# Nonlinear Elliptic Partial Differential Equations and Their Applications

Our primary goal in these notes will be a complete[1] proof of the existence (and uniqueness) of solutions to the Dirichlet problem for a large class of quasilinear and fully nonlinear elliptic equations, many of which arise naturally in engineering, geometry, materials science, physics and topology.

We follow quite closely the programme laid out in the marvellous text of Gilbarg and Trudinger [**2**]: by establishing *a priori* estimates in Hölder spaces for linear equations, the existence problem is reduced to the establishment of *a priori* estimates for first or second derivatives of solutions to the nonlinear problems. The latter can be achieved directly in many instances (by exploiting barriers and the maximum principle, say).

We first develop the theory of quasilinear equations, such as the minimal surface or mean curvature flow translator equations, where the Schauder theory and the Harnack inequality of de Giorgi, Nash and Moser for linear elliptic equations in divergence form reduces the existence problem to the establishment of a global $C^1$ estimate. We present the latter for a quite

[1] Insofar as completeness is concerned, we naturally fail in achieving our goal; the main gaps concern a complete treatment of regularity up-to-the-boundary (namely, Theorems 3.3, 6.8, 10.8 and 11.4, and Proposition 7.7).

general class of equations of mean curvature type (which includes the aforementioned examples).

We then consider concave fully nonlinear Hessian equations, such as the equation of prescribed Gauss curvature or the translator equations of Gauss curvature flows, where the Schauder theory and the Harnack inequality of Krylov and Safanov for linear elliptic equations not in divergence form reduces the existence problem to the establishment of a global $C^2$ estimate. We present the latter for certain equations of Monge–Ampère type (which includes the above examples) assuming the presence of suitable barriers.

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## 0. Motivating problems

We shall begin by presenting some examples of (nonlinear) partial differential equations which arise in various applications. We provide a brief description of their motivating problem and how they arise, but we shall not attempt a thorough discussion.

**0.1. Newtonian gravity.** Newton's law of gravity may be formulated in terms of the *gravitational potential* $u$ and the *mass distribution* $\rho$. The mass distribution $\rho$ describes the density of matter (units of mass per unit of volume) at each point of space and the gravitational potential $u$ is an unobservable quantity with units of energy which gives rise to the *gravitational force* $\mathbf{f}$ via the equation (expressed in appropriate units)

$$\mathbf{f} = -\operatorname{grad} u\,.$$

The gravitational force can be measured via Newton's second law of motion. Conservation of energy demands that its divergence is, in appropriate units, $4\pi$ times the mass distribution. So the gravitational potential satisfies *Poisson's equation*

$$\Delta u = 4\pi\rho\,, \tag{0.1}$$

where $\Delta\cdot \doteqdot \operatorname{div}(\operatorname{grad}\cdot\,)$ is the Laplacian.

As observed by Newton, if the mass density $\rho$ is compactly supported and integrable, then Poisson's equation is solved by the (three dimensional) NEWTONIAN POTENTIAL

$$\gamma_\rho(x) \doteqdot 4\pi(\Gamma * \rho)(x) \doteqdot 4\pi\int_{\mathbb{R}^3}\Gamma(x-y)\rho(y)\,dy\,,$$

where

$$\Gamma(x) \doteqdot -\frac{1}{4\pi|x|}$$

is the (three dimensional) FUNDAMENTAL SOLUTION to the Laplace equation. This solution is not unique, however, since (1.1) is satisfied by $u = \gamma_\rho + h$ for any harmonic function $h$.

For suitably regular $\Omega \subset \mathbb{R}^3$ and $\phi : \partial\Omega \to \mathbb{R}$, the Dirichlet problem

$$\begin{cases} \Delta u = 4\pi\rho \ \text{ in } \ \Omega \\ \quad u = \phi \ \text{ on } \ \partial\Omega \end{cases}$$

admits the unique solution $u = \gamma_\rho + h$, where $h$ is the unique harmonic function taking boundary values $\psi \doteqdot \phi - \gamma_\rho|_{\partial\Omega}$. When $\Omega$ is, for example, the unit ball $B$, $h$ admits the simple representation formula

$$h(x) \doteqdot \int_{\partial B}\psi(y)K(x,y)\,d\sigma(y)\,,$$

where

$$K(x,y) \doteqdot \frac{1-|x|^2}{|\partial B|}\frac{1}{|x-y|^3}$$

is the POISSON KERNEL (for $B$) and $\sigma$ is the standard measure on $\partial B$.

Poisson's equation also models a number of further phenomena. For example, in electrostatics, $u$ becomes the *electrostatic potential* and $4\pi\rho$ is replaced by the *charge density*. This is a common theme in the study of partial differential equations — very often, a given PDE or class of PDE will arise as a model for a number of apparently unrelated phenomena.

