The Discontinuous Galerkin Finite Element Method

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Overview

- What is DG?
- DG Formulation
- Data Structures
- Solvers
- Adaptivity

What is DG?

- The Discontinuous Galerkin (DG) Finite Element Method (FEM) is a variant of the Standard (Continuous) Galerkin (SG) FEM.
- SG-FEM requires continuity of the solution along element interfaces (edges).
- DG-FEM does not require continuity of the solution along edges.
- DG methods have more degrees of freedom (unknowns) to solve for than SG methods.

DG Advantages

- DG methods have a number of advantages over SG methods:
  - Assembly of stiffness matrix is easier to implement.
  - Refinement of triangles is easier to implement.
  - Adaptive methods are more flexible.
  - *Natural Hierarchy* allows for multilevel methods to be integrated into solvers.
- DG methods can support high order local approximations that can vary nonuniformly over the mesh.
- DG methods are readily parallelizable.

Model Problem

Let $\Omega \subset \mathbb{R}^d, d = 2, 3$ be a bounded open polyhedral domain with Lipshitz continuous boundary.

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g_D & \text{on } \Gamma_D \\ \nabla u \cdot n = g_N & \text{on } \Gamma_N \end{cases} \qquad \text{(MP)}$$

where $\partial\Omega := \Gamma = \Gamma_D \cup \Gamma_N$ and n is the unit normal vector exterior to Ω. We also assume that $\mu_{d-1}(\Gamma_D) > 0$, $f \in L^2(\Omega)$, $g_N \in L^2(\Gamma_N)$.

Notation

- Let $\mathcal{T}_h = \{K_i : i = 1, 2, \ldots, m_h\}$ be a family of star-like partitions of Ω parameterized by $0 < h \leq 1$.
- The elements of $\mathcal{T}_h$ satisfy the minimal angle condition.
- $\mathcal{T}_h$ is locally quasi-uniform.
- $\mathcal{E}^I = \{e = \partial K_j \cap \partial K_l : \mu_{d-1}(\partial K_j \cap \partial K_l) > 0\}$
- $\mathcal{E}^B = \{e = \partial K_j \cap \partial\Omega : \mu_{d-1}(\partial K_j \cap \partial\Omega) > 0\}$
- $\forall e \in \mathcal{E}^B$, either $e \subset \Gamma_D$ or $e \subset \Gamma_N$ and $\mathcal{E} = \mathcal{E}^I \cup \mathcal{E}^B$, where $\mathcal{E}^B = \mathcal{E}^B_D \cup \mathcal{E}^B_N$ and $\mathcal{E}^B_D \cap \mathcal{E}^B_N = \emptyset$.
- If $e \in \mathcal{E}^I$, then $e = \partial K^+ \cap \partial K^-$ for $K^+, K^- \in \mathcal{T}_h$.
- If $e \in \mathcal{E}^B$, then $e = \partial K^+ \cap \partial\Omega \equiv \partial K \cap \partial\Omega$.
- n^+ is the unit normal to e that points outward from K^+.
- On $\mathcal{T}_h$, for $r \geq 2$, define the energy space E_h and finite element space V_h^r by

$$E_h = \prod_{K \in \mathcal{T}_h} H^2(K), \quad V_h^r = \prod_{K \in \mathcal{T}_h} P_k(K)$$

where $P_k(K)$ denotes the space of polynomials of total degree $r - 1 \equiv k \geq 1$.

DG Formulation

- First obtain weak formulation by multiplying (MP) by $v \in V_h^r$ and integrating over Ω:

$$-\int_\Omega (\Delta u) v \, dx = \int_\Omega f v \, dx$$

- Now decompose integrals into element contributions and integrate by parts:

$$\sum_{K \in \mathcal{T}_h} -\int_K (\Delta u) v \, dx = \sum_{K \in \mathcal{T}_h} \int_K f v \, dx$$

$$\sum_{K \in \mathcal{T}_h} \int_K \nabla u \cdot \nabla v \, dx - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial u}{\partial n} v \, ds = \sum_{K \in \mathcal{T}_h} \int_K f v \, dx$$

DG Formulation, contd.

- Splitting Edge integrals:

$$\sum_{K\in\mathcal{T}_h}\left\langle\frac{\partial u}{\partial n},v\right\rangle_{\partial K}=\sum_{e\in\Gamma_D}\left\langle\frac{\partial u}{\partial n},v\right\rangle_e+\sum_{e\in\Gamma_N}\left\langle\frac{\partial u}{\partial n},v\right\rangle_e$$
$$+\sum_{e\in\mathcal{E}^I}\left(\left\langle\frac{\partial u^+}{\partial n^+},v\right\rangle_e+\left\langle\frac{\partial u^-}{\partial n^-},v\right\rangle_e\right)$$

- Resulting in:

$$\sum_{K\in\mathcal{T}_h}(\nabla u,\nabla v)_K-\left\langle\frac{\partial u}{\partial n},v\right\rangle_{\Gamma_D}-\sum_{e\in\mathcal{E}^I}\left(\left\langle\frac{\partial u^+}{\partial n^+},v\right\rangle_e-\left\langle\frac{\partial u^-}{\partial n^+},v\right\rangle_e\right)$$
$$=\sum_{K\in\mathcal{T}_h}(f,v)_K+\langle g_N,v\rangle_{\Gamma_N}$$

DG Formulation, contd.

- Two different ways of working with above internal edge integrals:
  - D. Arnold: $ac - bd = \frac{1}{2}(a+b)(c-d) + \frac{1}{2}(a-b)(c+d)$.
  - G. Baker: $ac - bd = a(c-d) + (a-b)d$.
- Define
  - $B(u,v) := \sum_{K \in \mathcal{T}_h} (\nabla u, \nabla v)_K$
  - $F(v) := \sum_{K \in \mathcal{T}_h} (f, v)_K + \langle g_N, v \rangle_{\Gamma_N}$
  - $J(u,v) := \left\langle \frac{\partial u}{\partial n}, v \right\rangle_{\Gamma_D} + \sum_{e \in \mathcal{E}^I} \left\langle \left\{ \frac{\partial u}{\partial n} \right\}, [v] \right\rangle_e$
  - where $\left\{ \frac{\partial u}{\partial n} \right\}\Big|_e = \frac{1}{2}\left(\frac{\partial u^+}{\partial n} + \frac{\partial u^-}{\partial n} \right)\Big|_e$ (Arnold) and,
  - $\left\{ \frac{\partial u}{\partial n} \right\}\Big|_e = \frac{\partial u^+}{\partial n}\Big|_e$ (Baker) , and
  - $[v]\big|_e = (v^+ - v^-)\Big|_e$.

SIPG Formulation

- Leads to the DG formulation of (MP): Find $u \in H^1 \cap E_h$ such that

$$B(u,v) - J(u,v) = F(v) \quad \forall v \in E_h$$

- Symmetric Interior Penalty Formulation (SIPG) involves modifi cations:
  - Symmetrization:

$$B(u,v) - J(u,v) - J(v,u) = F(v) - \left\langle \frac{\partial v}{\partial n}, g_D \right\rangle_{\Gamma_D}$$

SIPG Formulation, contd.

- Penalization of *jump* terms:
  - Let $\sigma > 0$ be a penalization parameter
  - Let $J^{\sigma}(u,v) := \sum_{e \in \mathcal{E}^I} \langle \sigma[u],[v] \rangle_e + \langle \sigma u, v \rangle_{\Gamma_D}$
- SIPG Formulation: Find $u \in H^1 \cap E_h$ such that

$$
\begin{aligned}
B(u,v) - J(u,v) - J(v,u) + J^{\sigma}(u,v) \\
= F(v) - \left\langle \frac{\partial v}{\partial n}, g_D \right\rangle_{\Gamma_D} + \langle \sigma g_D, v \rangle_{\Gamma_D} \quad \forall v \in E_h
\end{aligned}
$$

DG FEM Formulation

Find $u_h^\gamma \in V_h^r$ such that

$$a_h^\gamma\left(u_h^\gamma, v\right) = F_h^\gamma(v), \qquad \forall v \in V_h^r$$

where

$$a_h^\gamma\left(u_h^\gamma, v\right) = \sum_{K\in\mathcal{T}_h} (\nabla u_h^\gamma, \nabla v)_K$$

$$- \sum_{e\in\mathcal{E}^I\cup\mathcal{E}_D^B} \left(\langle\{\partial_n u_h^\gamma\}, [v]\rangle_e + \langle\{\partial_n v\}, [u_h^\gamma]\rangle_e - \gamma h_e^{-1} \langle[u_h^\gamma], [v]\rangle_e \right)$$

and

$$F_h^\gamma(v) = \sum_{K\in\mathcal{T}_h} (f, v)_K - \left\langle g_D, \partial_n v - \gamma h_e^{-1} v\right\rangle_{\Gamma_D} + \langle g_N, v\rangle_{\Gamma_N}$$

Energy Norm

- The bilinear form $a_h^\gamma(\cdot,\cdot)$ induces the following norm on E_h:

$$\|v\|_{1,h} = \Bigg(\sum_{K\in\mathcal{T}_h} \|\nabla v\|_{0,K}^2 + \sum_{e\in\mathcal{E}^I\cup\mathcal{E}_D^B} \left(h_e^{-1}\|[v]\|_{0,e}^2 + h_e \,|\{\partial_n v\}|_{0,e}^2 \right) \Bigg)^{1/2}$$

- Note that $a_h^\gamma(\cdot,\cdot)$ is symmetric, coercive for $\sigma > \sigma_0 > 0$ for σ_0 large enough.
- Note that $\sigma = \sigma(\gamma, r, h)$. Common to take $\sigma = \gamma(r-1)^2 h_e^{-1}$, and use the condition $\gamma > \gamma_0$ for γ_0 large enough.

