KEY

MATH 4023 **Name** KEY

Exam 2 **July 12, 2010** **Score:** ____________

Instructions: This is a closed book, closed notes exam. Please read all instructions carefully and complete all problems. Be sure to show your work in order to receive full credit, an answer with no supporting work will receive no credit.

1. (a) Let G be a group. If $a \in G$, then denote the set of all elements of G that commute with a by $C(a)$. That is $C(a) = \{g \in G : ga = ag\}$. Prove that $C(a)$ is a subgroup of G.

If $a = e$, then $C(a) = G$, a subgroup
If $a \neq e$, then $C(a) \neq \phi$.
Let $a, b \in C(a)$. Then we claim
$ab^{-1} \in C(a)$, i.e. $ab^{-1}g = gab^{-1}$
But $ab^{-1}g = ab^{-1}(g^{-1})^{-1} = a(g^{-1}b)^{-1}$
$= a(bg^{-1})^{-1}$
$= agb^{-1}$
$= gab^{-1}$.

Or, • $e \in C(a)$.
• If $k, h \in C(a)$, then
$khg = kgh = gkh, \forall g$
$\Longrightarrow kh \in C(a)$
• Let $h \in C(a)$. Then $h^{-1} \in C(a)$
Indeed, $h^{-1}g = h^{-1}(g^{-1})^{-1} = (g^{-1}h)^{-1}$
$= (hg^{-1})^{-1}$
$= gh^{-1}$.

(b) Recall $S_3 = \{\epsilon, \tau, \sigma, \sigma^2, \tau\sigma, \sigma\tau\}$, the symmetric group, where $\tau = (12)$ and $\sigma = (132)$. Find $C(\tau\sigma)$.

After some computation you get

$C(\tau\sigma) = \{\epsilon, \tau\sigma\}$.

(c) For the subgroup $C(\tau\sigma)$ in part b) find all of its distinct left cosets in S_3.

$C(\tau\sigma) = \{\epsilon, \tau\sigma\}$
$\tau C(\tau\sigma) = \{\tau, \sigma\}$
$\sigma^2 C(\tau\sigma) = \{\sigma^2, \sigma^2\tau\sigma\} = \{\sigma^2, \sigma\tau\}$.

(d) What is the index of $C(\tau\sigma)$ in S_3?

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2. (a) State Lagrange's Theorem. Make sure you clearly state the hypothesis and the conclusion.

If G is a finite group & H is a subgroup of G, then $|G| = [G:H]\,|H|$.

(b) If G is a group of order 54, what are the possible orders of elements of G?

Possible order of elements of G are divisors of 54

$1, 2, 3, 6, 9, 18, 27, 54$

(c) Suppose that G is a group of order 4. Show that either G is cyclic or every element a of G satisfies $a^2 = e$.

Since $|G| = 4$, the order of any element $a \in G$ is either $1, 2,$ or 4.

If $\exists\, a \in G$, $o(a) = 4$, then $G = \langle a \rangle$, cyclic.

If not, all elements of are of order 1 or 2. In either case $a^2 = e$.

3. (a) Use part c) of problem 2 to prove that any group of order 4 is necessarily Abelian (commutative).

Any group of order 4 is either cyclic or $a^2 = e \quad \forall a \in G$.

If cyclic, then clearly abelian.

If not, we take $a, b \in G$, and show $(ab) = (ba)$.

Since $a^2 = e$, then $a^{-1} = a$.

and $(ab)^2 = e \Rightarrow (ab)^{-1} = ab$

$\Leftrightarrow b^{-1}a^{-1} = ab$

$\Leftrightarrow ba = ab$. Since $b^{-1} = b$ & $a^{-1} = a$.

(b) Let G be the set $\{e, a, b, c\}$. Complete the following multiplication table for G so that G with the operation $*$ is a group.

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

Here because of Cancellation Law in every row no element appears more than once.

(c) For the group in part b) what are the orders of a, b and c?

$a^2 = e, \quad o(a) = 2$

$b^2 = a$, and $(b^2)^2 = a^2 = e \Rightarrow o(b) = 4$

$c^2 = a, \quad (c^2)^2 = e \Rightarrow o(c) = 4$.

