Introductory Lecture 2

Suzanne Lenhart

University of Tennessee, Knoxville

Departments of Mathematics

Supported by NIH grant

Exercise

$$\max_u \int_0^1 (x + u)\, dt$$

$x' = 1 - u^2, \quad x(0) = 1, \quad u$ control. $\quad H =$

$$\frac{\partial H}{\partial u} = 0$$

$\lambda' =$ $\qquad \lambda =$

$\lambda(1) =$

$u^* =$ $\qquad x^* =$

Example

$$\max_u \int_0^1 (x+u)\,dt$$

$$x' = 1 - u^2, \quad x(0) = 1 \quad u \text{ control}$$

$$H(t,x,u,\lambda) = x + u + \lambda(1-u^2)$$

$$\frac{\partial H}{\partial u} = 1 - 2\lambda u = 0 \quad \Rightarrow u = \frac{1}{2\lambda} \quad \& H_{uu} = -2\lambda \leq 0,$$

$$\lambda' = -\frac{\partial H}{\partial x} = -1, \quad \lambda(1) = 0, \quad \Rightarrow \lambda = 1 - t$$

$$x' = 1 - u^2 = 1 - \frac{1}{4(1-t)^2}$$

$$x^*(t) = t - 1/4(1-t) + 5/4, \quad u^*(t) = 1/2(1-t).$$

Contd.

$$\int_0^1 [x^*(t) + u^*(t)]\, dt = \infty$$

There is not an “Optimal Control" in this case.

Want finite maximum.

Here unbounded optimal state
unbounded OC

$$\max \phi(x(T)) + \int_0^T f(t, x(t), u(t))\, dt$$

$$x' = g(t, x, u) \qquad x(0) = x_0$$

where $\phi(x(T))$ is final payoff. What change results?

$$J(a) = \int_0^T f(t, y(t,a), u^* + ah)\, dt + \phi(y(T,a))$$

$$\vdots$$

$$\frac{\partial J}{\partial a}(0) = 0 = \int_0^T [\ldots]\, dt \text{ same as before}$$

$$- \lambda(T) \frac{\partial y}{\partial a}(T,0) + \phi'(x^*(T)) \frac{\partial y}{\partial a}(T,0)$$

Only change

$$\lambda(T) = \phi'(x^*(T))$$

Example

$$\max 5x^2(T) + \int_0^T f(t, x, u)\, dt$$

$$\phi(x) = 5x^2 \qquad \phi' = 10x$$

$$\lambda(T) = 10x^*(T)$$

Example

$x(t)$ Number of cancer cells at time t (exponential growth)
State

$u(t)$ Drug concentration **Control**

$$\frac{dx}{dt} = \alpha x(t) - u(t)$$

$$x(0) = x_0 \quad \text{known initial data}$$

$$\min x(t) + \int_0^T u^2(t)\, dt$$

where the first term is number of cancer cells at final time T and the second term is the harmful effects of drug on body.

$$H = u^2 + \lambda(ax - u)$$

$$\frac{\partial H}{\partial u} = 2u - \lambda = 0 \text{ at } u^* \Rightarrow u^* = \frac{\lambda}{2}$$

$$\lambda' = -\frac{\partial H}{\partial x} = -a\lambda \Rightarrow \lambda = \lambda_0 e^{-at}$$

$\lambda(T) = 1$ transversality condition

$$\phi(x) = x, \quad \phi'(x) = 1$$

$x(T) + \int_0^T u^2(t)\, dt$ here $\phi(x) = x$.

