Exercise

\[
\max_u \int_0^1 (x + u) \, dt
\]

\[x' = 1 - u^2, \quad x(0) = 1, \quad u \text{ control.} \quad H = \]

\[
\frac{\partial H}{\partial u} = 0
\]

\[
\lambda' = \quad \lambda =
\]

\[
\lambda(1) =
\]

\[u^* = \quad x^* =\]
Example

\[
\max_u \int_0^1 (x + u) \, dt
\]

\[x' = 1 - u^2, \quad x(0) = 1 \quad u \text{ control}\]

\[H(t, x, u, \lambda) = x + u + \lambda(1 - u^2)\]

\[
\frac{\partial H}{\partial u} = 1 - 2\lambda u = 0 \quad \Rightarrow \quad u = \frac{1}{2\lambda} \quad \& \quad H_{uu} = -2\lambda \leq 0,
\]

\[\lambda' = -\frac{\partial H}{\partial x} = -1, \quad \lambda(1) = 0, \quad \Rightarrow \quad \lambda = 1 - t\]

\[x' = 1 - u^2 = 1 - \frac{1}{4(1 - t)^2}\]

\[x^*(t) = t - 1/4(1 - t) + 5/4, \quad u^*(t) = 1/2(1 - t).\]
\[ \int_{0}^{1} [x^*(t) + u^*(t)] \, dt = \infty \]

There is not an "Optimal Control" in this case.

Want finite maximum.

Here unbounded optimal state unbounded OC.
\[
\max \phi(x(T)) + \int_0^T f(t, x(t), u(t)) \, dt
\]

\[
x' = g(t, x, u) \quad x(0) = x_0
\]

where \( \phi(x(T)) \) is final payoff. What change results?

\[
J(a) = \int_0^T f(t, y(t, a), u^* + ah) \, dt + \phi(y(T, a))
\]

\[
\frac{\partial J}{\partial a}(0) = 0 = \int_0^T \ldots \, dt \text{ same as before}
\]

\[
- \lambda(T) \frac{\partial y}{\partial a}(T, 0) + \phi'(x^*(T)) \frac{\partial y}{\partial a}(T, 0)
\]
Only change

\[ \lambda(T) = \phi'(x^*(T)) \]

Example

\[ \max 5x^2(T) + \int_0^T f(t, x, u) \, dt \]

\[ \phi(x) = 5x^2 \quad \phi' = 10x \]

\[ \lambda(T) = 10x^*(T) \]
**Example**

\[ x(t) \] Number of cancer cells at time \( t \) \( \text{ (exponential growth) } \)

**State**

\[ u(t) \] Drug concentration \textbf{Control}

\[
\frac{dx}{dt} = \alpha x(t) - u(t)
\]

\[ x(0) = x_0 \quad \text{known initial data} \]

\[
\min x(t) + \int_0^T u^2(t) \, dt
\]

where the first term is number of cancer cells at final time \( T \) and the second term is the harmful effects of drug on body.
\[ H = u^2 + \lambda (a - u) \]

\[
\frac{\partial}{\partial u} \frac{H}{u} = 2u - \lambda = 0 \quad \text{at} \quad u^* \implies u^* = \frac{\lambda}{2}
\]

\[ \lambda' = -\frac{\partial H}{\partial x} = -a\lambda \implies \lambda = \lambda_0 e^{-at} \]

\[ \lambda(T') = 1 \quad \text{transversality condition} \]

\[ \phi(x) = x, \quad \phi'(x) = 1 \]

\[ x(T') + \int_0^T u^2(t) \, dt \quad \text{here} \quad \phi(x) = x. \]
\[ \lambda = \lambda_0 e^{-at}, \quad \lambda(T) = 1 \Rightarrow \lambda_0 = e^{aT} \]

\[ \lambda = e^{-a(t-T)} \]

\[ x' = \mathbf{a} - u = ax - \frac{e^{-a(t-T)}}{2} \]

\[ x' - ax = -\frac{e^{-a(t-T)}}{2} \]

\[ (e^{-at}x)' = -\frac{e^{-2at}e^{aT}}{2} \]

\[ x^*(t) = e^{at}x_0 + e^{aT}\frac{(e^{-at} - e^{at})}{4a} \]
Well Stirred Bioreactor