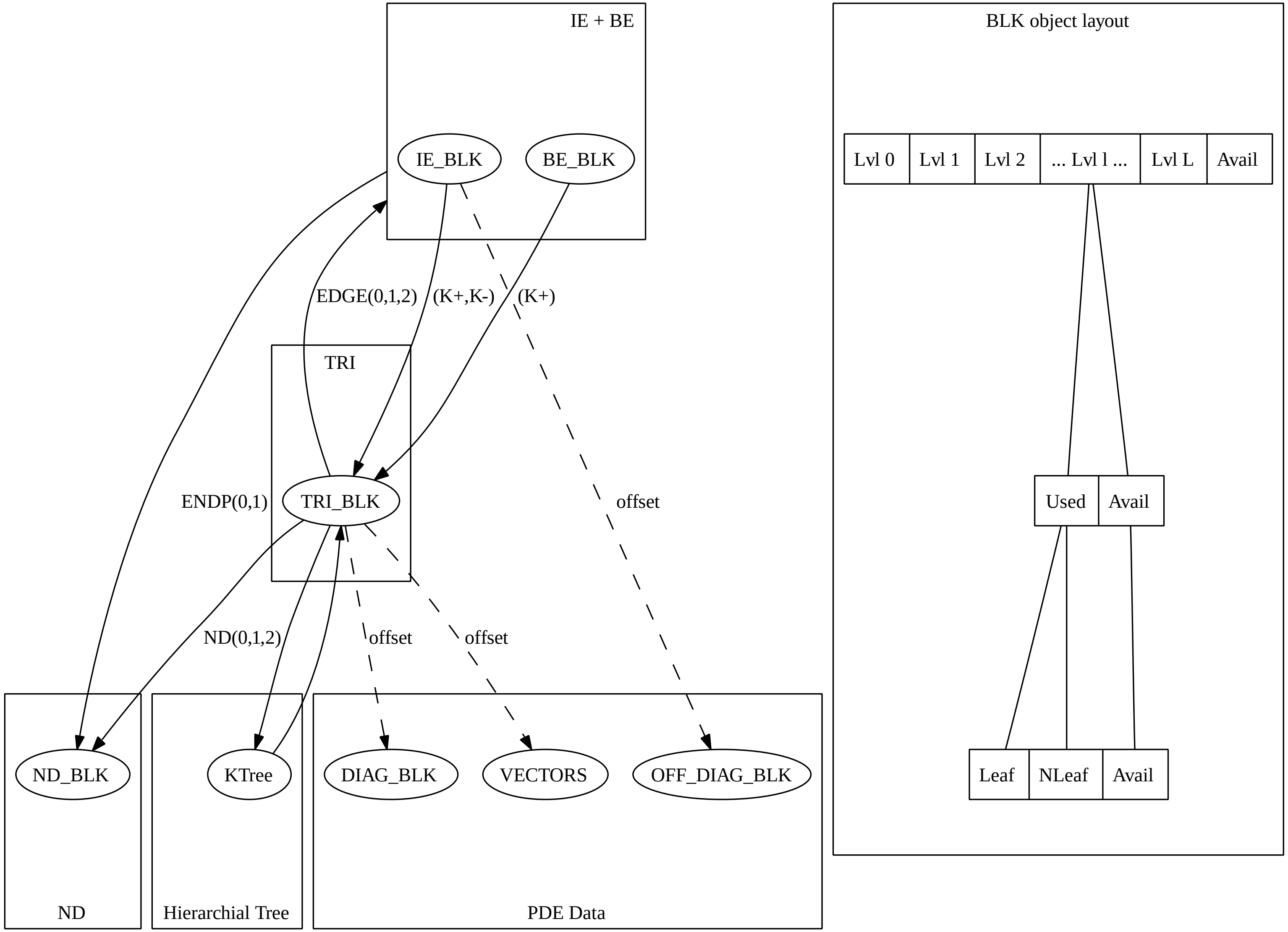
Stiffness Matrix Assembly

- The stiffness matrix has a very nice sparse block structure, consisting of two types of matrix subblocks
  - *Diagonal Blocks*, which describe interaction of an elements degrees of freedom with itself.
  - *Off Diagonal Blocks*, which describe interactions of K^+ dof with K^- dof through edge e.
- The triangulation $\mathcal{T}_h$ has imposed on it the constraint that any element K can at most have 2 neighboring elements K_1, K_2 along edge e. This is the case where one has a *hanging node* on an edge, it is also called a *1-irregular* mesh, or the *two-neighbor* condition.
- This results in a maximum block bandwidth of 6 for the stiffness matrix.

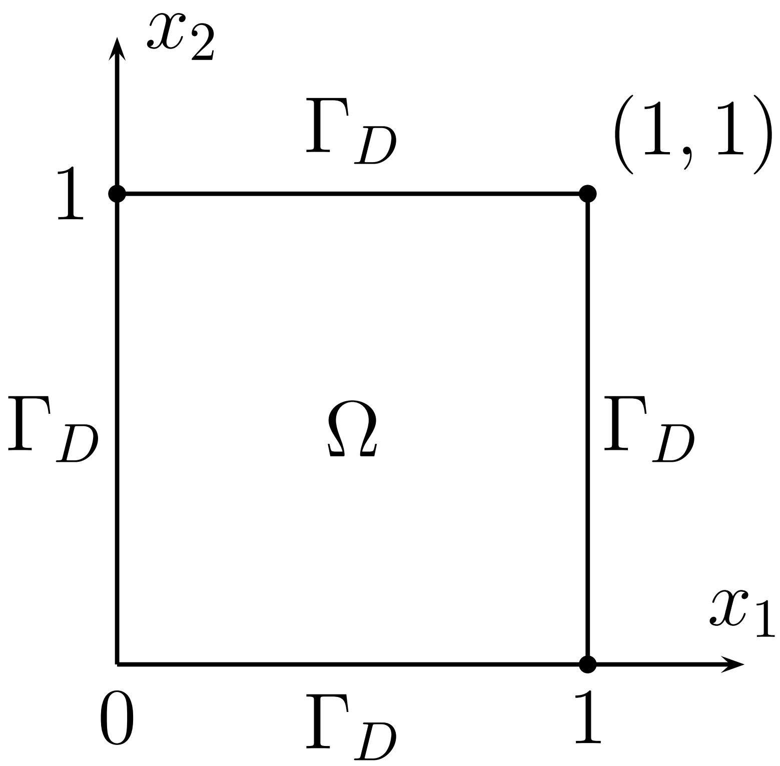
Data Structures

- Data objects include TRIANGLE, EDGE, and NODE.
- Objects stored in one long array of objects for each type via doubly linked list structures.
- Pointers are used to identify relations between objects.
- Hierarchial relations are stored in a 4-ary tree structure.
- PDE data (vectors, stiffness matrix blocks) are stored separately from geometric data.

Data Structure Relations



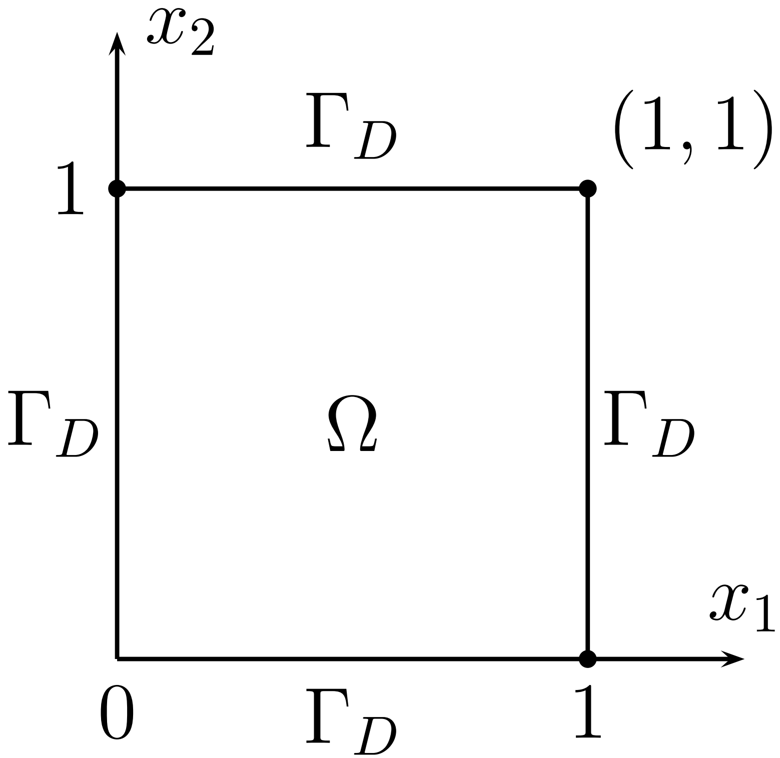
Test Problems - f3



$$\begin{cases} -\Delta u = 2\pi^2 \sin(\pi x)\sin(\pi y) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \end{cases}$$

Exact solution: $u = \sin(\pi x)\sin(\pi y)$.

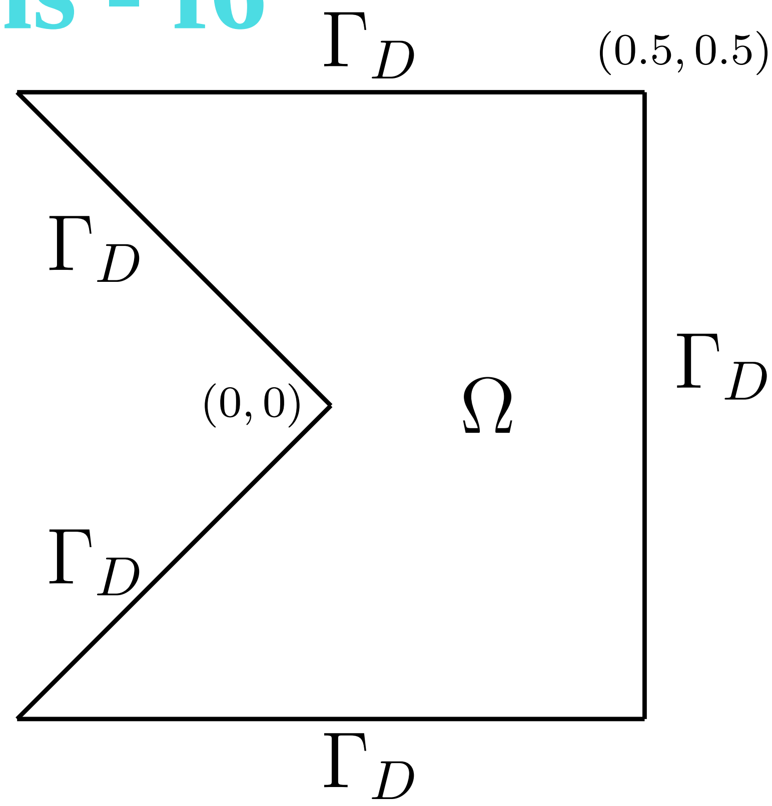
Test Problems - f4



$$\begin{cases} -\Delta u = 128\pi^2 \sin(8\pi x)\sin(8\pi y) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \end{cases}$$

Exact solution: $u = \sin(8\pi x)\sin(8\pi y)$.