Contd.

$$\lambda = \lambda_0 e^{-at}, \quad \lambda(T) = 1 \Rightarrow \lambda_0 = e^{aT}$$

$$\lambda = e^{-a(t-T)}$$

$$x' = ax - u = ax - \frac{e^{-a(t-T)}}{2}$$

$$x' - ax = -\frac{e^{-a(t-T)}}{2}$$

$$\left(e^{-at}x\right)' = -\frac{e^{-2at}e^{aT}}{2}$$

$$x^*(t) = e^{at}x_0 + e^{aT}\frac{(e^{-at} - e^{at})}{4a}$$

Well Stirred Bioreactor

Contaminant and bacteria present in spatially uniform time varying concentrations

$$z(t) = \text{ concentration of contaminant}$$
$$x(t) = \text{ concentration of bacteria}$$

bioreactor rich in all nutrients except one

$$u(t) = \text{ concentration of input nutrient}$$

bacteria degrades contaminant via co-metabolism.

$$x'(t) = G(u)x(t) - D(x(t))^2 \quad \text{where } G(u) = \frac{Gu}{H+u}$$

$$z'(t) = -Kz(t)x(t)$$

where $u(t)$ is control and $x(0), z(0)$ are known.

Objective functional:

$$J(u) = \int_0^T (Kx(t) - u(t))\, dt$$

Find u^* to maximize J

$$J(u^*) = \max J(u)$$

maximize bacteria and minimize input nutrient cost.

$$z(t) = z_0 \exp\left(-\int_0^t Kx(s)\,ds\right)$$

$$\int_0^t Kx(s)\,ds = -\ln\left(\frac{z(T)}{z_0}\right)$$

$J(u)$ penalizes large values of z at final time T.

Can eliminate z variable and work with $x(t)$ only.

$$H = Kx - u + \lambda \left(\frac{Gux}{H+u} - Dx^2 \right)$$

$$\frac{\partial H}{\partial u} = -1 + \lambda x \frac{\partial}{\partial u} \left(\frac{Gux}{H+u} \right) = 0 \quad \text{at } u^*$$

$$-1 + \lambda x \frac{GH}{(H+u)^2} = 0 \qquad \Rightarrow \lambda x G H = (H+u)^2$$

$$(\lambda x G H)^{1/2} = H + u$$

$$u^* = (\lambda x G H)^{1/2} - H$$

$$\lambda' = -\frac{\partial H}{\partial x} = -\left[\lambda\left(\frac{Gu}{H+u} - 2Dx\right) + K\right]$$

$$\lambda(T) = 0$$

$$\lambda' = -\left[\lambda\left(\frac{G\left\{(\lambda x G H)^{1/2} - H\right\}}{H + \left\{(\lambda x G H)^{1/2} - H\right\}} - 2Dx\right) + K\right]$$

$$x' = \frac{G\left\{(\lambda x G H)^{1/2} - H\right\}}{H + (\lambda x G H)^{1/2} - H} - Dx^2$$

$x(0) = x_0$ known .

Solve for x, λ numerically.

Problems

$$u^* = (\lambda x G H)^{1/2} - H$$

What if:

$$(\lambda x G H)^{1/2} = 0?$$

$$\lambda x G H \leq 0?$$

$$(\lambda x G H)^{1/2} - H < 0?$$

Need additional constraint

$$0 \leq u(t) \leq M.$$

Fishery Model

$$x' = Kx(M - x) - ux$$

$x(t)$ population level of fish

$u(t)$ harvesting control

Maximizing net profit:

$$\int_0^T e^{-\delta t} \left(p_1 ux - p_2 (ux)^2 - c_1 u \right) \, dt$$

where $e^{-\delta t}$ is discount factor, p_1, p_2, c_1 terms represent profit from sale of fish, diminishing returns when there is a large amount of fish to sell and cost of fishing. M, p_1, p_2, c_1 are positive constants.

Contd.

$$H = e^{-\delta t}\left(p_1 ux - p_2(ux)^2 - c_1 u\right) + \lambda(Kx(M-x) - ux)$$

$$\lambda' = -\frac{\partial H}{\partial x} = -\left[e^{-\delta t}(p_1 u - 2p_2 u^2 x) + \lambda(KM - 2Kx - u)\right]$$

$$\frac{\partial H}{\partial u} = e^{-\delta t}(p_1 x - 2p_2 ux^2 - c_1) + \lambda(-x) = 0$$

$$u^* = \frac{-\lambda x^* + e^{-\delta t}(p_1 x^* - c_1)}{2e^{-\delta t} p_2 (x^*)^2}$$

Contd.

Solve for u^*, x^*, λ numerically.

