus μ may be approximated.

The ideas are fairly standard, but for the sake of completeness we describe briefly the approximation of μ_i by finite computations. Consider the space of sequences $\Omega = \{\omega = (z_1, z_2, \ldots) : z_k \in \mathbf{S}^1\}$ with the product measure induced by m. Denote by δ_x the unit atomic measure at $\{x\}$. That is, if S is any subset of $\mathbb{R}$

$$\delta_x(S) = \begin{cases} 1, & \text{if } x \in S , \\ 0, & \text{otherwise} \end{cases}$$

Definition 5. Given $(z_1, z_2, \ldots) \in \Omega$, define a measure μ_{ij} on the subsets of $\mathbb{R}$ by

$$\mu_{ij} = \frac{1}{j}\sum_{k=1}^{j} \delta_{\rho_i(z_k)}\,.$$

Proposition 2. *Suppose $m(C) = 0$. Given $i \in \mathbb{N}$, then $\mu_{ij} \rightharpoonup \mu_i$ (weakly) $j \to \infty$, almost surely with respect to the product measure on Ω.*

Thus to approximate μ_i, it is appropriate to randomly choose a large number of initial conditions $\{z_k\}$ and calculate $\rho_i(z_k)$ for each.

Proof. Given $\psi \in C_b$ then

$$\int_{\mathbb{R}} \psi d\mu_{ij} = \frac{1}{j}\sum_{k=1}^{j} \psi(\rho_i(z_k))\,.$$

Denote $\alpha_k = \rho_i(z_k)$ and consider the space Ω_i of sequences of the form $\{\alpha_k\}$, along with the product measure induced by $\mu_i = m_{\rho_i} \equiv m \circ \rho_i^{-1}$. It is clear that $(\rho_i)^{-1}$ induces a measure preserving map from Ω_i to Ω. It is a well-known result that

$$\frac{1}{j}\sum_{k=1}^{j} \psi(\alpha_k) \to \int \psi d\mu_i, \quad \text{as} \quad j \to +\infty\,,$$

almost surely with respect to the measure on Ω_i. But, that implies almost sure convergence with respect to the product measure on Ω. ■

The set $O_{ik} \equiv \{\rho_i(z_j) : 0 \le j < k\}$ we call the *observed rotation set*. This set converges topologically to the likely rotation set almost surely as i, k go to infinity.

Of course an obstacle to real numerical computations is that chaotic orbits may be calculated accurately for only a few iterations. However, there is evidence that for a large set of parameter values the map f_a has either a periodic measurable attractor or an absolutely continuous measure (see [Jakobson, 1981; Graczyk & Sweitek, 1997]). In both cases, the sensitivity on initial conditions is irrelevant to the calculation; either because orbits are asymptotically stable, or shadowing occurs. In practice it is the case that quantities of complex systems are often studied by studying those characteristics along a large ensemble of orbits (see e.g. [Afraimovich & Zaslavsky, 1998]). Results similar to Proposition 1 may provide some justification for such studies, at least in spirit.

In light of Proposition 1, it was proposed that a satisfactory extension of the rotation distribution to the case $m(C) \neq 0$ should agree with Definition 1 when $m(C) = 0$ and should be such that whenever μ_i converges weakly, it agrees with the limit. In a profound sense, if μ_i does not converge weakly, then the dynamical system $(\mathbf{S}^1, f)$ is very badly behaved with respect to ρ. If there is no weak convergence of μ_i, then any number of finite calculations, even with exact precision, are meaningless. Following the terminology of Young [1998] we make the definitions:

Definition 6. If the sequence of measures μ_i converges weakly, then we say that $(\mathbf{S}^1, f)$ is *statistically rotation predictable*. In this case we call the limit the *rotation distribution* of f and denote it by μ. The support of the measure μ we call the *likely rotation set* of f.

3. The Standard Family of Circle Maps

By the standard family of maps we mean the maps on the circle defined by

$$T_{a,b}(x) = x + a + b\sin 2\pi x \qquad \text{mod } 1\,, \tag{5}$$

When

$$b > b_0 \equiv \frac{1}{2\pi} \approx 0.1591549431\ldots\,. \tag{6}$$

then $T_{a,b}$ is not invertible and the rotation interval is potentially, but not necessarily, nontrivial.

For $b > b_0$, $T_{a,b}(x)$ has two critical points, c_1 and c_2, which depend on b. At c_1 the function has a local maximum and at c_2 a local minimum. The points c_1 and c_2 are given by

$$c_i = \frac{1}{2\pi}\arccos\frac{-1}{2\pi b}, \tag{7}$$

where we give arccos the first two positive values.

One can easily show that for any b, the map $T_{b,b}$ has a saddle-node fixed point at $x = 3/4$, i.e. $T_{a,b}$ undergoes a saddle-node fixed point bifurcation at $a = b$. For b slightly larger than b_0, $T_{b,b}$ is invertible in a neighborhood of the saddle-node point. Thus dynamical orbits may not cross the saddle-node point and so all orbits are homoclinic to $x = 3/4$. However, for b larger, we find that $T_{b,b}(c_1) > 3/4$, and so the map is not invertible at $x = 3/4$. The transition occurs at a value b_1, at which the family

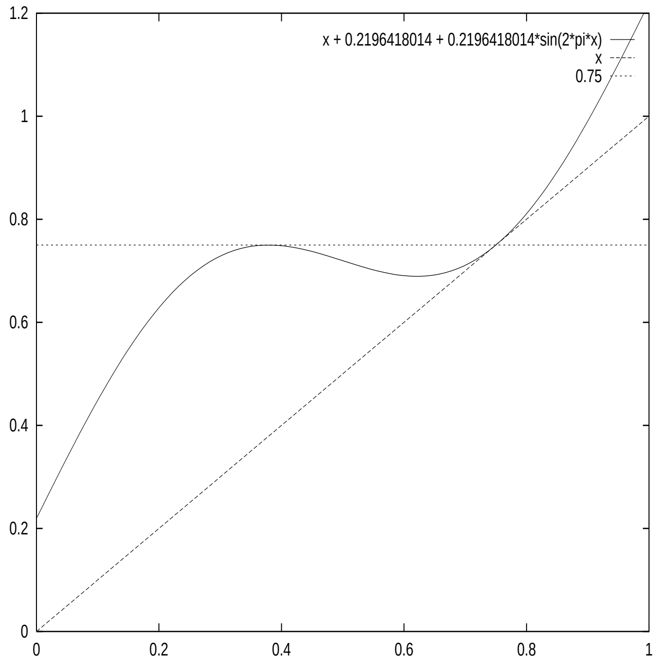

Fig. 1. Graph of $T_{b_1,b_1}(x) = x + b_1 \sin 2\pi x$. For any $b < b_1$, the rotation interval for $T_{a,b}$ is trivial before the saddle-node bifurcation at $a = b$.