Test Problems - f6



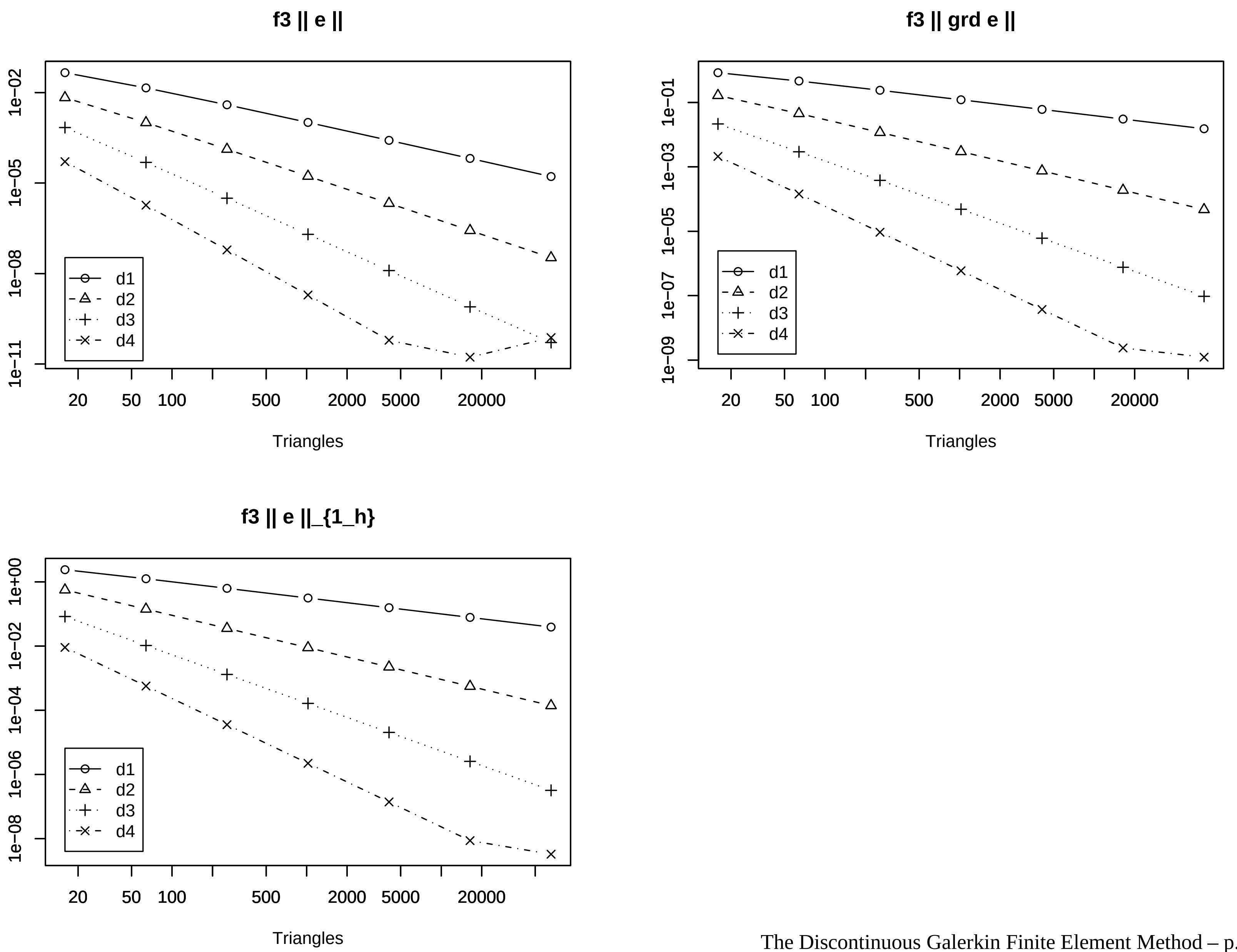
$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = r^{2/3}\sin(2/3\theta) & \text{on } \Gamma_D \end{cases}$$

Exact solution: $u = r^{2/3}\sin(2/3\theta)$.

FEM Error

- A quick way of determining if a FEM is working properly is if one obtains expected reductions in error as one uniformly refines a mesh.
- For $h \to h/2$ uniformly in a mesh with elements of degree p, one expects that
  - $\|u - u_{h/2}\|_{L^2(\Omega)} \approx \left(\frac{1}{2}\right)^{p+1} \|u - u_h\|_{L^2(\Omega)}$
  - $\|u - u_{h/2}\|_{H^1(\Omega)} \approx \left(\frac{1}{2}\right)^{p} \|u - u_h\|_{H^1(\Omega)}$
- As one can see in the following graphs, this is indeed the case for the smooth functions (f3, f4), but not necssarily the case for the point singularity problem (f6).

Uniform Refinement Error - f3



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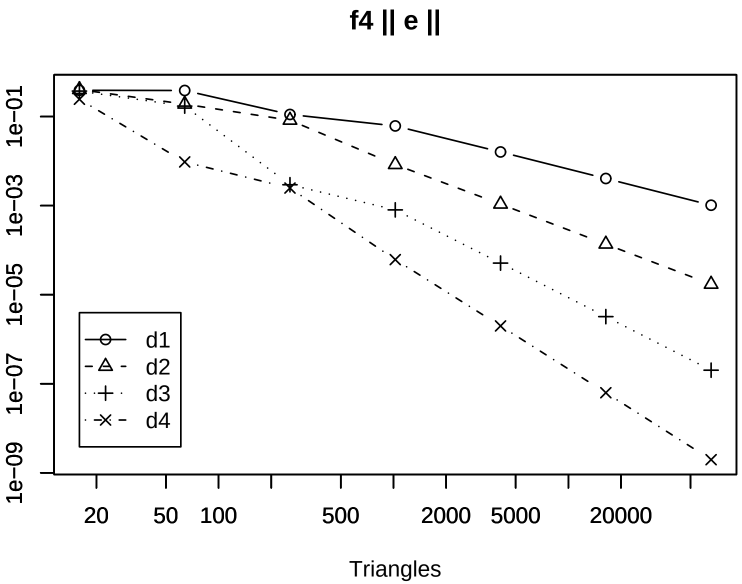
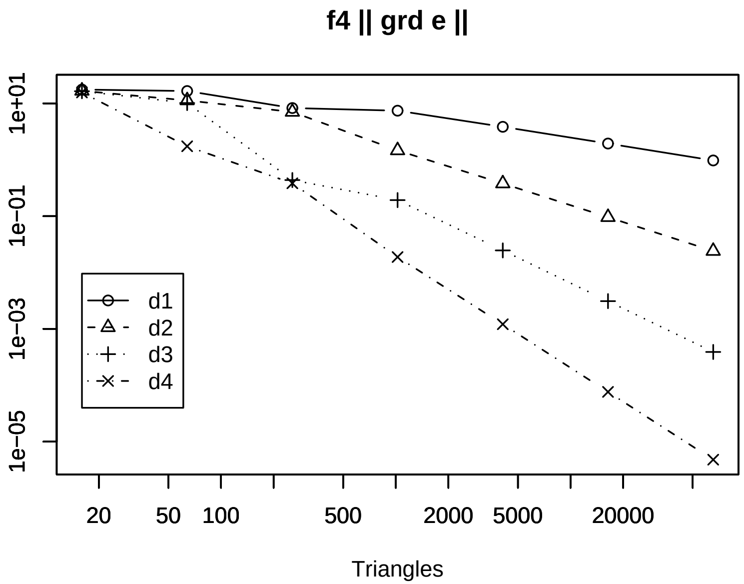
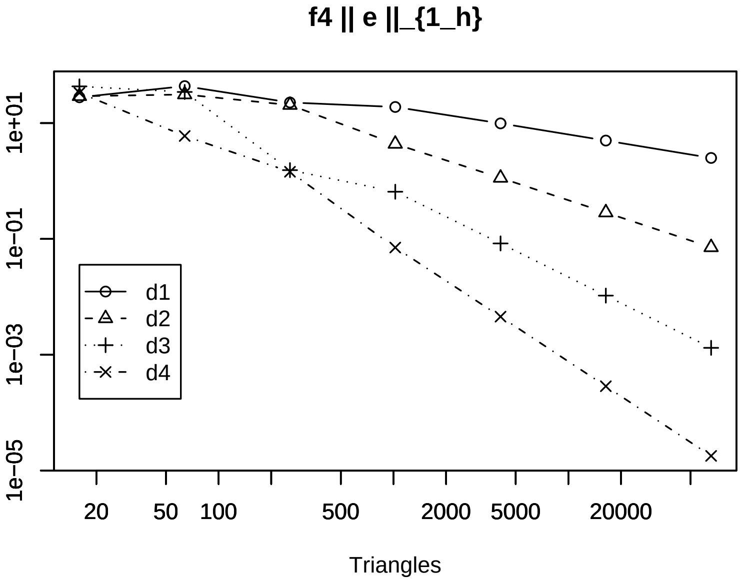
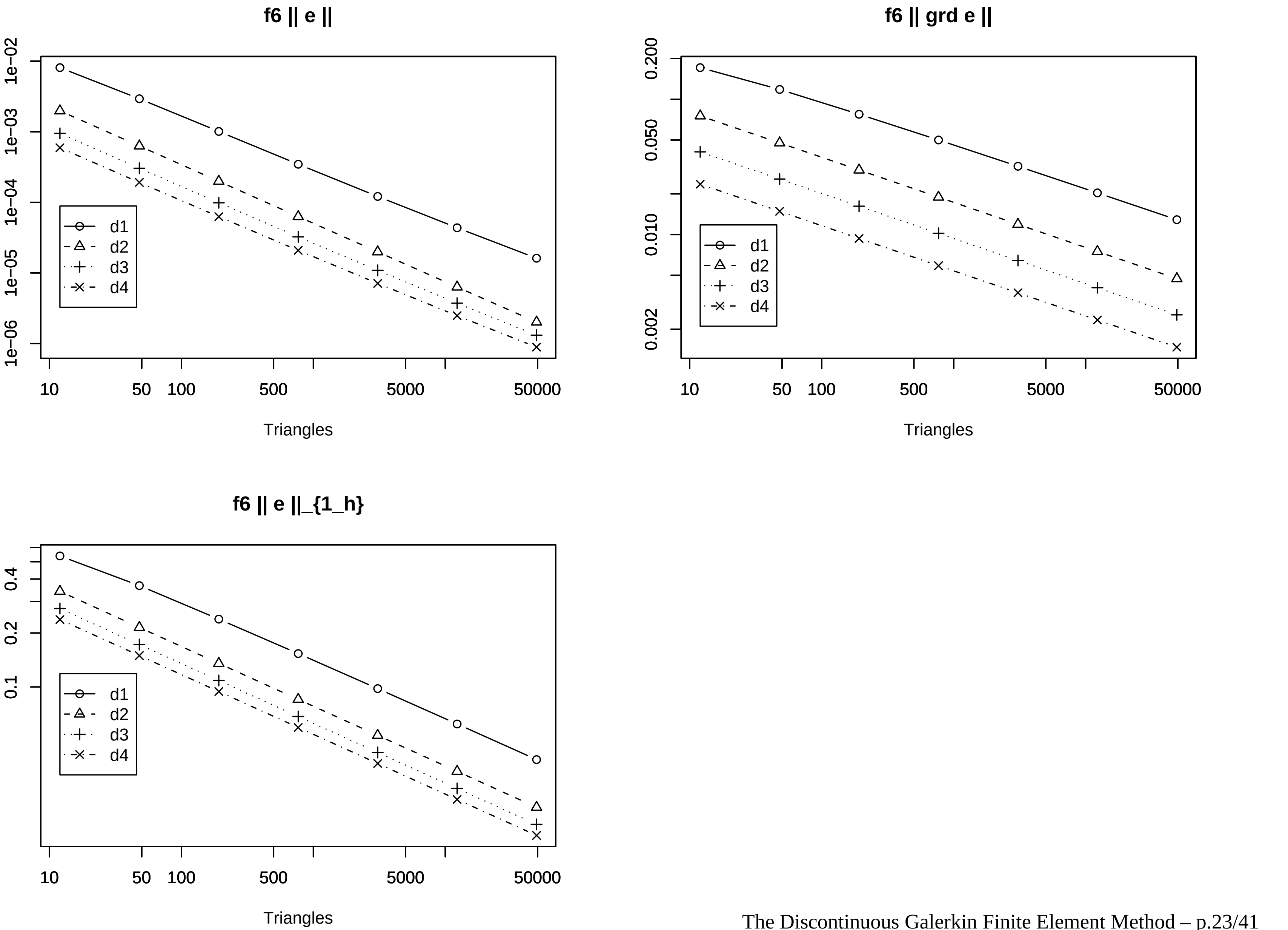
Uniform Refinement Error - f4