(d) Is the group in part b) cyclic? If yes, find a generator. If no, explain why.

Yes, G is cyclic, $G = \langle b \rangle = \langle c \rangle$.

4. Consider the ring $(\mathbb{Z}_9, +_9, \cdot_n)$, where $\mathbb{Z}_9 = \{\overline{0}, \overline{1}, \cdots, \overline{8}\}$ and the binary operations are the addition and multiplication mod 9.

(a) What is the additive inverse of $\overline{7}$?

$\overline{2}$, since $\overline{2} + \overline{7} = \overline{0}$

(b) Find all the units (*elements with multiplicative inverse*) of the ring. Denote the set of all units by $\mathbb{Z}_9^*$.

$\overline{1}, \overline{2}, \overline{4}, \overline{5}, \overline{7}, \overline{8}$.

$\overline{1}^{-1} = 1$

$\overline{2}^{-1} = \overline{5}$

$\overline{4}^{-1} = 7$

$\overline{8}^{-1} = \overline{8}$

(c) Construct a multiplication table for $(\mathbb{Z}_9^*, \cdot_9)$.

	$\overline{1}$	$\overline{2}$	$\overline{4}$	$\overline{8}$
$\overline{1}$	$\overline{1}$	$\overline{2}$		
$\overline{2}$	$\overline{2}$			
$\overline{4}$			$\overline{7}$	$\overline{2}$
$\overline{5}$			$\overline{2}$	

(d) Show that $(\mathbb{Z}_9^*, \cdot_9)$ is a group.

- $(\mathbb{Z}_9, \cdot_9)$ is semigroup
- $\overline{1}$ is the identity
- Inverse are given in (b).

(e) Is $(\mathbb{Z}_9, +_9, \cdot_9)$ a field? Explain?

No! Not all elements of $\mathbb{Z}_9$ are units.

$\overline{3}$ is a zero divisor

5. Consider the finite-state machine given by the table

	Next State		Output
Present State	0	1	
σ_0	σ_3	σ_0	0
σ_1	σ_2	σ_2	1
σ_2	σ_0	σ_3	1
σ_3	σ_0	σ_3	0

(a) Find Π_0, Π_1.

$$\Pi_0 = \{ \{\sigma_1, \sigma_2\}, \{\sigma_0, \sigma_3\} \}$$ since $\omega(\sigma_1) = 1$, $\omega(\sigma_2) = 1$, $\omega(\sigma_0) = 0$, $\omega(\sigma_3) \Rightarrow$

To find Π_1, we evaluate & check

$\tau(\sigma_1, 0) = \sigma_2$ $\tau(\sigma_1, 1) = \sigma_2$ | $\tau(\sigma_0, 0) = \sigma_3$ $\tau(\sigma_0, 1) = \sigma_0$

$\tau(\sigma_2, 0) = \sigma_0$ $\underbrace{\tau(\sigma_2, 1) = \sigma_3}_{\text{not in } \Pi_0}$ | $\tau(\sigma_3, 0) = \sigma_0$ $\tau(\sigma_3, 1) = \sigma_3$

$\in \Pi_0$.

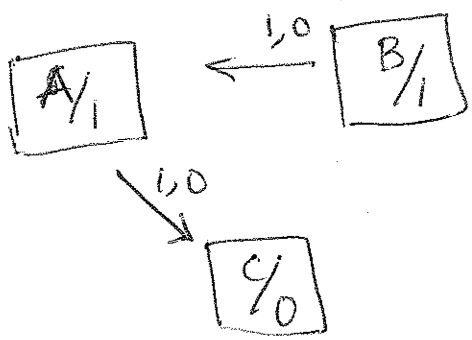
$$\Pi_1 = \{ \{\sigma_1\}, \{\sigma_2\}, \{\sigma_0, \sigma_3\} \}.$$

(b) Find the minimal quotient machine that is equivalent to the above one.

Check also that $\Pi_2 = \Pi_1$ & so

$$\Pi = \Pi_1$$

The minimal machin is



$A = \{\sigma_2\}$

$B = \{\sigma_1\}$

$C = \{\sigma_0, \sigma_3\}$