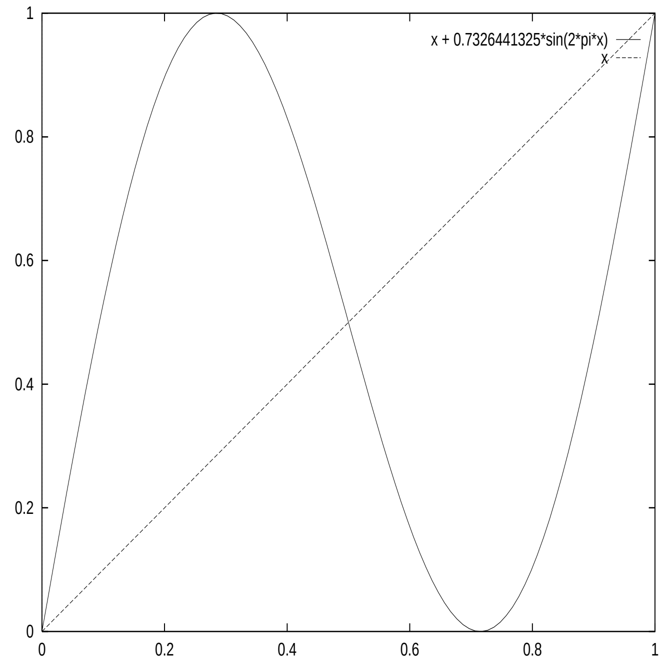

Fig. 2. Graph of $T_{0,b_2}(x) = x + b_2 \sin 2\pi x$. For any $b > b_2$ and all a, the rotation interval for $T_{a,b}$ is nontrivial.

of maps $T_{b,b}$ experiences a homoclinic bifurcation of the form

$$T_{b_1,b_1}(c_1) = \frac{3}{4}\,. \tag{8}$$

The value of b_1 is approximated numerically by

$$b_1 \approx 0.2196418014\ldots.$$

For $b > b_1$ the saddle-node bifurcation at $a = b$ will be of a different character than for $b < b_1$ because orbits may "jump over" the saddle-node fixed point. As we will show in the discussion, the effect of the saddle-node bifurcation on the observed rotation number will differ for the cases $b_0 < b < b_1$ and $b_1 < b$. See Fig. 1.

One can easily show that there is a parameter value b_2, such that the rotation interval is necessarily nontrivial for b greater than b_2. The parameter b_2 is the solution of the equation

$$T_{0,b_2}(c_1) = c_1 + b_2 \sin 2\pi c_1 = 1\,. \tag{9}$$

In light of (7), we may write Eq. (9) explicitly as:

$$\cos\sqrt{4\pi^2 b_2^2 - 1} = \frac{-1}{2\pi b_2}\,. \tag{10}$$

Solving this equation numerically we find:

$$b_2 \approx 0.7326441325\ldots. \tag{11}$$

See Fig. 2.

4. Numerical Studies

In this section we describe how computations were performed.

First of all we note that the calculation of the observed rotation numbers lends itself exceptionally well to distributed computation. For a given map, one must calculate the ith partial rotation number starting from several different initial conditions which are chosen randomly using a uniform distribution (see Sec. 2.2). Since the computation for each initial condition is completely independent of all the others, each can be carried out on a separate processor. In fact many calculations in dynamical systems where computations are carried out for multiple parameter values, or multiple initial conditions stand to benefit greatly from a distributive computing approach (see e.g. [de Figueiredo & Melta, 1998; Monti *et al.*, 1999]). Our calculations were performed in a distributive manner on clusters of PC's at Ohio University and the University of Tennessee using PVM (Parallel Virtual Machine) software.

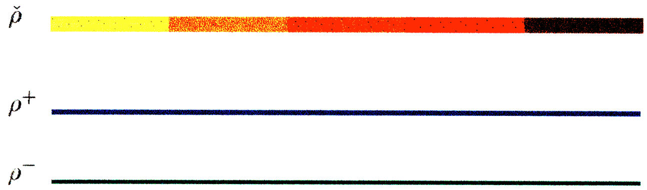

Fig. 3. Color scheme for all the plots of observed rotation numbers below. The upper and lower functions $\rho^+(a)$ and $\rho^-(a)$ are colored blue and green respectively. Observed rotation numbers $\breve{\rho}$ are indicated by yellow to brown colors. Darker shades indicate more frequent occurrence of observed rotation numbers near a given value.

To obtain observed rotation number data, for each a, 100–200 initial points are chosen at random. For each plot, 3000–10 000 values of a were studied. Each initial condition is followed for 50 000–150 000 iterations. The first 20 000–40 000 iterations are discarded. Increasing the number of iterations beyond 150 000 seems to make no difference in the plots. For each a value, the ρ axis was divided into small bins and the occurrences of observed rotation numbers in each bin was counted. In the plots bins with observed rotation numbers appear in colors from yellow to red to brown. Yellow indicates one observed rotation number in the bin, darker shades indicate more occurrences. See Fig. 3.

Because of symmetry, we only plot for a in the interval $(0, 1/2)$, rather than $(0, 1)$.

In the calculation of $\rho^+(a)$ and $\rho^-(a)$, we use a formulation of the rotation interval that was given in [Chenciner *et al.*, 1984] which is very convenient for our purposes. It was shown that

$$I(F) = [\rho(F^-),\, \rho(F^+)]\,, \tag{12}$$

where F^+ is the monotonic upper bound of F and F^- is the monotonic lower bound. That is,

$$F^+(x) = \sup_{y\le x}(F(y)), \quad \text{and} \quad F^-(x) = \inf_{y\ge x}(F(y)). \tag{13}$$

For a given F, the functions $F^\pm$ are easily programmed.

Because $F^\pm$ have constant regions, they typically have rational rotation numbers and orbits fall into the periodic behavior rather quickly. Thus $\rho^\pm$ can usually be calculated very quickly. However as functions, $\rho^\pm(a)$ are Cantor-like functions, they are constant almost everywhere, but difference quotients become unbounded. Thus to get a reasonable representation of $\rho^\pm(a)$, calculations must be performed for very many a values. In the plots, $\rho^+(a)$ appears as blue and $\rho^-(a)$ appears as green.

5. Discussion of Results

5.1. *The observed rotation set is very narrow*

In all cases which are presently understood, i.e. when f has a finite number of absolutely continuous ergodic measures or f has a measurable periodic attractor, the rotation distribution will be a finite sum of atomic measures. Our numerical studies support the hypothesis that for almost all a the distribution is at least very narrowly supported and most likely the finite sum of atomic measures.

Two important results for the quadratic map of the interval given by $p_\lambda(x) = \lambda x(1-x)$ suggest that possibly for most parameter values there is in fact an absolutely continuous invariant measure, or a hyperbolic periodic attractor. In [Jakobson, 1981] it was shown that for the quadratic family there is a set of parameters of positive measure for which there exist unique absolutely continuous ergodic measures. Recently the Dense Hyperbolicity Conjecture was proved [Graczyk & Sweitek, 1997] which states that for the quadratic map the set of parameter values for which the map has a hyperbolic attracting orbit is dense in $(0, 4)$. Although the methods used in the proofs of both of these results were very specialized and at present there is no hope of proving such results for general one-parameter families of maps, the results are very suggestive. The numerical evidence in the present study seems to support the conjecture that such results might be true in general.

Although the observed rotation set is very narrow in all our experiments, we also observed that there are regions on which it appears to not collapse to a single point. This could be a residual effect of the finiteness of our calculations, but could also be due to the following effect. For hyperbolic attractors of very high period, the immediate basins tend to be very narrow. As a result, the average time required to fall into the immediate basin will be longer than the number of iterations considered.

