## *A quick introduction to Dirichlet domains*

Let's consider the familiar covering map from the Euclidean plane $\mathbb{R}^2$ to the torus $S^1 \times S^1$:

$$f : (x, y) \mapsto (e^{2\pi i x}, e^{2\pi i y})$$

which wraps each horizontal grid line around the blue curve on the torus and which wraps each vertical grid line around the purple curve on the torus. The preimage of the point of intersection of the blue and purple curves is the set of integer lattice points on the plane, and the group of covering transformations consists of the Euclidean translations

$$\{\tau_{m,n} : (x, y) \mapsto (x+m, y+n) \ (m, n \in \mathbb{Z})\} .$$

To construct a Dirichlet domain for the torus, we first pick a point $x_0$ in the plane, say the central navy blue point illustrated. Then, for each of its translates $\tau_{m,n}(x_0)$, we draw the perpendicular bisector $\ell$ of the line segment joining $x_0$ with its translate, and discard the complementary (open) half-plane of $\ell$ not containing $x_0$. It's easily seen that after this process we're left with the yellow square region; this is the Dirichlet domain with basepoint $x_0$. Analogously we could start with the universal covering map of the 3-torus, $f : \mathbb{R}^3 \to S^1 \times S^1 \times S^1$, $f(x, y, z) = (e^{2\pi i x}, e^{2\pi i y}, e^{2\pi i z})$, and obtain a Dirichlet domain that's a unit cube in $\mathbb{R}^3$.

Note that it didn't matter where we chose the basepoint $x_0$; we'd always get a Dirichlet domain of the same shape. The smallest distance from the basepoint $x_0$ to any of its translates (namely $1$ in each of the above cases) is twice the so-called injectivity radius of the base manifold at $f(x_0)$. The injectivity radius is related to the question of how powerful a telescope is needed for someone to see the back of his or her head: see Jeff Weeks's article "Detecting topology in a nearly flat hyperbolic universe" at http://arxiv.org/abs/astro-ph/0212006.

A precisely analogous procedure can be carried out for a hyperbolic 3-manifold $M$, with universal covering map $f : \mathbb{H}^3 \to M$. We select a basepoint

$x_0 \in \mathbb{H}^3$, take the geodesic segment $\gamma$ joining $x_0$ with one of its translates, and then chop out one of the two complementary half-spaces of the bisecting geodesic plane of $\gamma$, specifically the half-space not containing $x_0$. After doing this for all translates of $x_0$, we're left with the Dirichlet domain for $M$ with basepoint $f(x_0)$.

For a hyperbolic 3-manifold The Dirichlet domain is a polyhedron whose faces are parts of hyperbolic planes. A convenient way of picturing such a polyhedron is to view it in the Klein or projective model, since hyperbolic lines (resp. planes) are represented in that model by Euclidean lines (resp. planes). Angles and distances are usually horribly distorted in the Klein model, though we can reduce the distortion by placing the centroid of the Dirichlet domain at the origin of the 3-ball modelling $\mathbb{H}^3$.

An interesting feature of hyperbolic manifolds is that the injectivity radius is not constant; it varies from place to place. Almost always the "nicest" Dirichlet domain, *i.e.* one displaying symmetries of the manifold (should it have any), is obtained by placing $x_0$ at a point where the injectivity radius achieves a local or global maximum; SnapPea has a function which finds such a point. A notable exception to this is the manifold Vol3 (so-called because it has the third-smallest known volume for closed hyperbolic 3-manifolds); Vol3 has a symmetry group of order $16$, yet the Dirichlet domain obtained by taking a point of maximum injectivity radius displays no symmetry whatever! However, close examination of these symmetries shows that there are two points in the manifold that are fixed by eight of them, and either of these yields the suitably symmetric polyhedron displayed on the main page.

For a cusped manifold, for example the complement of a hyperbolic link in the 3-sphere, some of the vertices of the Dirichlet domain will be "ideal points", situated on the boundary of $\mathbb{H}^3$. In the context of the Klein model, this just means that they will be at distance 1 from the origin.