Uniform Refinement Error - f6



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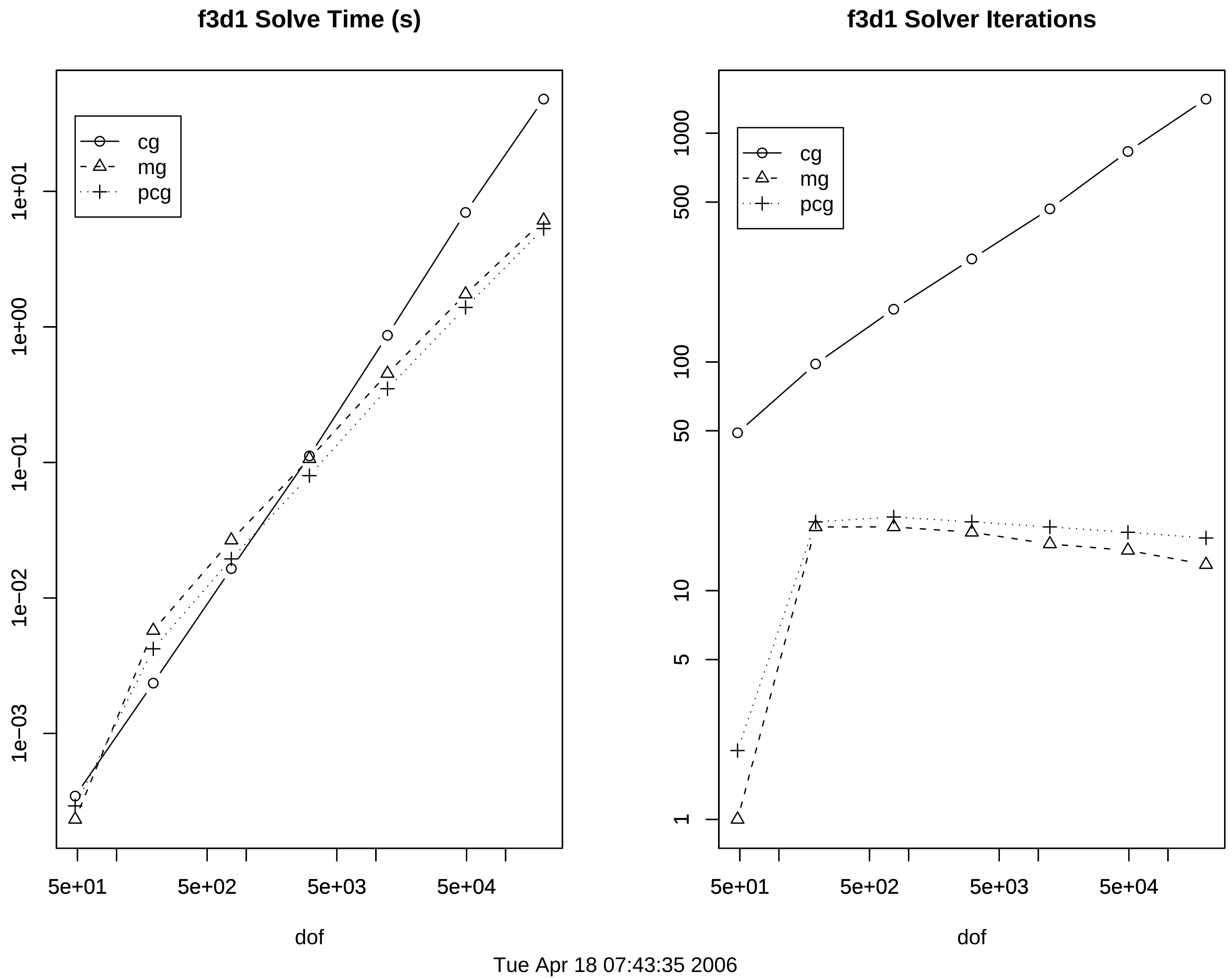
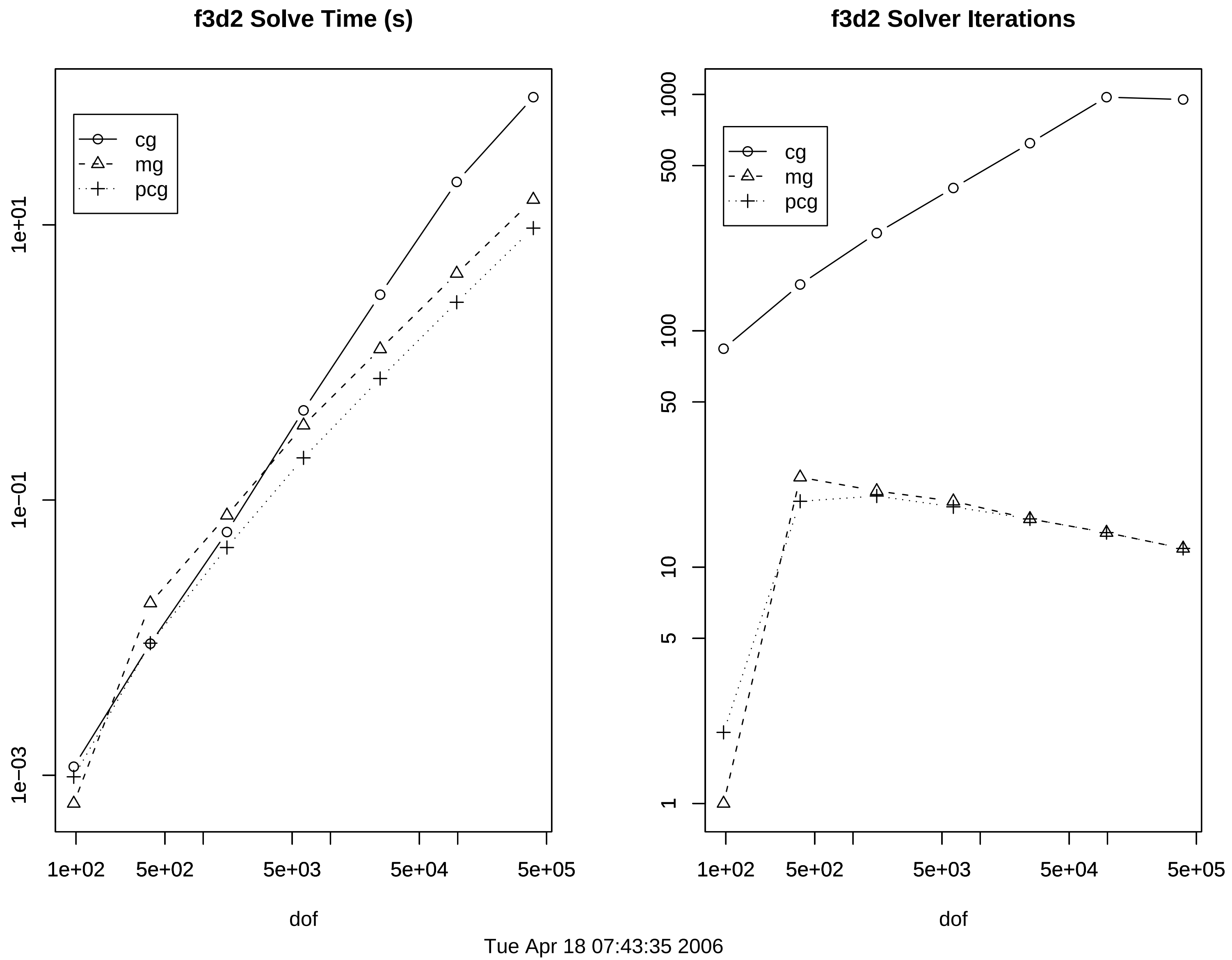
Linear Solvers

- Since SIPG produces a symmetric, positive definite linear system to solve, CG and PCG can be used.
- Due to the natural level based tree hierarchy produced, multigrid can also be used.
- PCG is used with MG as preconditioner.
- The previous solution obtained is embedded into the new triangulation to obtain the initial solution for each solve.
- Point Gauss-Seidel is used as the MG smoother.
- Local smoothing is implemented to improve solve time, i.e., on a particular level ℓ only dof's associated with levels up to $\ell - n$ are smoothed.
- Capability exists to implement either V or W cycles.