5.2. *Locking intervals*

Very much related to the Dense Hyperbolicity Theorem referenced above (Sec. 5.1), we observed intervals on which the rotation number locks. These locking intervals occur on all scales we are able to resolve. For instance in Fig. 6 the interval given approximately by $(0.11, 0.14)$ is a locking interval. In

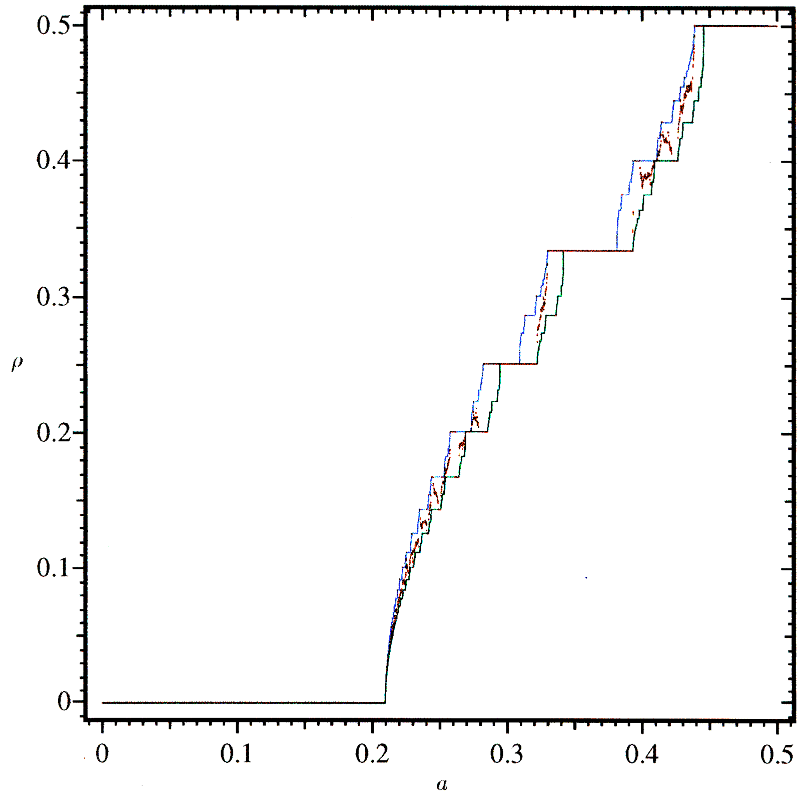

Fig. 4. Observed rotation numbers and $\rho^{\pm}(a)$ for $T_a(x) = x + a + \pi/15 \sin 2\pi x$. Refer to Fig. 3 for an explanation of the color scheme.

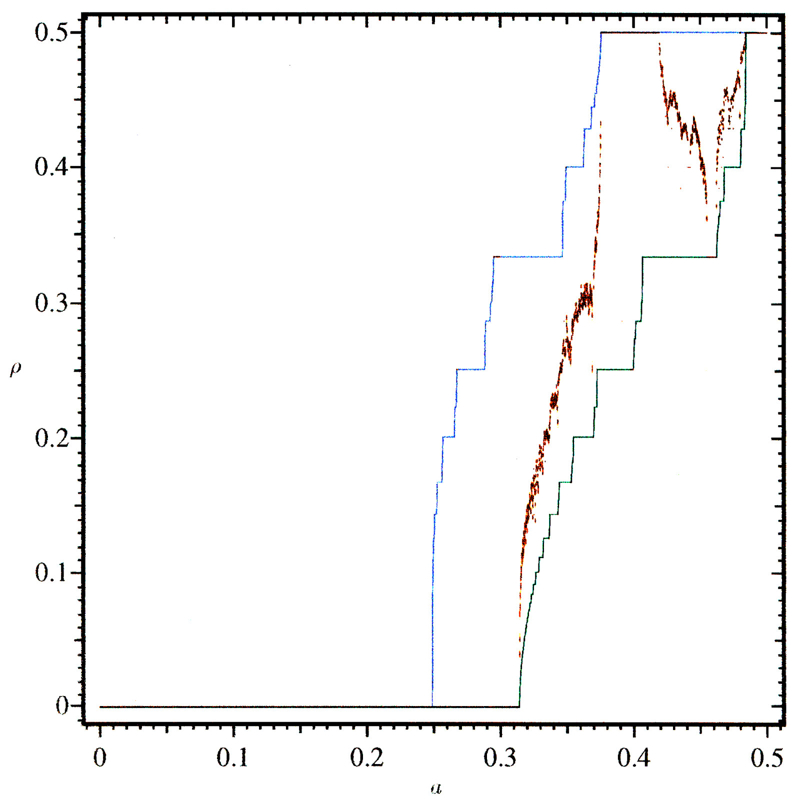

Fig. 5. Observed rotation numbers and $\rho^{\pm}(a)$ for $T_a(x) = x + a + \pi/10 \sin 2\pi x$.

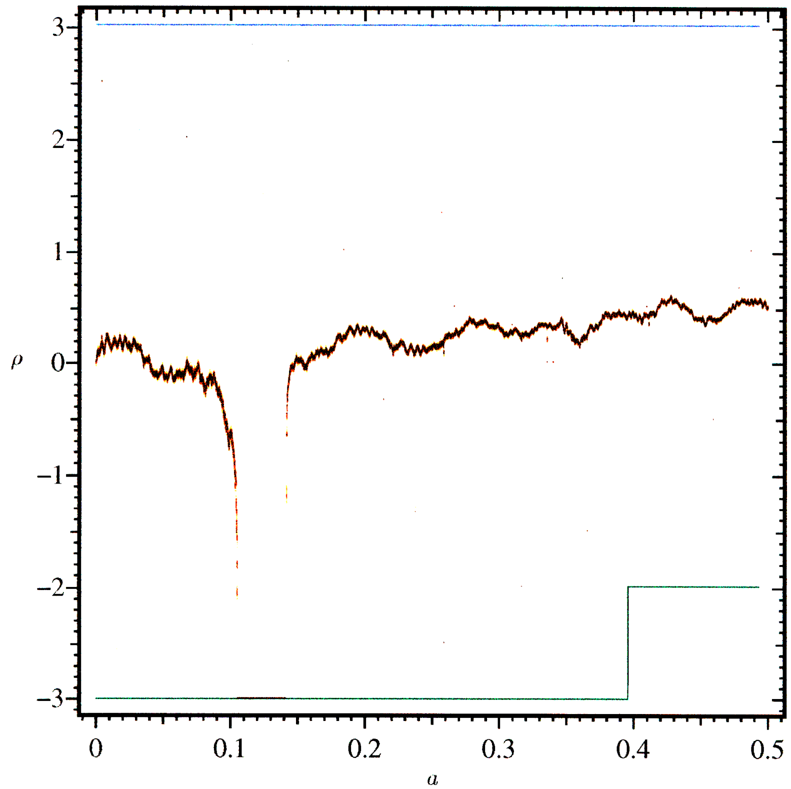

Fig. 6. Observed rotation numbers and $\rho^{\pm}(a)$ for $T_a(x) = x + a + \pi \sin 2\pi x$.

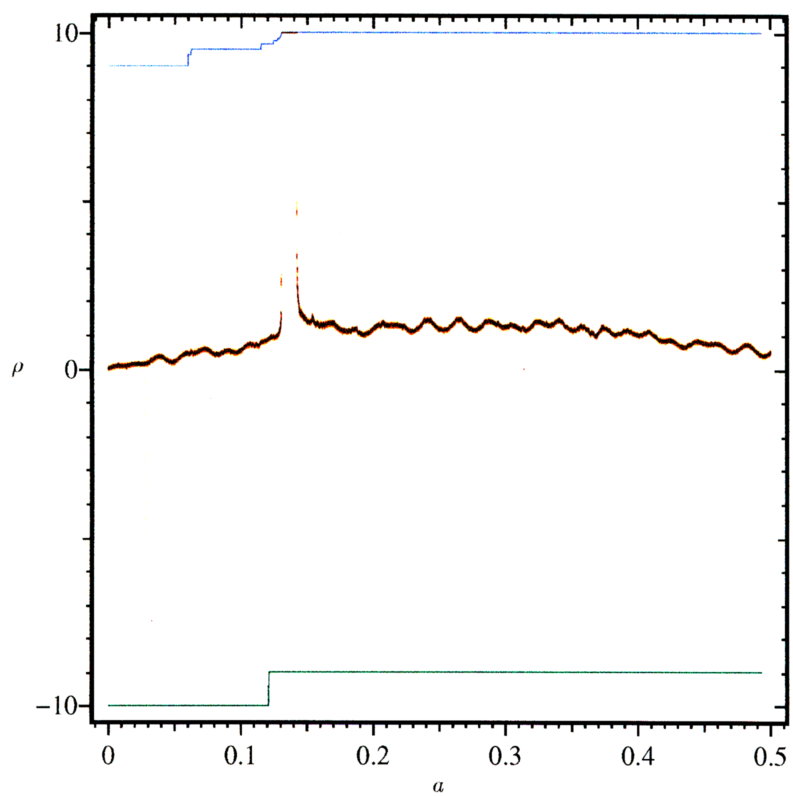

Fig. 7. Observed rotation numbers and $\rho^{\pm}(a)$ for $T_a(x) = x + a + \pi^2 \sin 2\pi x$.