**0.2. Diffusion.** In the absence of sources and sinks, Fourier's theory of heat diffusion may be formulated in terms of the *temperature* function $u$. The temperature is a scalar function which depends on space and time. It takes units of energy per unit volume and plays an analogous role to the gravitational potential in Newton's law of gravity. Fourier's Law states that the rate of flow of heat energy, the *heat flux* $\mathbf{q}$, is proportional to the negative temperature gradient. That is, in appropriate units,

$$\mathbf{q} = -\nabla u\,.$$

Conservation of energy demands that the total heat energy

$$Q(\Omega,t) \doteqdot \int_{\partial\Omega} u\,dV$$

contained in a region $\Omega\subset\mathbb{R}^3$, where $dV$ is the volume element of $\Omega$, can (in the absence of sources and sinks) only be gained or lost via flux through its boundary; that is,

$$Q(\Omega,t_2)-Q(\Omega,t_1) = -\int_{t_1}^{t_2}\!\!\int_{\partial\Omega}\langle\mathbf{q},\nu\rangle\,dA\,dt\,,$$

where $\nu$ and $dA$ are the outward unit normal and area element of $\partial\Omega$, respectively. From these postulates, we derive

$$\int_\Omega \partial_t u\,dV = \frac{d}{dt}Q(\Omega,t) = -\int_{\partial\Omega}\langle\mathbf{q},\nu\rangle\,dA = \int_{\partial\Omega}\langle\nabla u,\nu\rangle\,dA = \int_\Omega \Delta u\,dV,$$

where $\partial_t \doteqdot \frac{\partial}{\partial t}$. Since the same argument applies to every subdomain of $\Omega$, we actually obtain the pointwise equation

$$(\partial_t-\Delta)u = 0\,.$$

The heat equation and its close relatives model more general diffusion phenomena, and therefore arise in a number of areas, from physics, chemistry, and biology (particle diffusion), to sociology, economics, and finance (diffusion of people, ideas, and prices).

Though we shall only consider elliptic equations in these notes, essentially all of the techniques and results we cover have analogues in the parabolic setting.

**0.3. The Minkowski problem.**

> *The importance of the Minkowski problem and its solution is to be felt both in differential geometry and in elliptic partial differential equations, on either count going far beyond the impact that the literal statement superficially may have. From the geometric viewpoint it is the Rosetta Stone, from which several other related problems can be solved.*
> — Eugenio Calabi

Let $\Sigma^2 \subset \mathbb{R}^3$ be the boundary of a convex open set $\Omega \subset \mathbb{R}^3$. If $\Sigma$ is smooth, then it admits a well-defined outward unit normal $\nu(p) \in S^2$ at each point $p \in \Sigma$. The map $p \mapsto \nu(p)$ is called the GAUSS MAP of $\Sigma$. Note that the tangent planes $T_p\Sigma$ to $\Sigma$ and $T_{\nu(p)}S^2$ to $S^2$ are parallel. Up to identification of these planes, the SHAPE OPERATOR of $\Sigma$ at $p$ is defined as the differential $A_p \doteqdot (D\nu)|_p : T_p\Sigma \to T_p\Sigma$ of the Gauss map at $p$. The GAUSS CURVATURE $K(p)$ at $p$ is the determinant of $A_p$.

The MINKOWSKI PROBLEM asks for the existence of a closed, convex surface $\Sigma$ whose Gaussian curvature is a prescribed positive function $f : S^2 \to \mathbb{R}$ of its outward unit normal. That is,

$$K(p) = f(\nu(p))\,.$$

If $K$ is positive everywhere, then $\nu$ is a local diffeomorphism. Since $S^2$ is simply connected, $\nu$ must in fact be a diffeomorphism, so we may parametrize $\Sigma$ by the inverse $\varphi \doteqdot \nu^{-1} : S^2 \to \Sigma \subset \mathbb{R}^3$ of its Gauss map. This parametrization is closely related to the SUPPORT FUNCTION $\sigma : S^2 \to \mathbb{R}$ of $\Sigma$, which is defined by

$$\sigma(z) \doteqdot \sup_{x\in\Sigma} x \cdot z\,.$$

Equivalently (and more geometrically) $\sigma(z)$ is the distance to the origin of the supporting hyperplane for $\Sigma$ with outward unit normal $z$. So we find that

$$\sigma(z) = z \cdot \varphi(z)\,.$$

Moreover, differentiating this identity, we find that $\varphi^{\top_{S^2}} = \overline{\mathrm{grad}}\,\sigma$, and hence

$$\varphi(z) = \sigma(z)\, z + \overline{\mathrm{grad}}\,\sigma|_z\,,$$

where $\overline{\mathrm{grad}}$ is the gradient operator with respect to the round metric $\overline{g}$ on $S^2$. The shape operator is given, with respect to the Gauss map parametrization, by

$$A^{-1} = \overline{\nabla}\,\overline{\mathrm{grad}}\,\sigma + \sigma\,\mathrm{I}\,,$$

where $\overline{\nabla}$ is the covariant derivative operator of $S^2$ and $\mathrm{I} : TS^2 \to TS^2$ is the identity map. So the Minkowski problem asks for a solution $\sigma : S^2 \to \mathbb{R}$ to the Monge–Ampère type equation

$$\det\left(\overline{\nabla}\,\overline{\mathrm{grad}}\,\sigma + \sigma\,\mathrm{I}\right) = f^{-1}$$

on $S^2$.