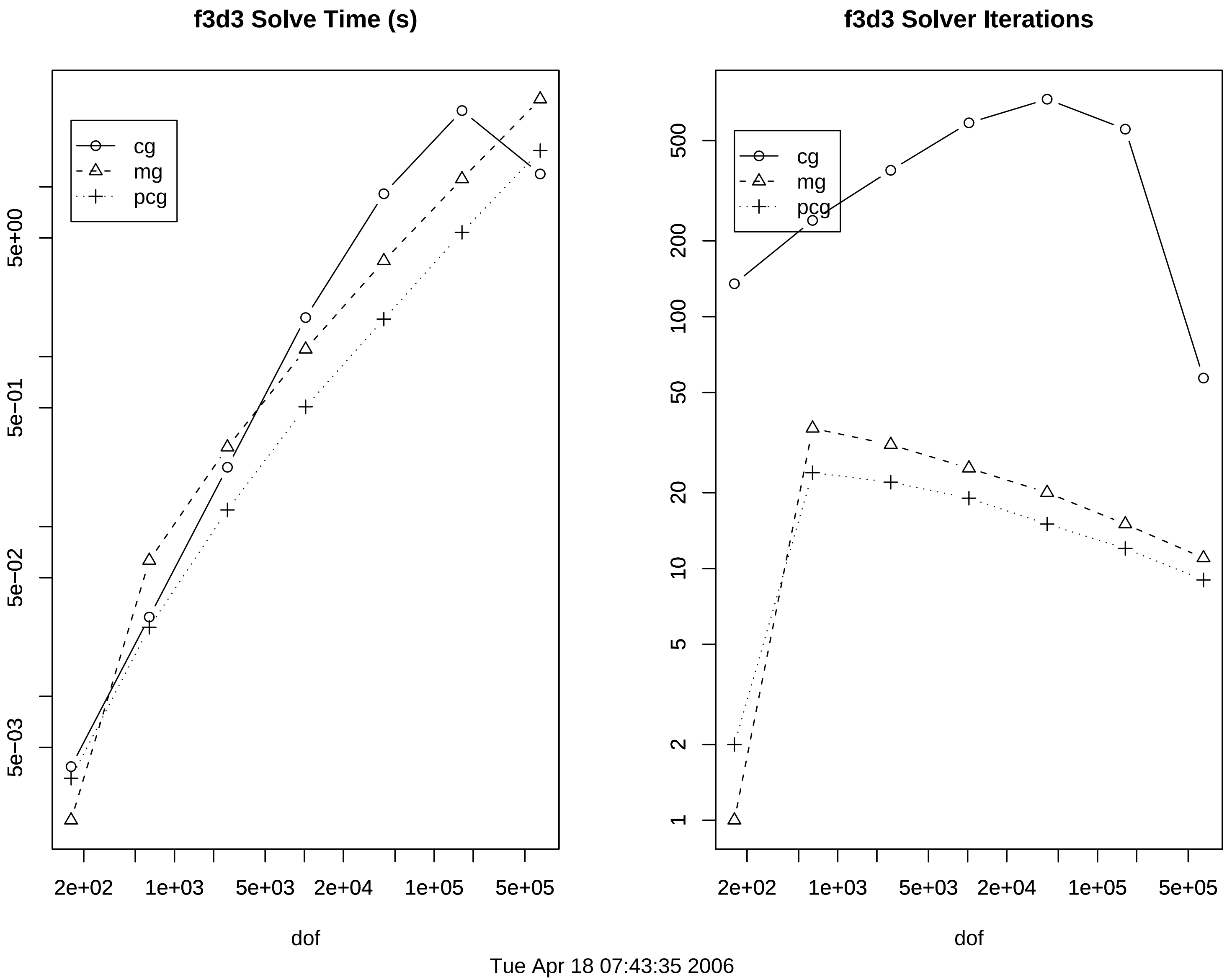
Solver Performance - f3d1



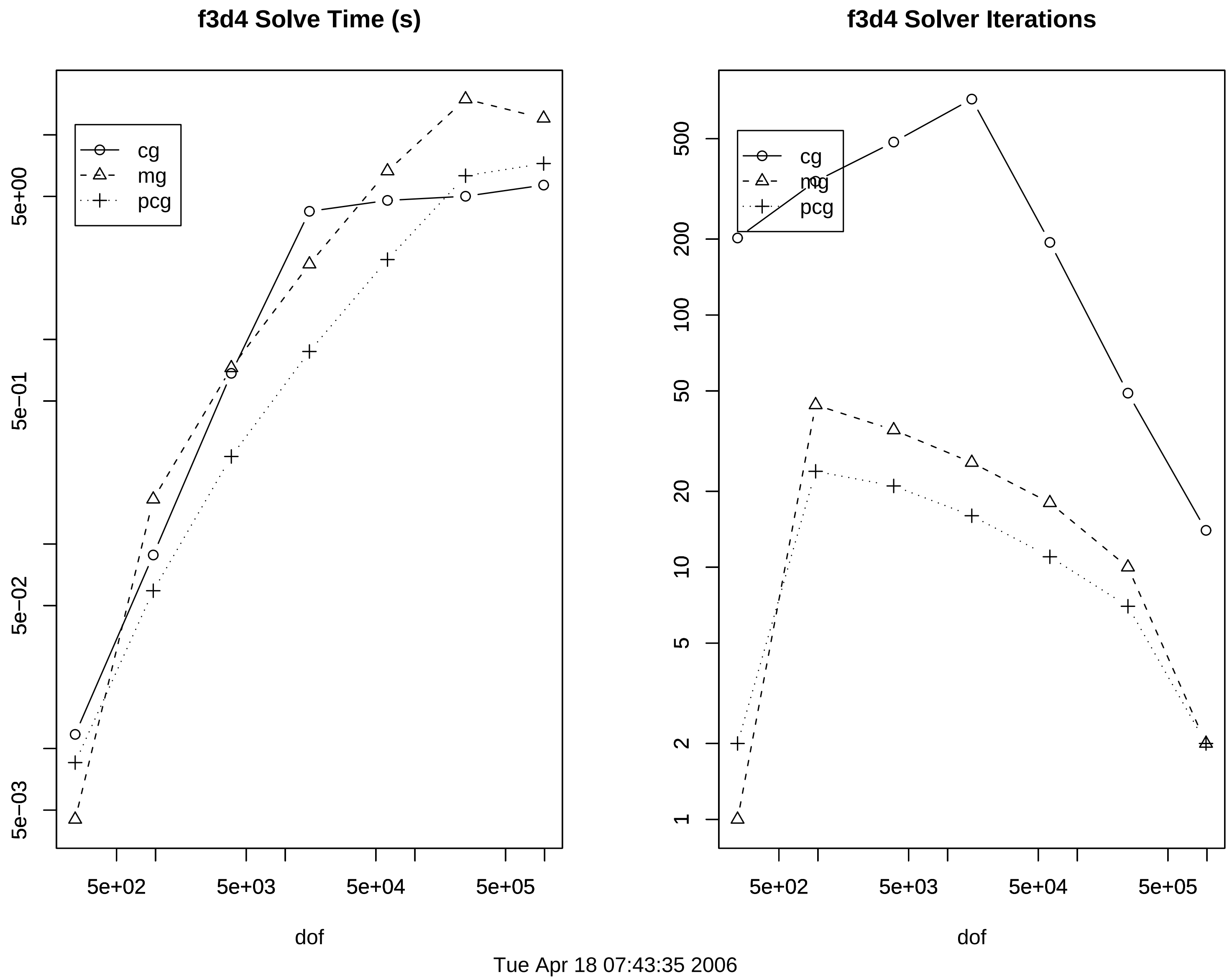
Solver Performance - f3d2



Solver Performance - f3d3



Solver Performance - f3d4



Adaptivity

- Uniform refinement is overkill for some problems. The idea of adaptive methods is to utilize some sort of *estimator* to selectively choose specific elements to refine.
- *Residual* based estimators utilize the previously obtained solution to identify candidates for refinement and coarsening.
- *Local Problem* based estimators solve local problems usually consisting of each element and its immediate neighbors to identify candidates for refinement and coarsening.
- An *Adaptive Iterations* consist of Solve-Estimate-Mark-Refine-Coarsen sequence.
- Adaptive iterations terminate when the desired tolerance is achieved.

Element Refinement

- DG allows a triangle to undergo *regular* refinement, i.e., each triangle is divided into four new triangles, each similar to its parent.
- We impose at most one hanging node per edge.
- SG doesn't allow hanging nodes to be present.
- DG refinement allows one to maintain area and normal orientation for the inital mesh triangles only; these quantities can be scaled appropiately for higher level (smaller) elements.
- Coarsening only occurs when all four children of a triangle are marked for coarsening.

A Posteriori Error Estimation

The following theorems stated without proof (see Karakashian and Pascal,2004) provide information on residual based a posteriori estimators used to aid in the determination of whether to refine or coarsen individual elements.