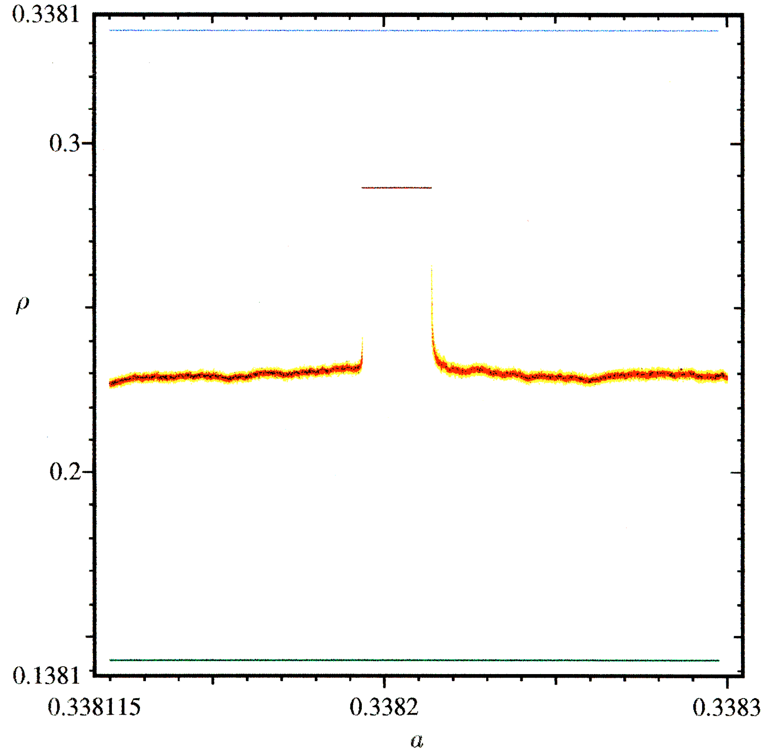

Fig. 8. The observed rotation numbers lock in windows of attracting periodicity for the family $T_a(x) = x + a + \pi/10 \sin 2\pi x$. The attracting window has period 7 and rotation number 2/7.

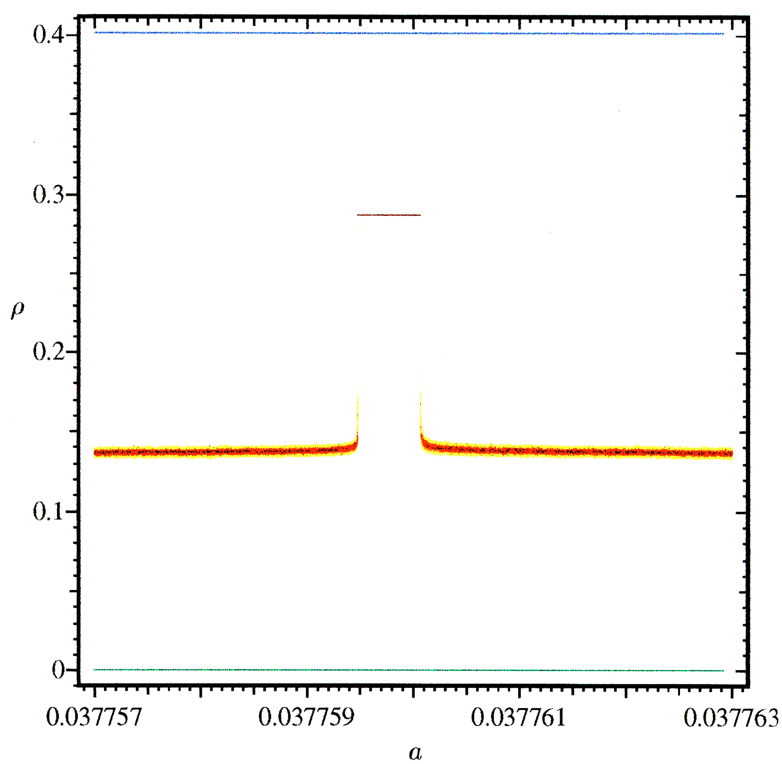

Fig. 9. The observed rotation numbers lock in windows of attracting periodicity for the family $T_a(x) = x + a + b_2 \sin 2\pi x$. The attracting window has period 7 and rotation number 2/7. Note that outside of the locking region the observed rotation numbers drop off very quickly to a "background" level.

addition, several seemingly isolated red dots appear in the plot. Upon closer investigation, these red dots in fact represent locking intervals. Figures 8 and 9 show close-ups of locking intervals.

Locking intervals are created and destroyed in the following way. A saddle node creates a stable periodic point, this undergoes period doubling and becomes a small chaotic attractor, and this attractor is eventually destroyed by a "boundary crisis", i.e. the attractor collides with its own basin boundary. This creation/destruction of an absorbing periodic region has been observed to be very common in one-dimensional dynamics [Grebogi *et al.*, 1983]. For parameter values inside the window of periodicity, the observed rotations numbers become locked at the (rational) rotation number ρ_L of the periodic region.

Even in regions mentioned earlier where the rotation set appears to be a narrow band, more careful calculations usually result in finding small locking intervals within the region.

In [Hunt *et al.*, 1997] the average transition time to periodic attractors was studied. There, it was determined that for a periodic window of width Δa the average time required to fall into the periodic attractor is of the order $(\Delta a)^{-\frac{1}{2}}$. In the graphs shown the smallest step size for a is of the order of 10^{-6} (in Fig. 9). Since we could expect to resolve periodic windows only slightly larger than this, the expected transition time for the smallest windows is of order 1000. Since in all our computations we discarded at least the first 2×10^4 steps, it is reasonable to assume that we were able to resolve most of the periodic windows.

Although we cannot prove that locking intervals are dense, we can prove a result which shows that the number of locking intervals is infinite. Consider the functions $\rho^{\pm}(a)$. As a varies from 0 to 1, each of these functions must increase continuously by 1. Typically, this function has a locking interval for each rational number. For each of these locking intervals for $\rho^{\pm}$, we can prove that there is in fact a subinterval which is an interval of attracting periodicity. Specifically,

Proposition 3. *Suppose that f has exactly two critical points c_1 and c_2 and consider the family $f_a = R_a \circ f$. Suppose that $[\alpha, \beta]$ is a maximal locking interval for ρ^+ (ρ^-), with rotation number p/q. Then there is a subinterval $[\alpha, \gamma]$ $([\gamma, \beta])$ for which f_a has an absorbing periodic interval with rotation number p/q.*

Proof. Suppose p/q is irreducible. Define $f^+ = \pi \circ F^+(\pi)^{-1}$. Let C_0 be the interval on $\mathbf{S}^1$ between c_1 and c_2 corresponding to where F^+ is constant. For $a \in [\alpha, \beta]$, the preimages $C_i = (f^+)^{-i}(C_0)$, $i = 0, \ldots, q-1$ are mutually disjoint. Otherwise f^+ would have a periodic point of period less than q, which would contradict the assumption that f^+ has rotation number p/q.