Contaminant and bacteria present in spatially uniform time varying concentrations

\[ z(t) = \text{concentration of contaminant} \]
\[ x(t) = \text{concentration of bacteria} \]

bioreactor rich in all nutrients except one

\[ u(t) = \text{concentration of input nutrient} \]

bacteria degrades contaminant via co-metabolism.
\[ x'(t) = G(u)x(t) - D(x(t))^2 \quad \text{where} \quad G(u) = \frac{Gu}{H + u} \]

\[ z'(t) = -K z(t) x(t) \]

where \( u(t) \) is control and \( x(0), z(0) \) are known.

Objective functional:

\[ J(u) = \int_0^T (K x(t) - u(t)) \, dt \]

Find \( u^* \) to maximize \( J \)

\[ J(u^*) = \max J(u) \]

maximize bacteria and minimize input nutrient cost.
\[ z(t) = z_0 \exp \left( - \int_0^t K x(s) \, ds \right) \]

\[ \int_0^t K x(s) \, ds = - \ln \left( \frac{z(T)}{z_0} \right) \]

\( J(u) \) penalizes large values of \( z \) at final time \( T \).

Can eliminate \( z \) variable and work with \( x(t) \) only.
\[ H = Kx - u + \lambda \left( \frac{Gux}{H + u} - Dx^2 \right) \]

\[ \frac{\partial H}{\partial u} = -1 + \lambda x \frac{\partial}{\partial u} \left( \frac{Gux}{H + u} \right) = 0 \quad \text{at } u^* \]

\[-1 + \lambda x \frac{GH}{(H + u)^2} = 0 \quad \Rightarrow \lambda x GH = (H + u)^2 \]

\[(\lambda x GH)^{1/2} = H + u\]

\[ u^* = (\lambda x GH)^{1/2} - H \]
\[
\lambda' = -\frac{\partial H}{\partial x} = - \left[ \lambda \left( \frac{G_u}{H + u} - 2Dx \right) + K \right]
\]

\[
\lambda(T) = 0
\]

\[
\lambda' = - \left[ \lambda \left( \frac{G \left\{ (\lambda x GH)^{1/2} - H \right\}}{H + \{ (\lambda x GH)^{1/2} - H \}} - 2Dx \right) + K \right]
\]

\[
x' = \frac{G \left\{ (\lambda x GH)^{1/2} - H \right\}}{H + (\lambda x GH)^{1/2} - H} - Dx^2
\]

\[
x(0) = x_0 \quad \text{known}.
\]

Solve for \(x, \lambda\) numerically.
Problems

\[ u^* = (\lambda x GH)^{1/2} - H \]

What if:

\[ (\lambda x GH)^{1/2} = 0? \]
\[ \lambda x GH \leq 0? \]
\[ (\lambda x GH)^{1/2} - H < 0? \]

Need additional constraint

\[ 0 \leq u(t) \leq M. \]
Fishery Model

\[ x' = Kx(M - x) - ux \]

\[ x(t) \quad \text{population level of fish} \]

\[ u(t) \quad \text{harvesting control} \]

Maximizing net profit:

\[ \int_0^T e^{-\delta t} \left( p_1ux - p_2(ux)^2 - c_1u \right) \, dt \]

where \( e^{-\delta t} \) is discount factor, \( p_1, p_2, c_1 \) terms represent profit from sale of fish, diminishing returns when there is a large amount of fish to sell and cost of fishing. \( M, p_1, p_2, c_1 \) are positive constants.
\[ H = e^{-\delta t} \left( p_1 u x - p_2 (u x)^2 - c_1 u \right) + \lambda (K x (M - x) - u x) \]

\[ \lambda' = -\frac{\partial H}{\partial x} = - \left[ e^{-\delta t} (p_1 u - 2 p_2 u^2 x) + \lambda (K M - 2 K x - u) \right] \]

\[ \frac{\partial H}{\partial u} = e^{-\delta t} (p_1 x - 2 p_2 u x^2 - c_1) + \lambda (-x) = 0 \]

\[ u^* = \frac{-\lambda x^* + e^{-\delta t} (p_1 x^* - c_1)}{2 e^{-\delta t} p_2 (x^*)^2} \]
Solve for $u^*, x^*, \lambda$ numerically.