Theorem. *Let* $e = u - u_h^\gamma$. *Then*

$$\sum_{K\in\mathcal{T}_h} \|\nabla e\|_K^2 \le c\Big(\sum_{K\in\mathcal{T}_h} h_K^2 \|f + \Delta u_h^\gamma\|_K^2 + \sum_{e\in\mathcal{E}^I} h_e |[\partial_n u_h^\gamma]|_e^2 + \sum_{e\in\mathcal{E}_N^B} h_e |g_N - \partial_n u_h^\gamma|_e^2 + \gamma^2 \sum_{e\in\mathcal{E}^I} h_e^{-1} |[u_h^\gamma]|_e^2 + \gamma^2 \sum_{e\in\mathcal{E}_D^B} h_e^{-1} |g_D - u_h^\gamma|_e^2 \Big)$$

A Posteriori Error Est., contd

Theorem. *Suppose f is a piecewise polynomial on $\mathcal{T}_h$. Then*

- $\forall K \in \mathcal{T}_h$

$$h_K^2 \|f + \Delta u_h^\gamma\|_K^2 \le c\|\nabla e\|_K^2$$

- $\forall e = K^+ \cap K^- \in \mathcal{E}^I$

$$h_e |[\partial_n u_h^\gamma]|_e^2 \le c \left(\|\nabla e\|_{K^+}^2 + \|\nabla e\|_{K^-}^2 \right)$$

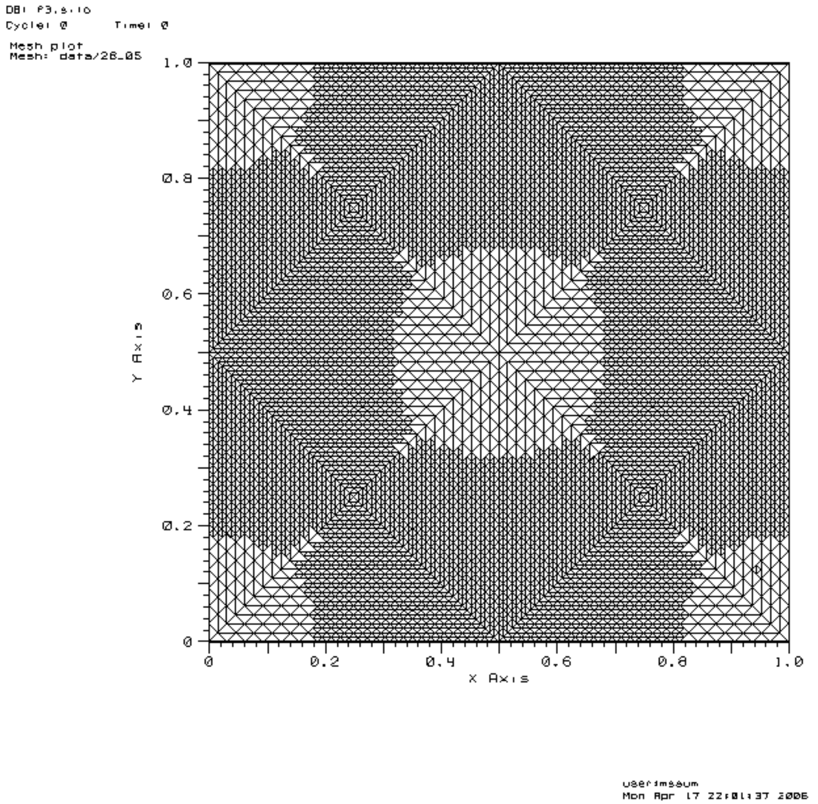
- $\forall e = K^+ \cap \partial\Omega \in \mathcal{E}_N^B$

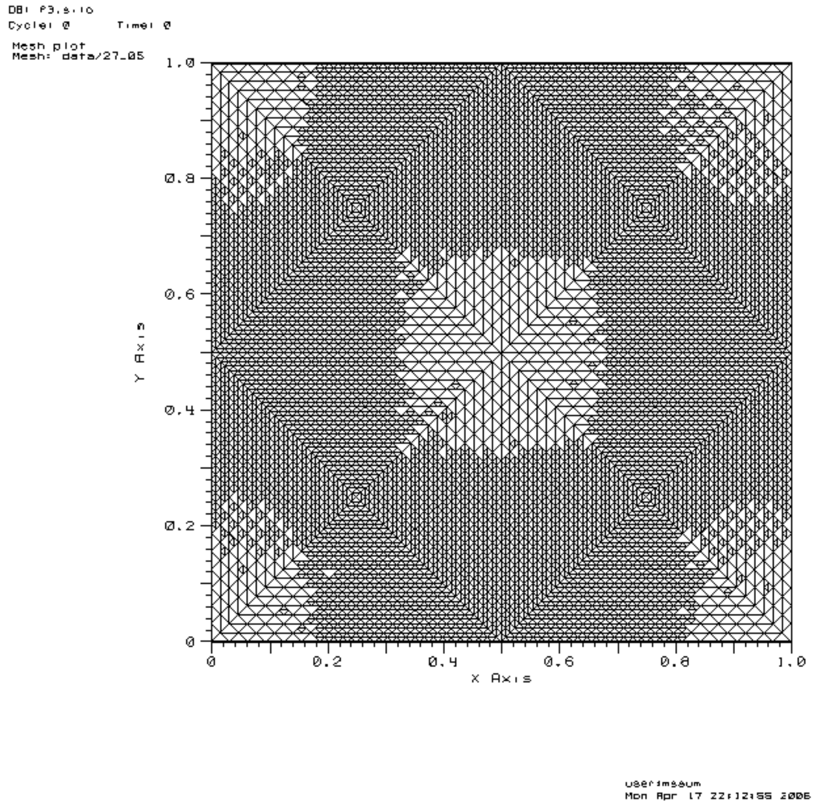
$$h_e |g_N - \partial_n u_h^\gamma|_e^2 \le c\|\nabla e\|_{K^+}^2$$

- *for γ large enough*

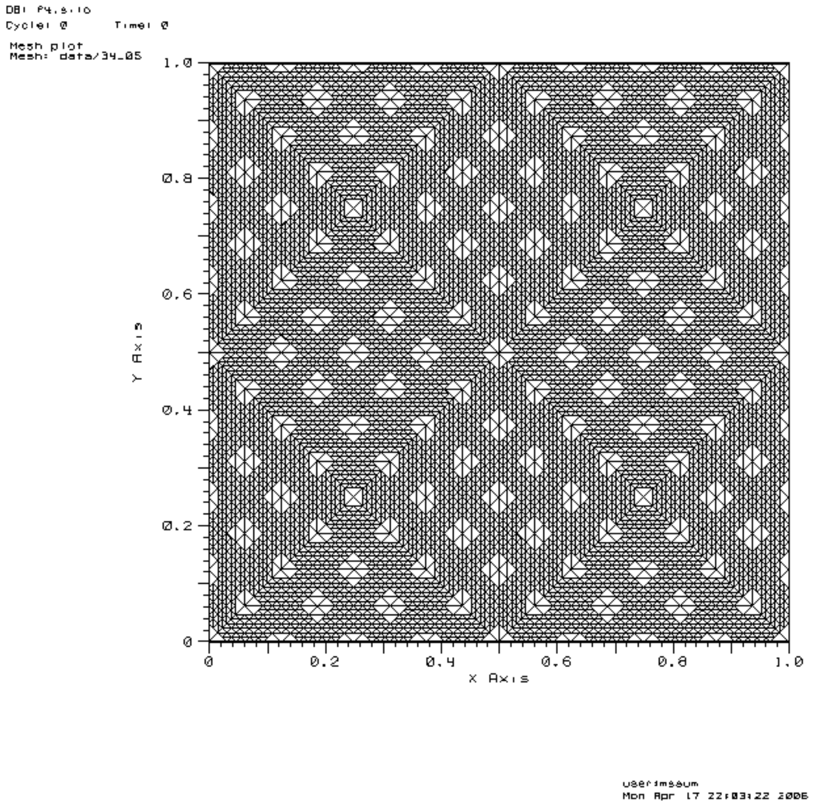
$$\gamma^2 \sum_{e \in \mathcal{E}^I} h_e^{-1} |[u_h^\gamma]|_e^2 + \gamma^2 \sum_{e \in \mathcal{E}_D^B} h_e^{-1} |[g_D - u_h^\gamma]|_e^2 \le c \sum_{K \in \mathcal{T}_h} \|\nabla e\|_K^2$$