Consider that F^+ is differentiable, with derivative 0 at the left endpoints of the intervals corresponding to C_0. $(F^+)^q$ is constant on each of the intervals C_i. Since $\{C_i\}$ are mutually disjoint, it follows that $(F^+)^q$ is differentiable at the left endpoints of each of these intervals and the derivatives there are 0.

Now consider $(F^+_\alpha)^q - p$. This function has a set of fixed points, but for all other points $(F^+_\alpha)^q(x) - p < x$ (because α in minimal). Further, the set of fixed points must be disjoint from the union of the intervals corresponding to $\{C_i\}$. Otherwise, by the derivative condition at the left endpoints of $\{C_i\}$, there would be points for which $(F^+_\alpha)^q(x) - p > x$.

Thus $(F^+_\alpha)^q - p$ has a set of fixed points which are disjoint and thus separated from the constant intervals and so f^+_α has a set of periodic orbits which are disjoint from C_0. This means that in fact f_α has such a set, since $f = f^+$ away from C_0. Furthermore for a in some interval, $[\alpha, \gamma']$, f^+_a must also have a periodic set which is disjoint from C_0 and thus corresponds to periodic orbits of f_a. By smoothness of f, for some interval $[\alpha, \gamma]$ these orbits must attract open sets. ■

Locking intervals of the type described in Proposition 3 may be seen in many of the figures. For instance such an interval occurs in Fig. 7, for a approximately in the interval [0.13, 0.143].

5.3. *Observed rotation numbers the edge of windows of periodicity*

We observe that at the two edges of a locking interval the observed rotation numbers appear to decay away from the locked value ρ_L in a very predictable way to a relatively constant background level which we denote by ρ_B. See Figs. 8 and 9.

For parameter values outside but near a window, the region where the periodic attractor existed continues to act as a pseudo-absorbing region in that orbits which fall into the region typically remain there for a relatively long time, giving the appearance of stability. The dynamics in the background region exhibit characteristics of hyperbolic dynamics and once an orbit exits the periodic region, it typically exhibits a chaotic-like trajectory until it happens to again fall into the pseudo-absorbing region. Phenomena having these general characteristics are often said to exhibit *intermittency*.

While the orbit is in the pseudo-trapping phase the partial sums (1) tend toward ρ_L, and while it is in the background region the partial sums tend toward ρ_B. The actual likely and observed rotation numbers will be approximated by a weighted average of these two values, the weightings being the average time orbits spent in each region.

Elsewhere we will give a detailed study of these effects [Homburg *et al.*, 2000].

5.4. *The likely rotation number as a function of a parameter seems to be either locally constant or varies wildly*

As seen in the figures, where the observed rotation numbers are not obviously locally constant, they appear to vary wildly with the parameter a. In particular, the observed rotation numbers do not have any monotonicity properties with respect to a and does not appear to have any differentiability away from the locking intervals. Certainly at the edges of locking intervals the observed rotation numbers are not a differentiable function. If locking intervals are indeed dense, the fact that the rotation numbers rise on one side and fall on the other would imply that there is no local monotonicity except inside the locking intervals.

Actually the situation is even worse. We will show in [Homburg *et al.*, 2000] that just beyond any saddle-node or boundary-crisis bifurcation, given any rational ρ in the interior of the rotation interval at the bifurcation, there are values of a arbitrarily close to the bifurcation value for which the map has ρ as a likely rotation number.

5.5. *Observed rotation numbers at various types of saddle-node bifurcations*

There are four types of saddle-node bifurcations with respect to the rotation interval and these four types have markedly different effects on the observed rotation numbers. Let f_a be a one-parameter family of circle maps which experience a saddle-node bifurcation for $a = a_{sn}$.

- *Type 1a.* The rotation interval for $f_{a_{sn}}$ is trivial and there exist $a > a_{sn}$ arbitrarily close to a_{sn} for which $\rho^+(a) = \rho^-(a)$.
- *Type 1b.* The rotation interval for $f_{a_{sn}}$ is trivial and there exists $\bar{a} > a_{sn}$ such that $\rho^+(a) > \rho^-(a)$ for all $a \in (a_{sn}, \bar{a})$.
- *Type 2.* The rotation interval of $f_{a_{sn}}$ is nontrivial, and the rotation number of the saddle-node point coincides with either $\rho^+(a_{sn})$ or $\rho^-(a_{sn})$. Further, whichever of ρ^+ or ρ^- that corresponds to the rotation number of the saddle-node point, that function increases on one side of a_{sn}.
- *Type 3.* All other cases.

The saddle-node bifurcations for the standard family at $a = b$ are of Type 1 when $b < b_1$ and of Type 2 when $b > b_2$. For bifurcations of Types 1, $\rho_a^\pm$ are both necessarily constant functions to one side of a_{sn} and increasing functions on the other side of a_{sn}. On the increasing side, Proposition 3 implies that there will be attracting periodic orbits with rotation numbers corresponding to both $\rho^\pm$ arbitrarily close to a_{sn}. This can be clearly seen in Figs. 10 and 12. For either type of bifurcation it is known that both ρ^+ and ρ^- follow repeating patterns on fundamental intervals in a [Afraimovich *et al.*, 1996]. For Type 1 bifurcations, these fundamental intervals for ρ^+ and ρ^- are synchronized, causing the observed rotation numbers to behave in approximately the same pattern on each fundamental interval. Both Types 1a and 1b bifurcations belong to the class of maps studied in [Young, 1999]. In the terminology used there Type 1a bifurcations occur when the passing number of the map is 1. Type 1b bifurcations occur when the passing number is 2 or greater. The passing number is defined roughly by the maximum number of iterations one orbit may jump ahead of another. That maps with passing number 1 are of Type 1a was shown in [Friedman & Tresser, 1986]. That maps with passing number 2, or higher, are of Type 1b is the essence of Proposition 9 of [Young, 2000].

For Type 2 bifurcations, suppose for the sake of clarity that the saddle-node periodic point has rotation number $\rho_{a_{sn}}^-$. Then ρ_a^+ is typically constant in a neighborhood of a_{sn} (in the sense that ρ^+ increases only on a set of measure zero, it is very likely that it is constant in a neighborhood of a_{sn}). It is also the case that ρ_a^- is an increasing function of a to the right of a_{sn}. Proposition 3 implies that the