Note that $f$ must satisfy the constraint equation

$$\int_{S^2} \frac{e \cdot z}{f(z)} d\mu_{S^2}(z) = 0$$

for all $e \in \mathbb{R}^3$. Indeed, since $\det D\varphi = K^{-1}$, the area formula and the divergence theorem yield

$$\int_{S^2} \frac{e \cdot z}{f(z)}\, d\bar{\mu}(z) = \int_{S^2} \frac{e \cdot z}{K(z)}\, d\bar{\mu}(z) = \int_{\Sigma} e \cdot \nu(x)\, d\mu(x) = \int_{\Omega} \operatorname{div} e\, d\mathscr{L} = 0\,,$$

where $\bar{\mu}$ is the area measure on $S^2$ induced by $\bar{g}$, $\mu$ is the area measure on $\Sigma$ induced by its embedding in $\mathbb{R}^3$, and $\mathscr{L}$ is the Lebesgue measure on $\mathbb{R}^3$.

Despite its purely geometric origin, the Minkowski problem appears in many applications. For example, the problem of radiolocation and the "inverse problem" of short-wave diffraction both reduce to the Minkowski problem.

The Minkowski problem was solved by Louis Nirenberg in 1953. The statement generalizes in a straightforward manner to higher dimensions, and this was solved by Pogorelov in 1978.

**0.4. The Weyl problem.** Gauss' famous *theorema egregium* states that the intrinsic curvature of a surface which is isometrically immersed in $\mathbb{R}^3$ is equal to its extrinsic (a.k.a. Gauss) curvature. The Hadamard theorem states that any properly immersed surface in $\mathbb{R}^3$ with positive Gauss curvature is the boundary of a convex body. Motivated by these two results, the Hermann Weyl asked whether a given closed Riemannian surface of positive intrinsic curvature is necessarily realized by the boundary of a convex body in Euclidean three-space (with its induced geometry).

Given an immersion $u : M^2 \to \mathbb{R}^3$ and a coordinate chart $(x^1, x^2) : U \to \mathbb{R}^2$ for $M^2$, the metric components and the Gaussian curvature are given in $U$ by

$$g_{ij} = \frac{\partial u}{\partial x^i} \cdot \frac{\partial u}{\partial x^i} \quad \text{and} \quad K = \det\left(g^{jk} \frac{\partial^2 u}{\partial x^i \partial x^k} \cdot \nu\right)\,,$$

respectively, where

$$\nu = \frac{\frac{\partial u}{\partial x^1} \times \frac{\partial u}{\partial x^2}}{|\frac{\partial u}{\partial x^1} \times \frac{\partial u}{\partial x^2}|}$$

is the (right-handed) unit normal field and $g^{ij}$ are the components of the inverse of the component matrix $g_{ij}$. That is, $g_{ik}g^{kj} = \delta_i^j$. So, given a closed Riemannian surface $(M^2, g)$ with positive intrinsic curvature $K = K[g]$, the Weyl problem asks for a map $u : M^2 \to \mathbb{R}^3$ which satisfies the coupled differential equations

$$\begin{cases} \dfrac{\partial u}{\partial x^i} \cdot \dfrac{\partial u}{\partial x^i} = g_{ij} \\ \det \left( g^{jk} \dfrac{\partial^2 u}{\partial x^i \partial x^k} \cdot \dfrac{\frac{\partial u}{\partial x^1} \times \frac{\partial u}{\partial x^2}}{\left|\frac{\partial u}{\partial x^1} \times \frac{\partial u}{\partial x^2}\right|} \right) = K \end{cases}$$

in any local coordinate chart $(x^1, x^2) : U \to \mathbb{R}^2$ for $M^2$.

The Weyl problem was resolved affirmatively by Alexandrov (1941) and Pogorelov (1952) and, independently, by Nirenberg (1953) (in the same paper in which he presented his solution to the two dimensional Minkowski problem).

The work of Nirenberg and Pogorelov on the Weyl and Minkowski problems are some of the most important developments in the theory of elliptic partial differential equations.

### 0.5. Surface tension.

> *"Make a soap bubble and observe it;*
> *you could spend a whole life studying it,"*
> Sir William Thomson, Lord Kelvin

A macroscopic consequence of statistical mechanics is the presence of *cohesive forces* within liquids. In an equilibrium state, these forces pull each molecule of the liquid in every direction with equal magnitude, resulting in a net force of zero. At a flat interface between two liquids (the respective cohesive forces within which being unequal) a net difference of force per unit area (pressure) normal to the interface results. Moreover, the tangential forces at the boundary have the effect of decreasing the area of the interface in the vicinity of every point. This causes distortion of the interface until equilibrium is restored.