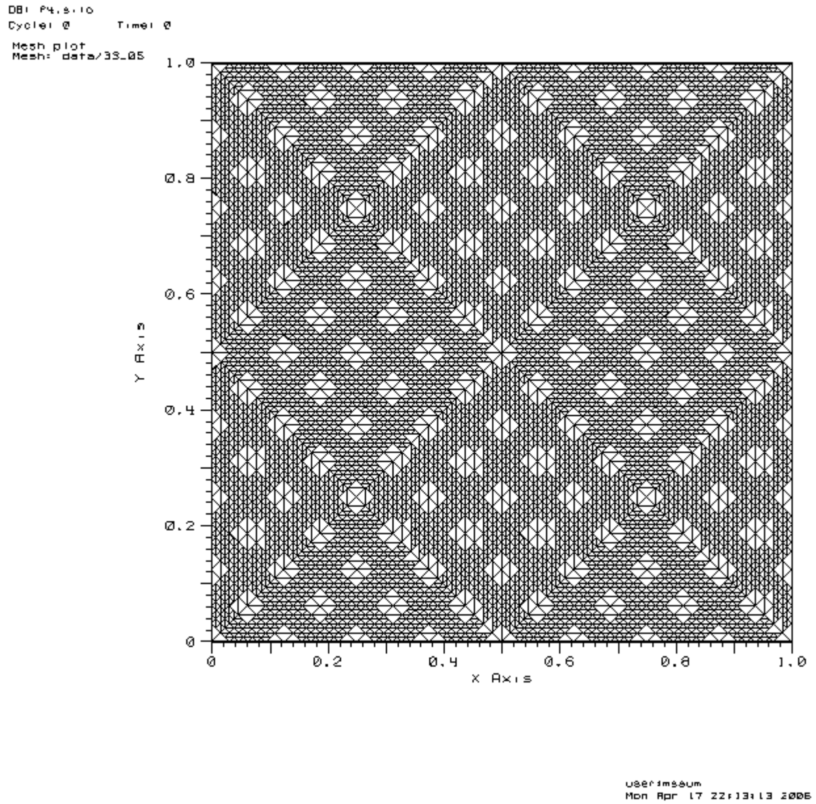
f3 Adaptive Meshes



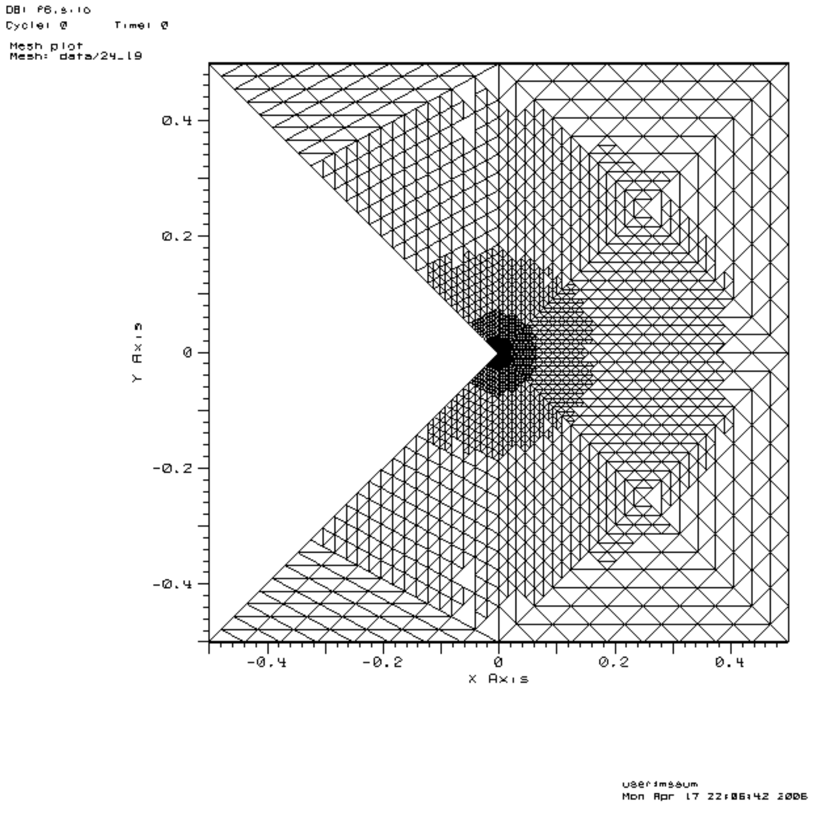


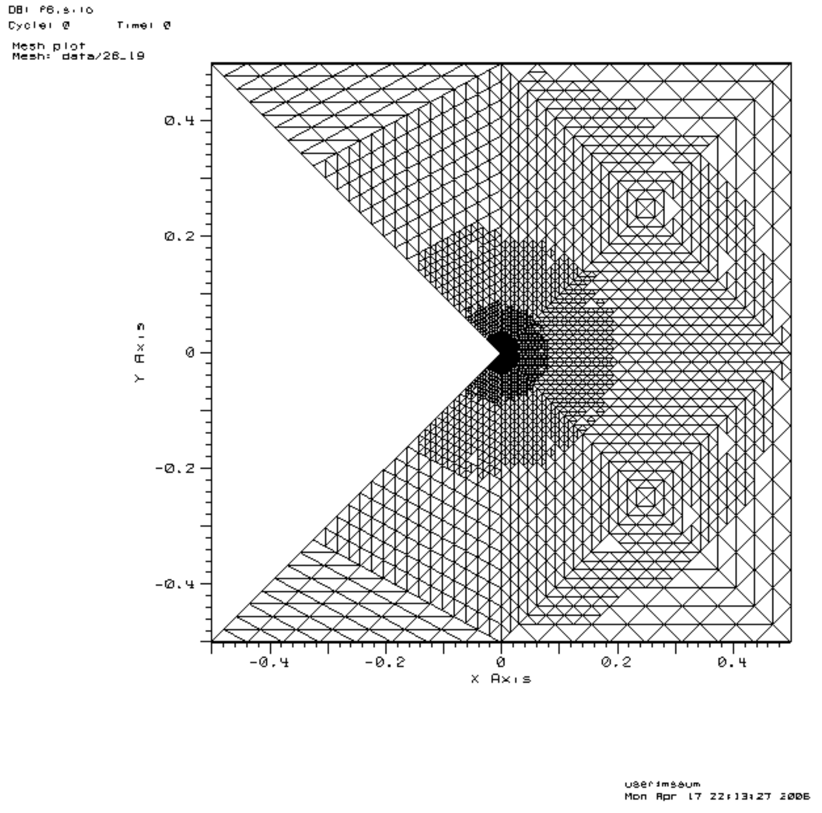
f4 Adaptive Meshes



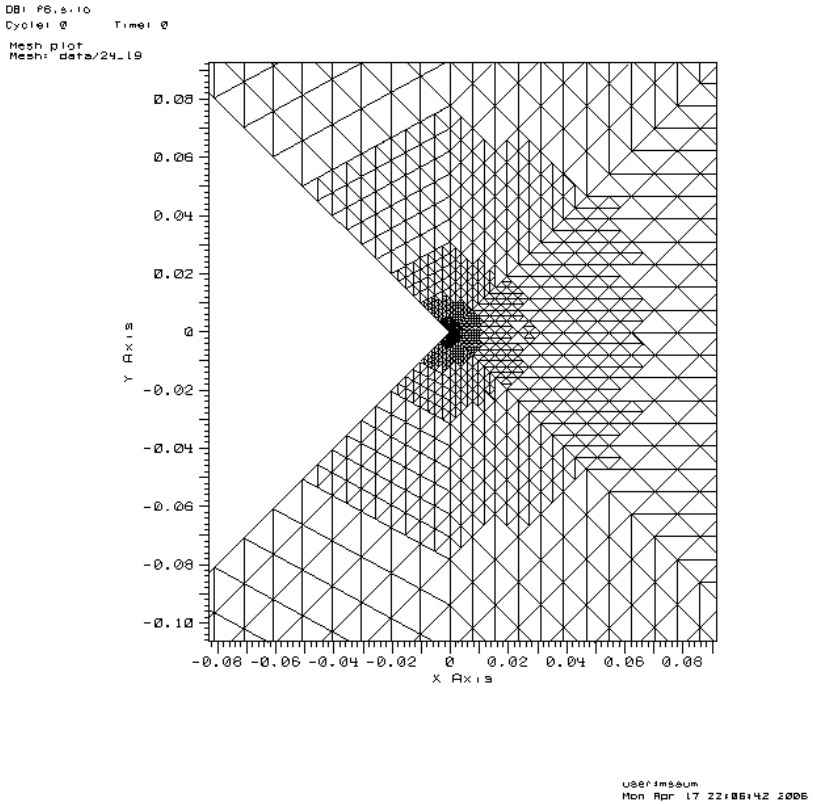


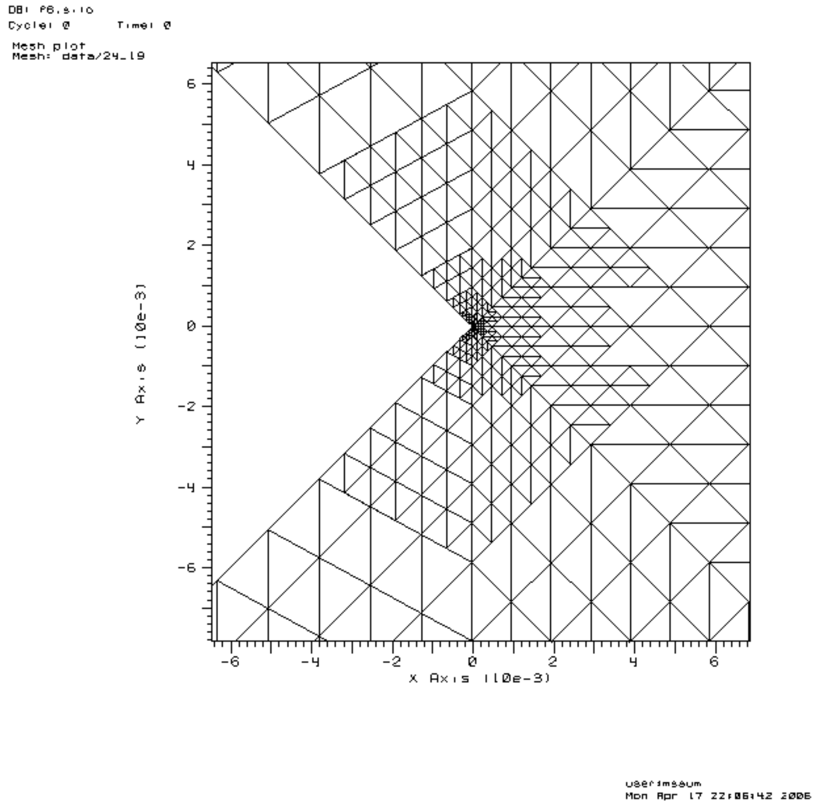
f6 Adaptive Meshes



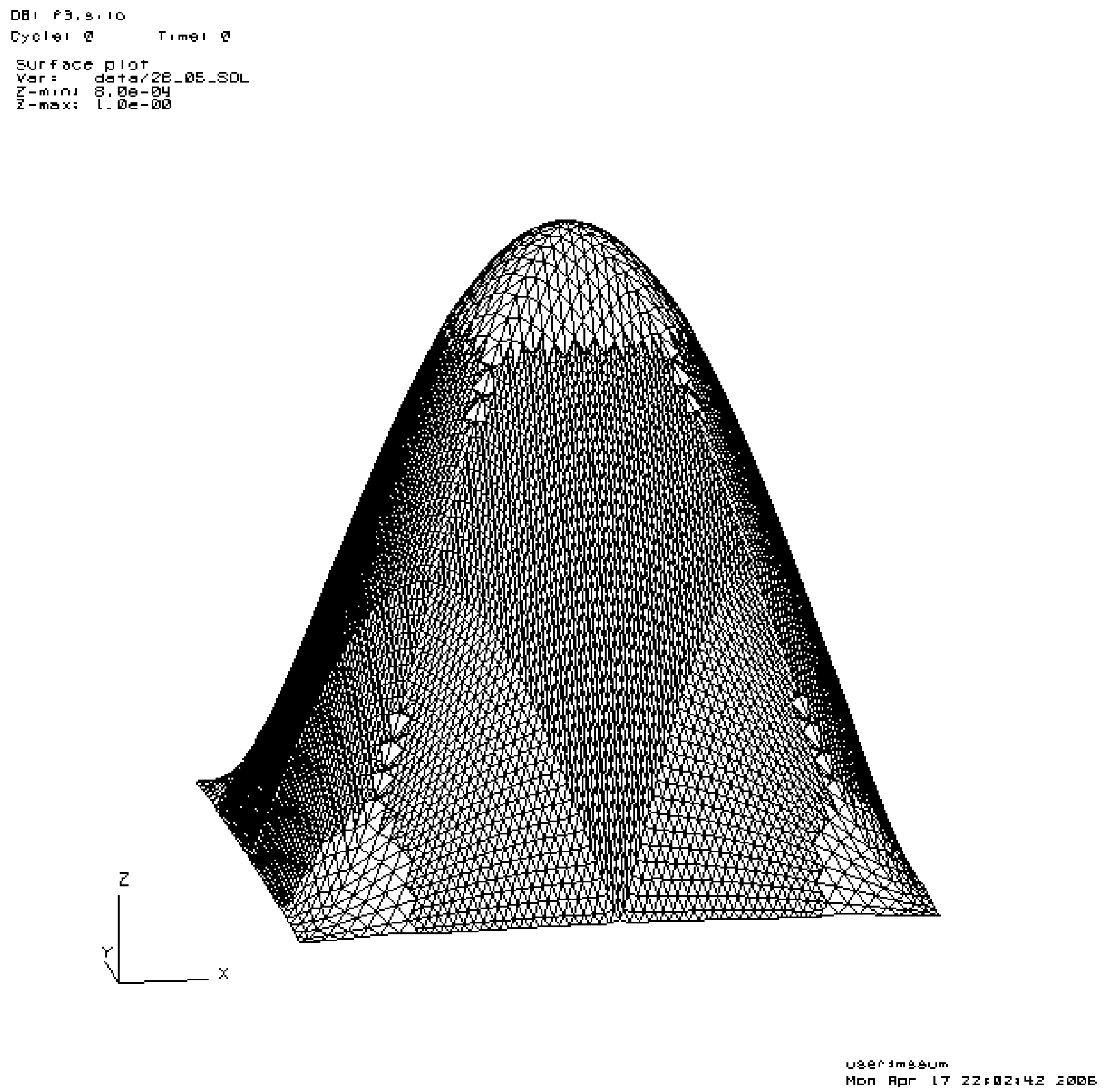


f6 Adaptive Meshes - Zoom

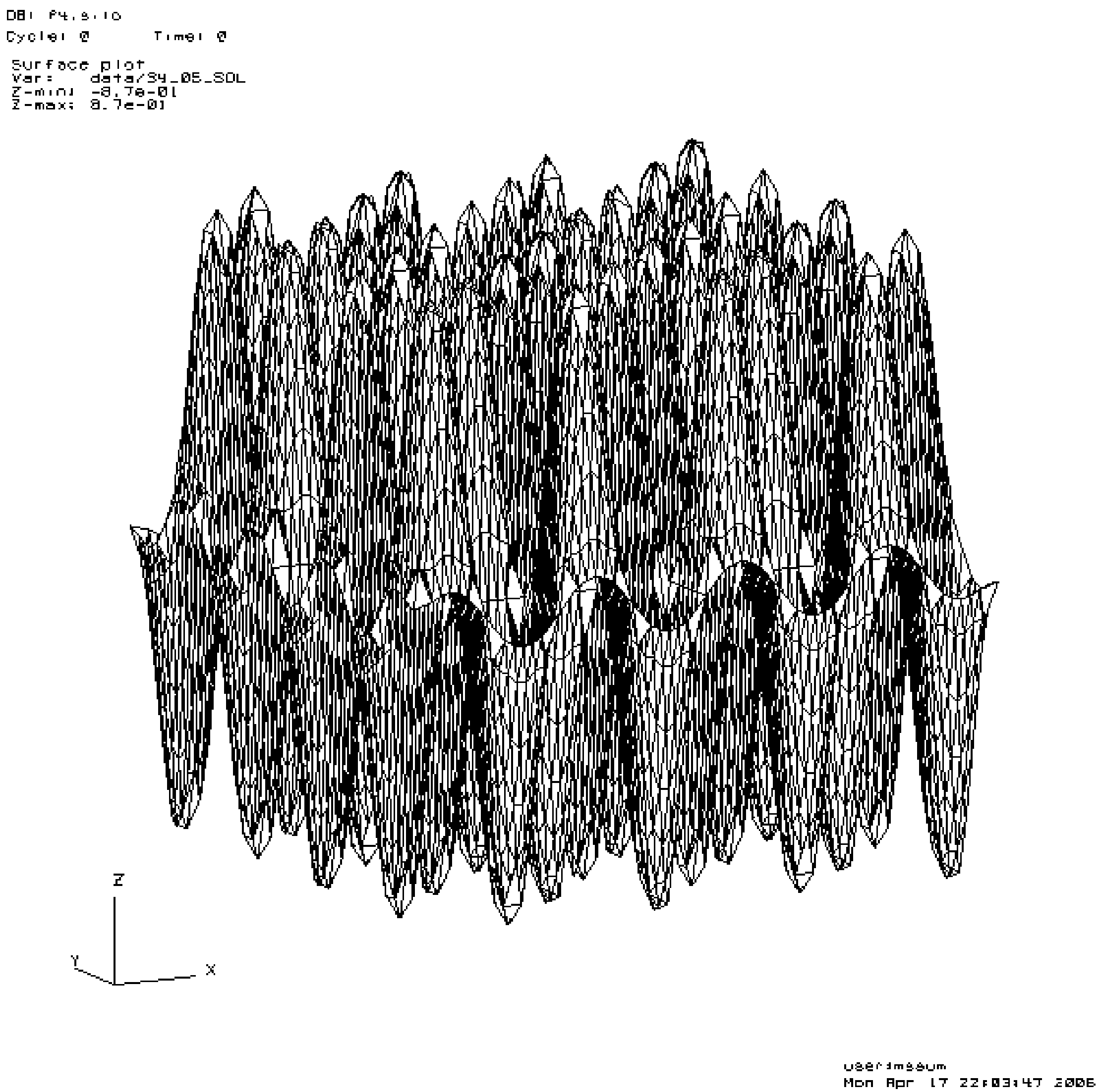




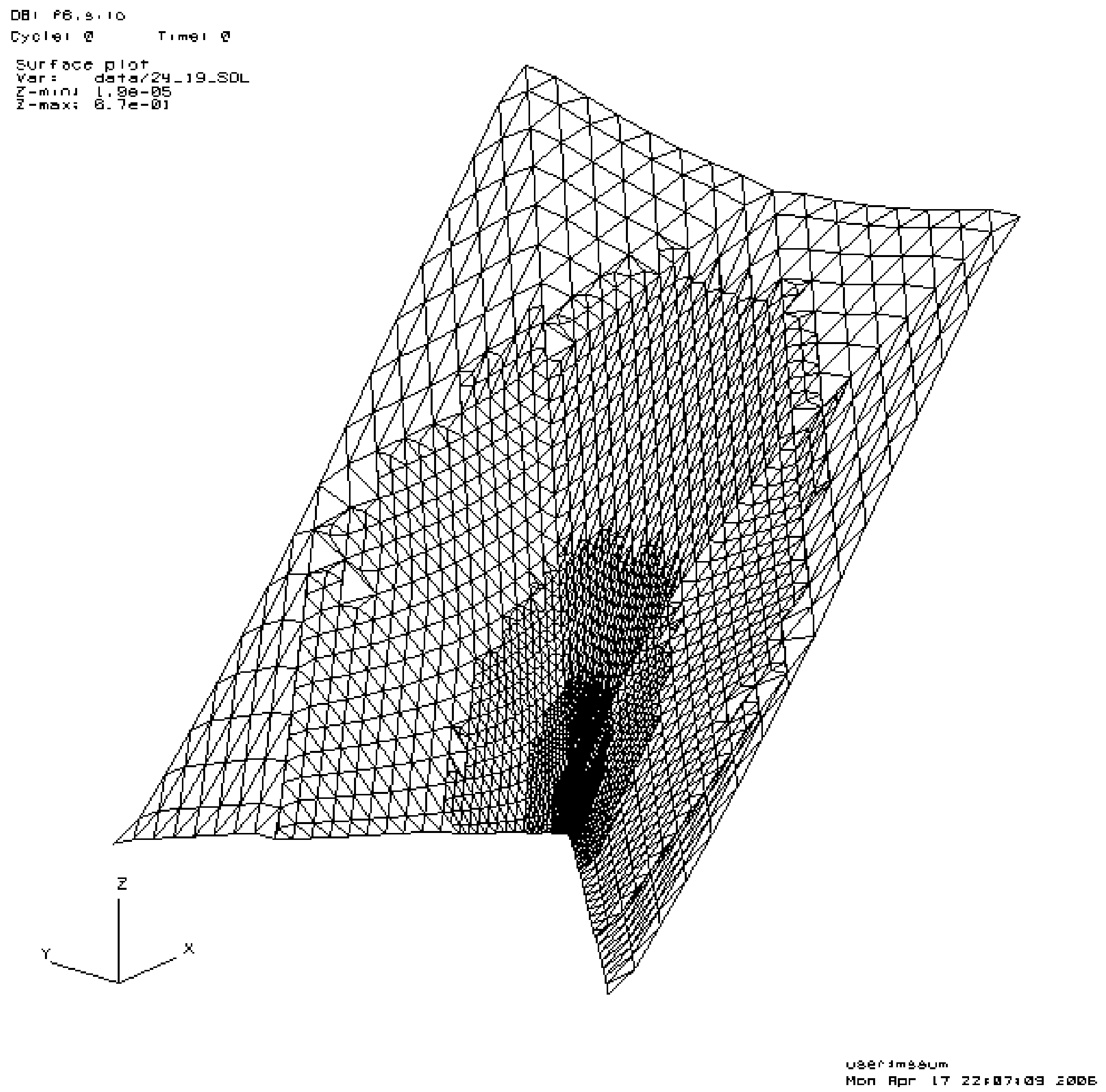
f3 Adaptive Solution



f4 Adaptive Solution



f6 Adaptive Solution



The Biharmonic Model Problem

Let $\Omega \subset \mathbb{R}^d, d = 2, 3$ be a bounded open polyhedral domain with Lipshitz continuous boundary.

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega \\ u = g_D & \text{on } \Gamma \\ \nabla u \cdot n = g_N & \text{on } \Gamma \end{cases}$$

where $\partial\Omega := \Gamma$ and n is the unit normal vector exterior to Ω. We also assume that $\mu_{d-1}(\Gamma) > 0$, $f \in L^2(\Omega)$, $g_N \in L^2(\Gamma)$.

We have created an SIPG implementation for this problem (a different bilinear form). The following solution was obtained using uniform refinement with $r = 5$ for $g_D = g_N = 0$ with exact solution: $u = (1 - \cos(2\pi x))(1 - \cos(2\pi y))$

Biharmonic Computed Solution