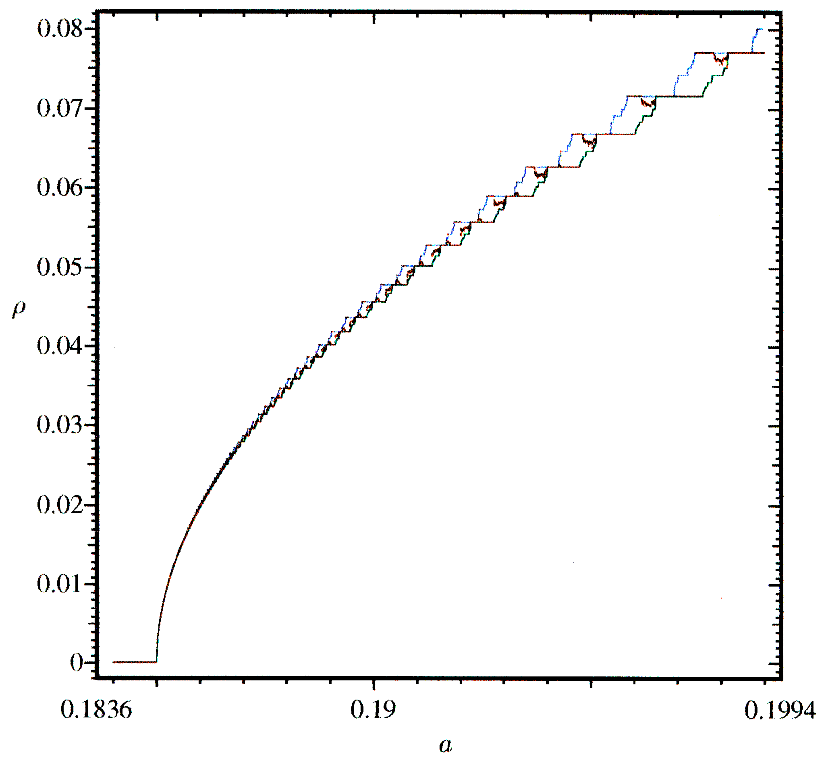

Fig. 10. Observed rotation numbers at a saddle-node bifurcation in the family $T_a(x) = x + a + 0.185 \sin 2\pi x$. The saddle-node occurs at the parameter value $a = 0.185$. Since $\rho^+(a_{sn}) = \rho^-(a_{sn})$, and there are repeated subintervals on which $\rho^+(a) = \rho^-(a)$, this is a Type 1a saddle-node. Refer to Fig. 3 for an explanation of the color scheme.

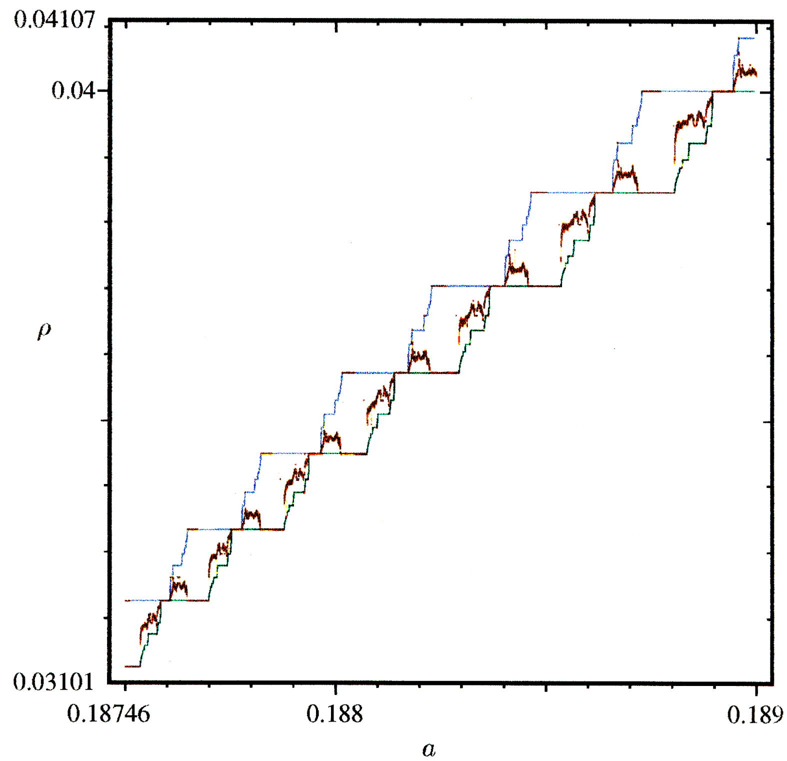

Fig. 11. A close up of Fig. 10. Observed rotation numbers near a saddle-node bifurcation in the family $T_a(x) = x + a + 0.185 \sin 2\pi x$. The saddle-node occurs at the parameter value $a = 0.185$. Note the repeated pattern. It is known that the map has a repeated universal structure as $a \searrow a_{sn}$.

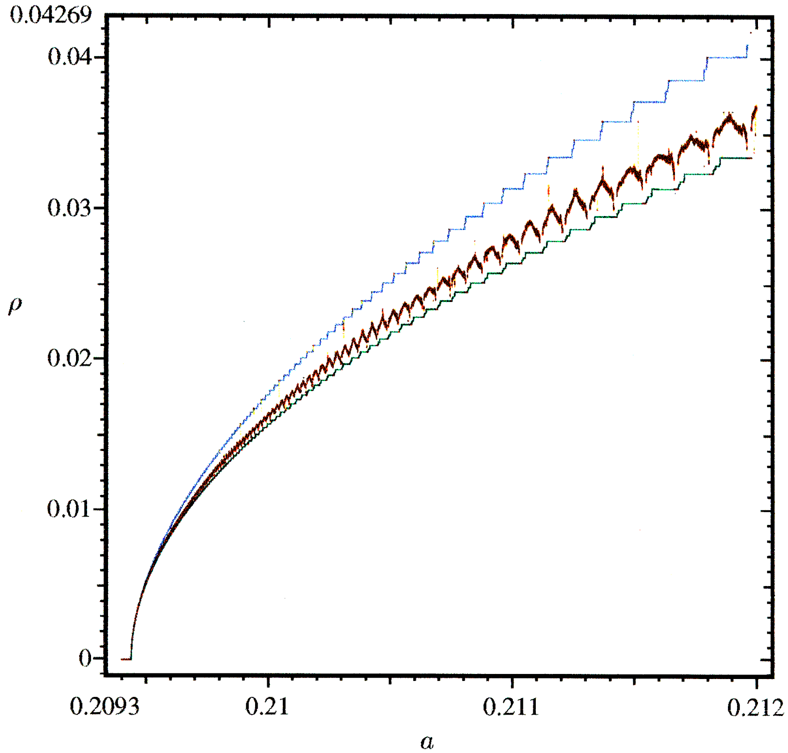

Fig. 12. Observed rotation numbers at a saddle-node bifurcation in the family $T_a(x) = x + a + \pi/15 \sin 2\pi x$. The saddle-node occurs at the parameter value $a_{sn} = b = \pi/15$. Since $\rho^+(a_{sn}) = \rho^-(a_{sn})$, and $\rho^+(a)$ is strictly greater than $\rho^-(a)$ for $a > a_c$, this is a Type 1b saddle-node.

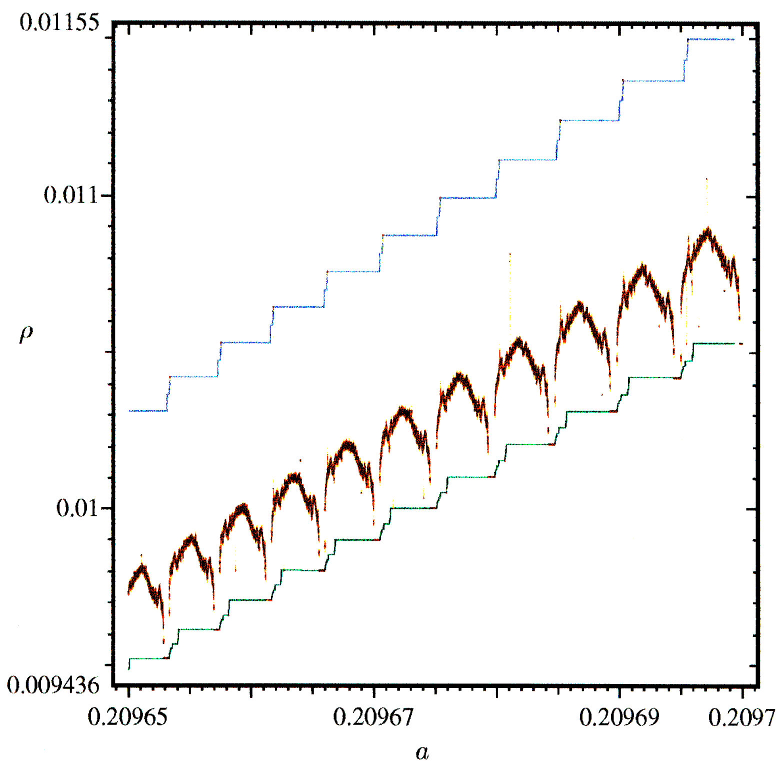

Fig. 13. A close-up of Fig. 12. Observed rotation numbers near a saddle-node bifurcation in the family $T_a(x) = x + a + \pi/15 \sin 2\pi x$. This is a Type 1b saddle-node. There is a universal pattern, but $\rho^+(a)$ is strictly greater than $\rho^-(a)$. That is, the rotation interval is nontrivial.