The Young–Laplace law asserts that the net difference in pressure between the two liquids at a point $p$ on the interface is proportional to the MEAN CURVATURE $H(p)$ of the interface at $x$ (twice the average of the curvatures at $p$ of all normal sections). In a neighbourhood of any point $p$ of a smooth interface $\Sigma$, we may represent $\Sigma$ as the graph of a function $u$ over the tangent plane $T_p\Sigma$ at $p$. That is, we can find an open subset $U \subset \mathbb{R}^3$ containing $p$ and a function $u : T_p\Sigma \to \mathbb{R}$ such that

$$U \cap \Sigma = \{x + u(x)\nu(p) : x \in T_p\Sigma\} \cap U\,,$$

where $\nu(p)$ is a unit normal to $\Sigma$ at $p$. The mean curvature of a point $x+u(x)\nu(p)$ in this region is the trace of the shape operator $A = D\nu$, which is equal to

$$H(x) = \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)\Bigg|_x .$$

Thus, if the pressure on both sides is balanced, then the interface will satisfy locally the (GRAPHICAL) MINIMAL SURFACE EQUATION

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0 .$$

If the pressure on both sides is unbalanced, then the interface will satisfy locally the (GRAPHICAL) PRESCRIBED MEAN CURVATURE EQUATION

$$-\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \psi(\cdot, u) .$$

Graphical minimal surfaces with prescribed Dirichlet boundary data model SOAP FILMS. Graphical surfaces of prescribed mean curvature with prescribed boundary contact angle model CAPILLARY SURFACES.

**0.6. Optimal transportation.** The field of optimal transportation was developed in order to minimize the cost or work associated with transportation of some quantity of a commodity from one specified location to another.

Given two domains, $\Omega$, $\Omega^* \subset \mathbb{R}^n$, and corresponding (non-negative) densities $f \in L^1(\Omega)$ and $f_* \in L^1(\Omega^*)$ satisfying

$$\int_\Omega f = \int_{\Omega^*} f_* ,$$

we denote by $\mathcal{I}$ the set of MEASURE PRESERVING TRANSFORMATIONS from $\Omega$ to $\Omega^*$. A map $T : \Omega \to \Omega^*$ is in $\mathcal{I}$ provided that $T$ is measurable and

$$\int_{T^{-1}(E)} f = \int_E f_*$$

for any Borel set $E \subset \Omega^*$.

For a given cost function $c : \Omega \times \Omega \to \mathbb{R}$, we consider the problem of determining a measure preserving transformation $T$ which minimizes the COST FUNCTIONAL

$$C(T) \doteqdot \int_\Omega c(x, T(x)) f(x) dx .$$

In the original problem formulated by Monge, the cost function $c$, given by

$$c(x, y) = |x - y| ,$$

corresponds to the work done in moving a mass distribution from $\Omega$ to $\Omega^*$.

Under mild hypotheses, a smooth optimal map $T$ (if it exists) can be realized as

$$T(x) = p(x, Du(x)),$$

where, denoting derivatives with respect to the $x$ and $y$ variables by corresponding subscripts, $p$ satisfies

$$c_x(x, p(x, v)) = v$$

and $u$ satisfies the equation

$$\det\left(D^2\gamma - D^2u\right) = |\det c_{x,y}| \frac{f}{f_* \circ T}$$

together with the concavity condition

$$D^2u \le D^2\gamma,$$

where $\gamma(x) \doteqdot c(x, T(x))$.

In the special case of a quadratic cost function

$$c(x, y) = -x \cdot y,$$

by replacing $u$ with $-u$ we obtain the Monge–Ampère type equation

$$\det D^2u = \psi(\cdot, u, Du)$$

and the convexity constraint

$$D^2u \ge 0,$$

where

$$\psi(\cdot, u, Du) \doteqdot \frac{f}{f_*(Du)}.$$

**0.7. Geometric optics.** Consider a non-isotropic light source positioned at a point $O$ in space $\mathbb{R}^3$. Let $S$ be a unit radius sphere with centre at $O$ and consider $\Omega$ an open subset of $S$. Denote by $\Gamma$ a surface which projects radially in a one-to-one fashion onto $\Omega$. The surface is supposed to have a perfect reflection property. That is, no loss of energy occurs when a beam of light is reflected by it. Suppose a ray is originated from $O$ in the direction $x$ and is reflected by $\Gamma$, producing a reflected ray in the direction $y$. If, we identify a direction with a point on $S^2$, then we get a mapping $u$ of $\Omega \subset S^2$ into $S^2$.

The REFLECTOR PROBLEM is the problem of constructing the reflecting surface $\Gamma$ in such a way that the reflected rays cover a prescribed region $\Omega^*$ of a "far-field sphere" and the density of the distribution of the reflected rays is a prescribed function of the incoming directions.

The reflector problem can be reduced to the Monge–Ampère type equation

$$\det_g\left(\nabla^2 u - \frac{|\nabla u|^2}{2u} g + \frac{1}{2} u g\right) = \psi(\cdot, u, \nabla u) \text{ on } \Omega \subset S^2$$

for the radial graph height $u : \Omega \to \mathbb{R}$ of $\Gamma$, equipped with an appropriate boundary condition, where $g$ and $\nabla$ are, respectively, the metric and covariant derivative on $S^2$, and $\psi$ is determined *a priori* by the illumination densities on the input and output domains $\Omega$ and $\Omega^*$.

**0.8. The Yamabe problem.** The famous uniformization theorem, resolved by Poincaré and Koebe in 1907, states that every Riemannnian surface admits a conformally equivalent metric of constant Gauss curvature. The YAMABE PROBLEM generalizes this to higher dimensions by asking whether a given Riemannian metric on a closed differentiable manifold of dimension at least three is conformally equivalent to a metric of constant scalar curvature.