Need control bounds

$$0 \leq u(t) \leq a_1$$

Ref:
B D Craven book
Control and Optimization
Interpretation of Adjoint

\[ \max_u \int_{t_0}^{t_1} f(t, x, u) \, dt \equiv V(x_0, t_0) \]

(Definition of value function)

\[ x' = g(t, x, u) \]

\[ x(t_0) = x_0 \]

\[ \frac{\partial V}{\partial x}(x_0, t_0) = \lambda(t_0) \]

\[ \lim_{a \to 0} \frac{V(x_0 + a, t_0) - V(x_0, t_0)}{a} \]

Units: money/unit item in profit problems.
\[ \lambda(t_0) = \text{marginal variation in the optimal objective functional value of the state value at } t_0. \]

“Shadow price”

* additional money associated with additional increment of the state variable

\[
\frac{\partial V}{\partial x}(x^*(t), t) = \lambda(t) \quad \text{for all } t_0 \leq t \leq t_1
\]

“If one fish is added to the stock, how much is the value of the fishery affected?”
\[
\frac{\partial V}{\partial x}(x_0, t_0) = \lambda(t_0)
\]

Approximate

\[
\frac{V(x_0 + 1, t_0) - V(x_0, t_0)}{1} \approx \lambda(t_0)
\]

\[
V(x_0 + 1, t_0) \approx V(x_0, t_0) + \lambda(t_0)
\]

New value \quad Original value + adjoint
Principle of Optimality

If $u^*, x^*$ is an optimal pair on $t_0 \leq t \leq t_1$ and $t_0 \leq \hat{t} \leq t_1$, then $u^*, x^*$ is also optimal for the problem on $\hat{t} \leq t \leq t_1$:

\[
\max_u \int_{\hat{t}}^{t_1} f(t, x, u) \, dt \quad x' = g(t, x, u)
\]

\[x(\hat{t}) = x^*(\hat{t})\]
“Sufficient conditions to guarantee existence of OC"
Suppose $u^*, x^*, \lambda$ satisfy

$$x' = g(t, x, u) \quad x(t_0) = x_0$$
$$\lambda' = -(f_x + \lambda g_x) \quad \lambda(t_1) = 0$$

$H$ is maximized w.r.t. $u$ at $u^*$

plus

set of controls compact

$f, g$ jointly concave in $x$ and $u$

bounded state functions
For details about existence of OC see Macki and Strauss book

Fleming and Rishel book

Back to example

\[
\int_{0}^{1} (x + u) \, dt
\]

\[
x' = 1 - u^2 \quad x(0) = 1
\]

To guarantee the maximum value of $J(u)$ would be finite, need a priori bound on state $x$, control $u$. 

Optimality System

State system coupled with adjoint system
- optimal control’s expressions substituted in

Uniqueness of Optimality System $\rightarrow$ Uniqueness of Optimal Control

Uniqueness of Optimality System - only for small time $T$ due to opposite time orientations

BUT Uniqueness of Optimal Control $\leftrightarrow$ Uniqueness of Solutions of Optimality System

To get uniqueness of OC directly, need strict concavity of $J(u, x(u))$. 
Optimality System

State system coupled with adjoint system
- optimal control’s expressions substituted in

Uniqueness of Optimality System - only for small time $T$
due to opposite time orientations

Numerical Solutions by Iterative Method
- with Runge Kutta 4, Matlab or favorite ODE solver

(Characterization of OC non-smooth)
- guess for controls, solve forward for states
- solve backward for adjoints
- update controls, using characterization
- repeat forward and backwards sweeps and control updates until convergence of iterates
For Lab example

Suppose $x(0) = x_0$ and $x(T) = x_1$ are BOTH GIVEN

Then $\lambda$ does not have a boundary condition.

We will discuss this in more details in following lectures.