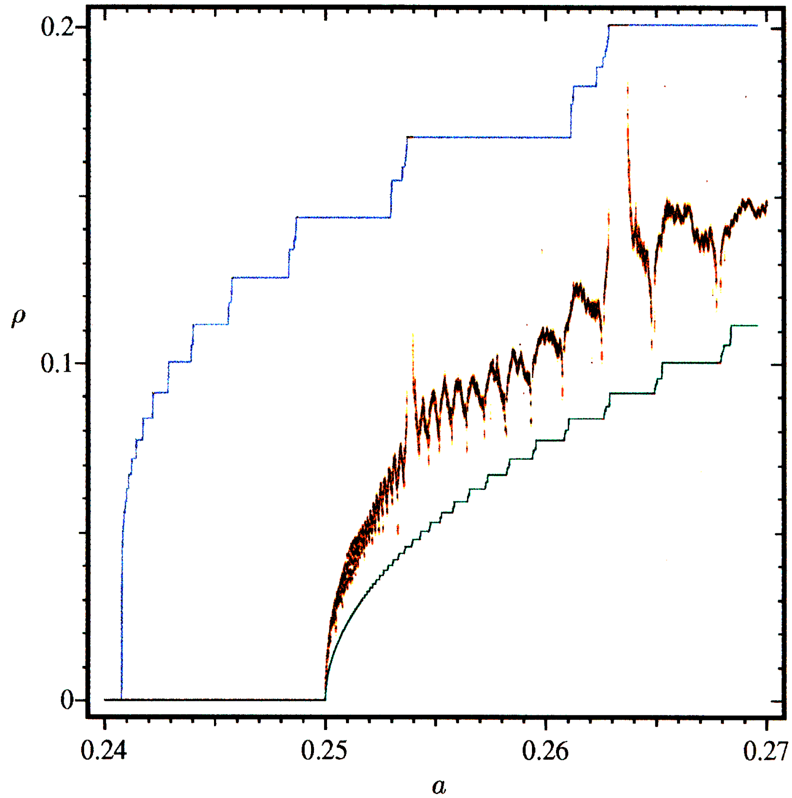

Fig. 14. Observed rotation numbers at a saddle-node bifurcation in the family $T_a(x) = x + a + 0.25 \sin 2\pi x$. The saddle-node occurs at the parameter value $a_{sn} = b = 0.25$. Since $\rho^+(a_{sn}) > \rho^-(a_{sn})$, the observed rotation number at a_{sn} coincides with $\rho^-(a_{sn})$, and $\rho^-(a)$ increases for $a > a_c$, this is a Type 2 saddle-node.

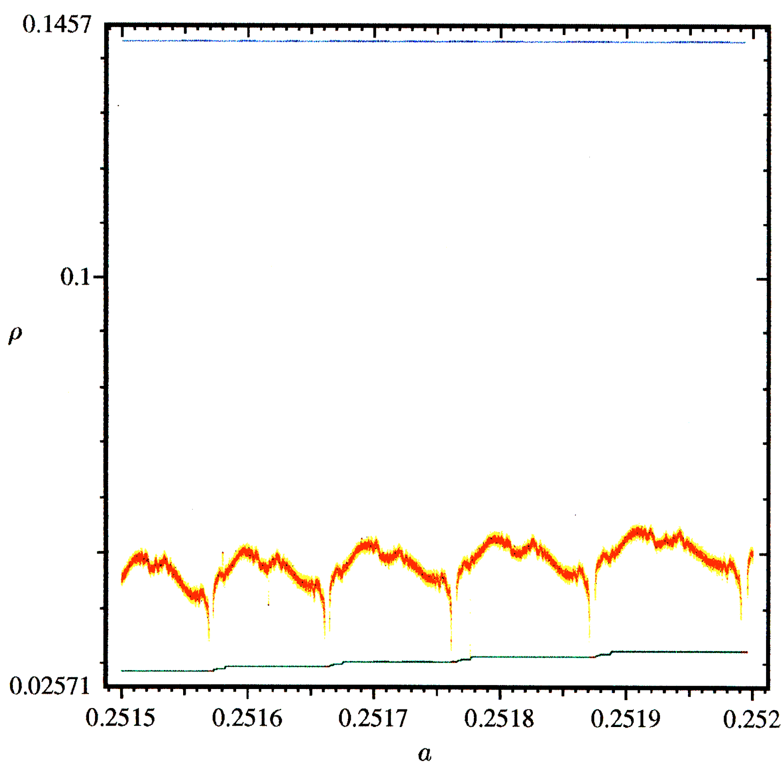

Fig. 15. A close-up of Fig. 14 near the saddle-node bifurcation. There seems to be a universal pattern.

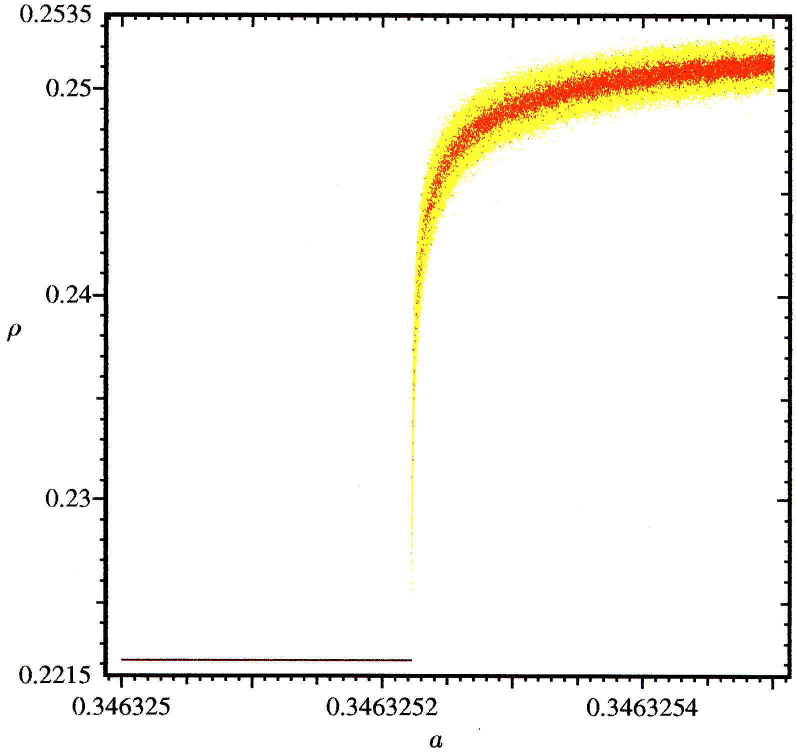

Fig. 16. Observed rotation numbers at a saddle-node bifurcation in the family $T_a(x) = x + a + \pi/10 \sin 2\pi x$. Since $\rho^-(a_{sn}) < \check{\rho}(a_{sn}) < \rho^+(a_{sn})$, this is a Type 3 saddle-node.