Need control bounds

$$0 \leq u(t) \leq a_1$$

Ref:
B D Craven book

Control and Optimization

Interpretation of Adjoint

$$\max_u \int_{t_0}^{t_1} f(t,x,u)\,dt \equiv V(x_0,t_0)$$

(Definition of value function)

$$x' = g(t,x,u)$$

$$x(t_0) = x_0$$

$$\frac{\partial V}{\partial x}(x_0,t_0) = \lambda(t_0)$$

$$\lim_{a\to 0} \frac{V(x_0+a,t_0) - V(x_0,t_0)}{a}$$

Units: money/unit item in profit problems.

$\lambda(t_0)$ = marginal variation in the optimal objective functional value of the state value at t_0.

"Shadow price"

$*$ additional money associated with additional increment of the state variable

$$\frac{\partial V}{\partial x}(x^*(t), t) = \lambda(t) \quad \text{for all } t_0 \leq t \leq t_1$$

"If one fish is added to the stock, how much is the value of the fishery affected ?"

$$\frac{\partial V}{\partial x}(x_0, t_0) = \lambda(t_0)$$

Approximate

$$\frac{V(x_0+1, t_0) - V(x_0, t_0)}{1} \approx \lambda(t_0)$$

$$V(x_0+1, t_0) \approx V(x_0, t_0) + \lambda(t_0)$$

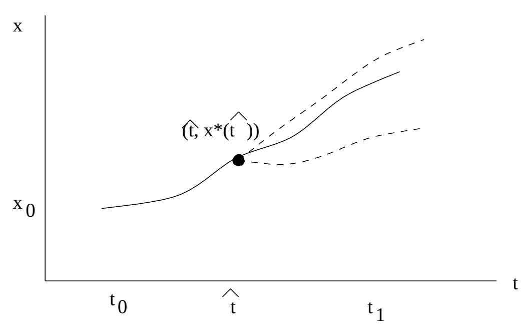
New value Original value + adjoint

Principle of Optimality

If u^*, x^* is an optimal pair on $t_0 \le t \le t_1$ and $t_0 \le \hat{t} \le t_1$, then u^*, x^* is also optimal for the problem on $\hat{t} \le t \le t_1$:

$$\max_u \int_{\hat{t}}^{t_1} f(t, x, u)\, dt \qquad x' = g(t, x, u)$$

$$x(\hat{t}) = x^*(\hat{t})$$



Existence of Optimal Controls

“Sufficient conditions to guarantee existence of OC"

Suppose u^*, x^*, λ satisfy

$$x' = g(t, x, u) \quad x(t_0) = x_0$$

$$\lambda' = -(f_x + \lambda g_x) \quad \lambda(t_1) = 0$$

H is maximized w.r.t. u at u^*

plus

set of controls compact

f, g jointly concave in x and u

bounded state functions

For details about existence of OC see Macki and Strauss book

Fleming and Rishel book

Back to example

$$\int_0^1 (x+u)\,dt$$

$$x' = 1 - u^2 \quad x(0) = 1$$

To guarantee the maximum value of $J(u)$ would be finite, need a priori bound on state x, control u.

Optimality System

State system coupled with adjoint system
- optimal control's expressions substituted in

Uniqueness of Optimality System $\rightarrow$ Uniqueness of Optimal Control

Uniqueness of Optimality System - only for small time T due to opposite time orientations

BUT Uniqueness of Optimal Control $\nrightarrow$ Uniqueness of Solutions of Optimality System

To get uniqueness of OC directly,

need strict concavity of $J(u, x(u))$.

Optimality System

State system coupled with adjoint system
- optimal control's expressions substituted in

Uniqueness of Optimality System - only for small time T due to opposite time orientations

Numerical Solutions by Iterative Method
- with Runge Kutta 4, Matlab or favorite ODE solver

(Characterization of OC non-smooth)
- guess for controls, solve forward for states
- solve backward for adjoints
- update controls, using characterization
- repeat forward and backwards sweeps and control updates until convergence of iterates

For Lab example

Suppose $x(0) = x_0$ and $x(T) = x_1$ are BOTH GIVEN

Then λ doesnot have a boundary condition.

We will discuss this in more details in following lectures.