Two Riemannian metrics $g$ and $\tilde{g}$ on a differentiable manifold $M$ are CONFORMALLY EQUIVALENT if there is a (positive) function $f$ such that $\tilde{g} = fg$. If the dimension $n$ of $M$ is at least three, then we may assume that $\tilde{g} = u^{\frac{4}{n-2}} g$ for some positive function $u$. A direct computation shows that the scalar curvature $\tilde{\mathrm{R}}$ of $\tilde{g}$ is related to the scalar curvature $\mathrm{R}$ of $g$ by the equation

$$\tilde{\mathrm{R}} = u^{1-2^*}\left(-\tfrac{4(n-1)}{n-2}\Delta u + \mathrm{R}\, u\right),$$

where $2^* \doteqdot \frac{2n}{n-2}$ and $\Delta \cdot \doteqdot \operatorname{div}_g(\operatorname{grad} \cdot)$ is the Laplacian on $M$ induced by $g$. (Note that $2^*$ is the critical exponent in the Sobolev embedding, a fact that plays a decisive role in the problem.) So the Yamabe problem is equivalent to finding a solution $(u, \lambda)$, $\lambda \in \mathbb{R}$, to the nonlinear eigenvalue problem

$$-\tfrac{4(n-1)}{n-2}\Delta u + \mathrm{R}\, u = \lambda u^{2^*-1}.$$

The Yamabe problem is closely connected with the positive mass theorem in general relativity. It was resolved through the combined works of Trudinger (1968), Aubin (1976), and Schoen (1984).

**0.9. Black hole horizons.** Spacetime in Einstein's general theory of relativity is modelled by a four dimensional Lorentzian manifold $(\overline{M}^4, \overline{g})$ for which $\overline{g}$ satisfies the Einstein equation,

$$\overline{\mathrm{Rc}} - \frac{1}{2}\overline{\mathrm{R}}\,\overline{g} = 8\pi T,$$

where $\overline{\mathrm{Rc}}$ and $\overline{\mathrm{R}}$ are, respectively, the Ricci and scalar curvatures of $\overline{g}$, and, in appropriate units, $T$ is the stress-energy tensor, a source term representing the configuration of matter and energy.

A TRAPPED SURFACE $\tau \subset \overline{M}^4$ in a Lorentzian manifold $(\overline{M}^4, \overline{g})$ is a smooth, closed, two-dimensional, spacelike submanifold of $\overline{M}^4$ whose mean curvature vector $\mathbf{H}$ is past pointing timelike. This condition means that both outward and inward directed causal curves are converging at $\tau$. Indeed, if $V \in \Gamma(T\tau)$ is a future pointing causal (timelike or null) vector field on $\tau$, then the first variation formula yields

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} A(\tau + \varepsilon V) = -\int g(\mathbf{H}, V) d\mu \leq 0$$

with strict inequality unless $V \equiv 0$, where $A$ is the area functional and

$$\tau + \varepsilon V \doteqdot \{\gamma_{(p,V(p))}(\varepsilon) : p \in \tau\}$$

is the surface obtained by moving each point $p \in \tau$ a parameter distance $\varepsilon$ along the geodesic $\gamma_{(p,V(p))}$ with initial data $(p, V(p))$. So the area is locally decreasing about any $p \in \tau$ in the direction of any nonzero future pointing causal vector field. This means that all causal curves emanating from the surface are "pulled inwards", even light emitted in the outwards normal direction.

Trapped surfaces are closely connected to black holes: under mild conditions on $(M, g)$, Penrose's singularity theorem[2] asserts that a spacetime which admits a trapped surface must also admit a "singularity". In a spacetime which contains a trapped surface, the OUTERMOST TRAPPED SURFACE $\Sigma$ is the boundary of the union of all the trapped surfaces. Outward directed lightrays are neither converging nor diverging. This is the *event horizon*. (Note that $\Sigma$ may have multiple connected components.)

On a spacelike surface $\tau$ in $M$, any normal vector can be written as a linear combination of two future-pointing null-vectors $k_\pm$ normalized by the equation

$$g(k_-, k_+) = -2\,.$$

In particular, the mean curvature vector $\mathbf{H}$ may be decomposed as

$$\mathbf{H} = \theta_- k_- + \theta_+ k_+\,.$$

So being trapped means precisely that both the INNER and OUTER EXPANSIONS $\theta_\pm$ are negative. On the outermost trapped surface, one of the two expansions vanishes, which we may take to be the outer expansion $\theta_+$. Such surfaces are called MARGINALLY OUTER TRAPPED SURFACES. They are Lorentzian analogues of minimal surfaces. To see why, we need to introduce *initial data sets*.

Locally, any observer $\gamma : I \to M^4$ gives rise to a splitting of spacetime into space+time: at any given time $t \in I$, which we without loss of generality take to be 0, the spacelike subspace $\gamma'(0)^\perp \subset T_{\gamma(0)}M^4$ may be locally

[2] Sir Roger Penrose shared the 2020 Nobel Prize in physics for this work.