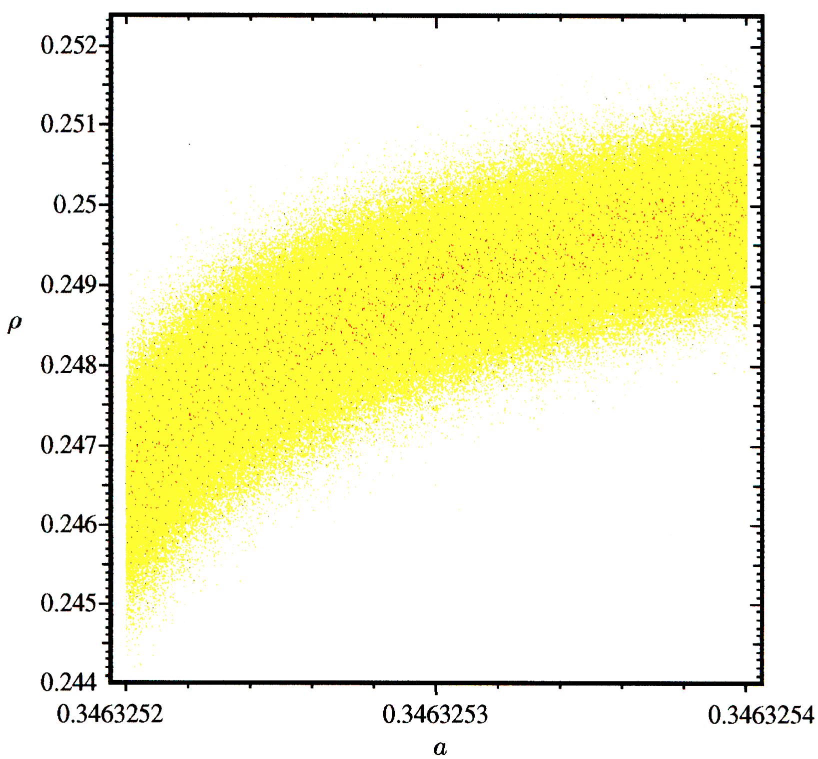

Fig. 17. A close-up of Fig. 16. Observed rotation numbers near a saddle-node bifurcation in the family $T_a(x) = x + a + \pi/10 \sin 2\pi x$. This is a Type 3 saddle-node. There is no detectable universal pattern.

likely rotation sets include values of ρ_a^- for some values of a arbitrarily close to a_{sn}. Furthermore, with ρ^+ constant, the observed rotation numbers form a universal pattern on fundamental intervals corresponding to fundamental intervals for ρ^-. See Figs. 14 and 15.

For all examples we have been able to study, the effect of Type 3 bifurcations on the observed rotation numbers is profoundly different from those seen in the other types of bifurcations. In the other types of saddle-node bifurcations, the nearby observed rotation numbers vary in a very nonsmooth way as a is varied. In Type 3 bifurcations, the effect on the observed rotation numbers is very smooth. See Figs. 16 and 17.

6. Some Questions for Further Study

As discussed above, for most f the observed rotation sets appear to be either very narrow or isolated points. It is not clear, and cannot be determined numerically, whether in fact the narrow interval which is often observed would actually collapse to a single point in the limit. The first question of importance is whether or not this is the case.

Question 1. Do there exist $f \in \mathrm{End}_1^r(\mathbf{S}^1)$ for which the likely rotation set is nonfinite.

At this point we have no reason to believe that either the affirmative or negative is true. A related question is whether there exist f for which μ has a nontrivial m-absolutely continuous part?

The condition that $m(C) = 0$ and the condition that $\{\mu_i\}$ converge play a large role in this theory. Thus we raise the following questions:

Question 2. For what classes of circle endomorphisms is it always, or at least generically true that $m(C) = 0$?

Question 3. For what classes of circle endomorphisms does $\{\mu_i\}$ converge?

In [Young, 1998] the condition that $m(C) = 0$ and the condition that $\{\mu_i\}$ converges were discussed for Birkhoff averages of observables in general dynamical systems. These conditions were shown to be related to very general conditions on the dynamical systems, independent of the observable in question. However, at this time it is not known whether or not those general conditions are typical in either a topological or measure sense.

References

Afraimovich, V., Liu, W.-S. & Young, T. [1996] "Conventional multipliers for homoclinic orbits," *Nonlinearity* **9**, 115–136.

Afraimovich, V. & Zaslavsky, G. [1998] *Sticky Orbits of Chaotic Hamiltonian Dynamics. Chaos, Kinetics and Nonlinear Dynamics in Fluids and Plasmas* (*Carry-Le Rouet, 1997*), *Lecture Notes in Physics*, Vol. 511 (Springer-Verlag, Berlin), pp. 59–82.

Bamon, R., Malta, I., Pacifico, M. & Takens, F. [1984] "Rotation intervals of endomorphisms of the circle," *Erg. Th. Dyn. Syst.* **4**, 493–498.

Block, L., Guckenheimer, J., Misiurewicz, M. & Young, L. [1980] "Periodic points and topological entropy of one-dimensional maps," in *Global Theory of Dynamical Systems, Lecture Notes in Mathematics* (Springer-Verlag), Vol. 819.

Boyland, P. [1986] "Bifurcations of circle maps: Arnol'd tongues, bistability and rotation intervals," *Commun. Math. Phys.* **106**, 353–381.

Chenciner, A., Gambaudo, J.-M. & Tresser, C. [1984] "Une remarque sur las structure des endomorphisms de degré 1 du cercle," *C.R. Acad. Sci. Paris* **299** série I, 145–148.

Cornfeld, I., Sinai, Y. & Fomin, S. [1982] *Ergodic Theory* (Nauka, Moscow); translated (Springer).

de Figueiredo, J. B. & Malta, C. [1998] "Lyapunov graph for two-parameters map: Application to the circle map," *Int. J. Bifurcation and Chaos* **8**, 281–294.

de Melo, W. & van Strien, S. [1993] *One-Dimensional Dynamics* (Springer-Verlag).

Friedman, B. & Tresser, C. [1986] "Comb structures in hairy boundaries: Some transition problems for circle maps," *Phys. Lett.* **A117**, 15–22.

Graczyk, J. & Sweitek, G. [1997] "Generic hyperbolicity in the logistic family," *Anal. Math.* **146**, 1–52.

Grebogi, C., Ott, E. & Yorke, J. [1983] "Crises, sudden changes in chaotic attractors and transient chaos," *Physica* **D7**, 181–200.

Homburg, A., Saum, M. & Young, T. [2000] "Intermittency near windows of attracting periodicity and decay of observed rotation numbers," preprint.

Hunt, B., Jacobs, J. & Ott, E. [1997] "Scaling of the durations of chaotic transients in windows of attracting periodicity," *Phys. Rev.* **E56**, 6508–6515.

Jakobson, M. [1981] "Absolutely continuous invariant measures for one-parameter families of one-dimensional maps," *Commun. Math. Phys.* **81**, 39–88.

Milnor, J. [1985] "On the concept of attractor," *Commun. Math. Phys.* **99**, 177–195.

Monti, M., Pardo, W., Walkenstein, J., Rosa, Jr., E. & Grebogii, C. [1999] "Color map of Lyapunov exponents of invariant sets," *Int. J. Bifurcation and Chaos* **8**, 1459–1464.

Newhouse, S., Palis, J. & Takens, F. [1983] "Bifurcations and stability of families of diffeomorphisms," *Publs. Math. Inst. Hautes Etud. Sci.* **57**, 5–71.

Shiryayev, A. [1979] *Probability*, *Graduate Texts in Mathematics*, Vol. 95 (Springer-Verlag).

Young, T. [1998] "Distributions of Birkhoff averages with respect to noninvariant measures," preprint.

Young, T. [2000] "Entropy and rotation intervals for circle maps near saddle-node bifurcations," *Math. Zeit.* **234**, 487–506.