"integrated" to obtain a spacelike hypersurface $M_0^3 \subset M^4$ by shooting out (spacelike) geodesics from $o \doteqdot \gamma(0)$ in directions $v \in \gamma'(0)^\perp$. By parallel translating $U_o \doteqdot \gamma'(0)$ along these geodesics, we may then shoot out (timelike) geodesics from each $p \in \Sigma$ in the direction of $U_p$ (the parallel translate of $U_o$ at $p$). We may interpret these geodesics as a family of freefallers which are instantaneously comoving with $\gamma$ at time 0. Denote by $U$ the vector field on the resulting open subset $O \subset M^4$ which gives at each $q \in O$ the tangent vector to the comoving freefaller at $q$, by $t : O \to \mathbb{R}$ the function which assigns to each $q \in O$ the proper time along the comoving freefaller which joins $q$ to $M_0^3$, and, abusing notation, by denote by $M_t^3 \doteqdot \{p \in O : t(p) = t\}$ the $t$-level set of the function $t$. By the Gauss lemma, the metric is of the form

$$-dt \otimes dt + h$$

on $O$, where $g(U, \cdot) = 0$ and $t \mapsto g(t) \doteqdot g|_{TM_t^3 \otimes TM_t^3}$ is a Riemannian metric on $M_t^3$ for each $t$. Having split spacetime locally into space + time in this manner, it is possible to view Einstein's equation as an evolution equation for $g$ with initial data $(M_t^3, g, \partial_t g)$ at $t = 0$ plus certain *constraint equations* for $(M_t^3, g, \partial_t g)$.

An INITIAL DATA SET is a triple $(M^3, g, A)$, where $(M^3, g)$ is a Riemannian three-manifold and $A$ is a symmetric bilinear form on $A$, which satisfy the CONSTRAINT EQUATIONS

$$\begin{aligned} \frac{1}{2}\left(\mathrm{R}_g + K^2 - |A|^2\right) &= \rho \\ \operatorname{div}_g A - dK &= J\,, \end{aligned}$$

where $\mathrm{R}_g$ is the scalar curvature of $g$, $K \doteqdot \operatorname{tr}_g A$, and the function $\rho$ and the one-form $J$ are source terms which are determined by the distribution of matter in $M^3$ via the energy-momentum tensor. Choquet-Bruhat proved that, given any initial data set $(M^3, g, A)$, we can find a solution $(\overline{M}^4, \overline{g})$ to Einstein's equation in which $(M^3, g)$ embeds isometrically with second fundamental form $A$.

Now suppose that our marginally outer trapped surface $\Sigma$ lies in an initial data set $(M^3, g, A)$. Then, since $(M^3, g)$ embedds in the spacetime $(\overline{M}^4, \overline{g})$, the Gauss equation implies that

$$\mathbf{H} = H\nu - \operatorname{tr}_{T\Sigma}(A) U\,,$$

where $\nu$ is the outer of unit normal field to $\Sigma$ in $M^3$ (so that $k_\pm = U \pm \nu$, where $U$ is the future pointing unit normal field to $M^3$ in $\overline{M}^4$), $H = \operatorname{div} \nu$ is the corresponding scalar mean curvature, and $U$ is the future directed timelike unit normal vector field of $M^3$ in $\overline{M}^4$. Thus,

$$\theta_+ = H + \operatorname{tr}_{T\Sigma}(A)\,.$$

In particular, if $(M^3, g, A)$ is TIME-SYMMETRIC, meaning that $A \equiv 0$, then

$$\theta_+ = H\,,$$

and hence marginally outer trapped surfaces in $(M^3, g)$ are just minimal surfaces in $(M^3, g)$.

**0.10. The Penrose inequality.** A Riemannian three-manifold $(M^3, g)$ is called ASYMPTOTICALLY FLAT if the metric $g$ approaches the Euclidean metric at infinity in a quantitative way. More precisely, we require that $(M^3, g)$ admits a compact set $K$ such that $M^3 \setminus K$ is the union of finitely many connected components (its "ends") each of which is diffeomorphic to the compliment $\mathbb{R}^3 \setminus B$ of a ball $B$ in $\mathbb{R}^3$ via a diffeomorphism $(x^1, x^2, x^3) : E \to \mathbb{R}^3 \setminus B$ which satisfies[3]

$$|g_{ij} - \delta_{ij}| \leq O(|x|^{-1})\,, \quad |\partial_{x^k} g_{ij}| \leq O(|x|^{-2})\,, \quad \text{and} \quad |\partial_{x^k}\partial_{x^l} g_{ij}| \leq O(|x|^{-3})$$

as $|x| \to \infty$.

The Arnowitt–Deser–Misner (ADM) mass of an end $E$ of an asymptotically flat three manifold $(M^3, g)$ is defined to be

$$m_{\mathrm{ADM}}(E) \doteqdot \frac{1}{16\pi} \lim_{r\to\infty} \int_{\partial B_r} (\partial_{x^j} g_{ii} - \partial_{x^i} g_{ij})\, n^j\, dx\,,$$

where $n(x) \doteqdot \frac{x}{|x|}$ is the outer unit normal to the coordinate sphere $\partial B_r$ at $x \in \partial B_r$. The ADM mass is a geometric invariant of the given end, despite being expressed in coordinates. The ADM mass of $(M^3, g)$ is defined to be the sum of the ADM masses of its ends.

For the Riemannian Schwarzschild metric

$$g_{\mathrm{Schwarzschild}} = \left(1 + \frac{m}{2r}\right)^4 g_{\mathbb{R}^3}$$

on the outer region $M^3_{\mathrm{Schwarzschild}} \doteqdot \mathbb{R}^3 \setminus B_{\frac{m}{2}}$, where $r(x) \doteqdot |x|$, the ADM mass is equal to

$$m_{\mathrm{ADM}}(M^3_{\mathrm{Schwarzschild}}, g_{\mathrm{Schwarzschild}}) = m = \sqrt{\frac{A}{16\pi}}\,,$$

where $A$ is the area of the horizon $\partial(\mathbb{R}^3 \setminus B_{\frac{m}{2}})$.

Penrose conjectured that the ADM mass of an asymptotically flat initial data set is bounded from below in terms of the areas of its black holes. More precisely, he conjectured that

$$m_{\mathrm{ADM}}(M^3, g) \geq \sqrt{\frac{A}{16\pi}}\,,$$

with equality only if $(M^3, g)$ is isometric to $(\mathbb{R}^3 \setminus B_{\frac{m}{2}}, g_{\mathrm{Schwarzschild}})$.

[3] There are different definitions requiring different rates or orders of decay, depending on the context.

Hawking observed that the ADM mass of $(M^3, g)$ can be recovered from the *quasi-local mass* (known now as the HAWKING MASS)

$$\Sigma^2 \subset M^3 \mapsto m_H(\Sigma) \doteqdot \sqrt{\frac{|\Sigma|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma H^2 \, d\mu\right)$$

by taking the limit as $r \to \infty$ of $m_{\mathrm{H}}(\Sigma_r)$ over coordinate spheres $\Sigma_r \doteqdot x^{-1}(B_r)$. Here, $H$ is the mean curvature of $\Sigma$, $|\Sigma| = \mu(\Sigma)$ is its area, and $\mu$ is the induced Riemannian measure.

Geroch (1973) observed that the Hawking mass $m_{\mathrm{H}}(\Sigma_t)$ is monotone non-decreasing along a family of surfaces $\Sigma_t$ which evolve according to the INVERSE MEAN CURVATURE FLOW

$$\partial_t X = H^{-1}\nu \,, \tag{0.2}$$

where $\nu$ is the outward pointing unit normal field. The equation (0.2) tends to expand initial surfaces satisfying $H > 0$, and diffusion tends to smooth out the curvature so that they become "rounder". If, given a time-symmetric initial data set $(M^3, g)$, we can find a family of surfaces $\Sigma_t$ which start from the outermost trapped surface $\Sigma_0 = \partial M^3$ and expand outwards, eventually approximating large coordinate spheres as $t \to \infty$, all the while moving according to inverse mean curvature flow, then we can conclude that

$$m_{\mathrm{ADM}}(M^3, g) = \lim_{t\to\infty} m_{\mathrm{H}}(\Sigma_t) \geq m_{\mathrm{H}}(\Sigma_0) = \sqrt{\frac{|\Sigma|}{16\pi}}\,.$$

This idea for proving Penrose's conjecture (in the time-symmetric case) was suggested by Jang and Wald (1977), and was eventually made rigorous by Huisken and Ilmanen (2001). The main difficulty was overcoming the fact that solutions to (0.2) starting at the horizon $\Sigma = \partial M^3$ generally run into singularities of the form $H \to 0$ before the surfaces are able to expand to infinity. To overcome this, Huisken and Ilmanen studied the *level set formulation* of the flow, which asks for a function $u : M^3 \to \mathbb{R}$ satisfying

$$\begin{cases} \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = |\nabla u| \\ u|_{\partial M^3} = 0\,. \end{cases} \tag{0.3}$$

The level sets $\Sigma_t \doteqdot \partial\{p \in M^3 : u(p) \leq t\}$ of a function $u$ satisfying (0.3) will be smooth and satisfy (0.2) wherever $|\nabla u| \neq 0$. The advantage of the level set formulation is that a global solution can be constructed (including possibly points where $|\nabla u|$ degenerates), which thereby allows the continuation of the inverse mean curvature flow through singularities. On the other hand, proving that the ADM mass is non-decreasing across critical times requires additional arguments.

The general (i.e. non time-symmetric) Penrose inequality remains an open problem.

**0.11. Curvature flow.** The CURVE SHORTENING FLOW drives an immersed curve $\gamma : M^1 \to M^2$ on a Riemannian surface $M^2$ with velocity equal to its curvature vector $\vec{\kappa}$,

$$\partial_t \gamma = \vec{\kappa}\,.$$

Its name comes from the fact that it is the $L^2$-gradient flow of the length functional, so it tends to decrease the length of the curve in the vicinity of any given point.

Appropriately viewed, the curve shortening flow can be seen as a parabolic partial differential equation. With respect to the arc-length parameter $s$, it takes the form

$$\partial_t \gamma = \partial_s^2 \gamma\,,$$

which looks an awful lot like the heat equation.

With respect to a local coordinate $x : U \subset M^1 \to \mathbb{R}$ for the 1-manifold $M^1$ (topologically either $S^1$ or $\mathbb{R}$), curve shortening flow (in the plane) becomes

$$\begin{aligned}\frac{\partial \gamma}{\partial t} &= \left|\frac{\partial \gamma}{\partial x}\right|^{-2}\left(\frac{\partial^2 \gamma}{\partial x^2}\right)^{\perp}\\ &= \left|\frac{\partial \gamma}{\partial x}\right|^{-2}\left(\frac{\partial^2 \gamma}{\partial x^2} - \left|\frac{\partial \gamma}{\partial x}\right|^{-2}\left\langle\frac{\partial^2 \gamma}{\partial x^2}, \frac{\partial \gamma}{\partial x}\right\rangle\frac{\partial \gamma}{\partial x}\right),\end{aligned}$$

a degenerate system of nonlinear parabolic equations.

If, instead, we view $\gamma$ locally as the graph of a family of functions $u(\cdot, t) : U \subset \mathbb{R} \to \mathbb{R}$, then curve shortening flow (in the plane) becomes

$$u_t = \frac{u_{xx}}{1 + (u_x)^2}\,.$$

If we view the (planar) curves $\gamma$, assumed now to be convex, as the $t$-level sets of a function $u : \mathbb{R}^2 \to \mathbb{R}$, then $u$ satisfies the LEVEL SET (CURVE SHORTENING) FLOW

$$-|Du|\,\mathrm{div}\left(\frac{Du}{|Du|}\right) = 1\,.$$

The curve shortening flow has much better convergence properties than other geometric flows. Indeed, Grayson (1987), building on work of Gage–Hamilton (1986), proved that every closed embedded planar curve converges to a single point (becoming round in the process) under curve shortening flow. In fact, this behaviour holds in more general ambient spaces (e.g. compact surfaces), leading to numerous geometric applications (e.g. simple proofs of the Lusternik–Schnirelmann theorem on the existence of at least three closed geodesics on any Riemannian two-sphere $(S^2, g)$, and Smale's theorem on the retractibility of $\mathrm{Diff}(S^2)$ to $SO(3)$).

The curve shortening flow also improves the isoperimetric ratio of an initial embedded curve, leading to further applications.

The curve shortening flow is also a model for grain boundaries in annealing metals and the shapes of worn stones (in two dimensions). It has found useful applications in image processing as it can be implemented to smooth the boundaries between shapes, making them easier for computer software to recognize.

There are many higher dimensional generalizations of the curve shortening flow, where the normal speed $\kappa$ is replaced by some function $F(A)$ of the shape operator $A$ satisfying certain structure conditions. Important examples are the MEAN CURVATURE FLOW of submanifolds (for which the velocity is the mean curvature vector) and the GAUSS CURVATURE FLOW of convex hypersurfaces (for which the inward normal speed is the Gauss curvature). The mean curvature flow is a model for many dynamical processes which involve surface tension (such as soap films and black-hole horizons); the Gauss curvature flow is a model for wearing processes (such as lens making, or the erosion of pebbles on a beach). These higher dimensional flows generally encounter singularities before smooth convergence (possibly after rescaling) to a model space, in contrast to curve shortening flow. So called "weak" formulations of these flows, such as their level set flows, are able to extend solutions beyond singularities, at the expense of losing topological information across singular times.

A powerful tool for studying singularities in curvature flows is the so-called "blow-up" method, whereby one "zooms in" at the singularity — rescaling so that the curvature becomes normalized. In the limit, we find a regular solution, often of a very special nature. Particularly important special solutions are the TRANSLATING SOLUTIONS (or TRAVELLING WAVE SOLUTIONS), which satisfy the TRANSLATOR EQUATION

$$F(A) = \langle \nu \, , \, e_{n+1} \rangle$$

at any particular time, and move through space by translation with constant velocity $e_{n+1}$. In the graphical case, the translator equation becomes the GRAPHICAL TRANSLATOR EQUATION

$$F(Du, D^2u) = \frac{1}{\sqrt{1+|Du|^2}} \, .$$

For example, when $F(A)$ is the mean curvature, we obtain

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{1}{\sqrt{1+|Du|^2}} \, .$$

**0.12. Ricci flow and the Poincaré conjecture.** The Ricci flow is an evolution equation for a Riemannian manifold $(M, g)$ which deforms the metric $g$ in the direction of its Ricci tensor Rc. That is,

$$\partial_t g = -2\,\mathrm{Rc}\,.$$

Appropriately viewed, this is a parabolic system of differential equations. In fact, it is a very natural analogue of the heat equation. Indeed, in *harmonic coordinates* $\{x^i\}_{i=1}^n$, the Ricci tensor takes the form

$$\mathrm{Rc}_{ij} = -\frac{1}{2} g^{kl}\partial_k\partial_l g_{ij} + Q_{ij}(g, \partial g)\,,$$

where $Q_{ij}(g, \partial g)$ denotes terms that are quadratic in the components of $g$ and their first partial derivatives.

In two dimensions, the Ricci curvature is equal to $\frac{1}{2}\,\mathrm{R}\,g$, where R is the scalar curvature. So the two-dimensional Ricci flow becomes

$$\partial_t g = -\,\mathrm{R}\,g\,.$$

Since the right hand side is, in this case, just a multiple of $g$, we can expect the flow to preserve the conformal structure. In a neighbourhood of any point of $M$, there exist "isothermal" coordinates $(x, y)$, which just means that the metric takes the form $g = \mathrm{e}^{2u}(dx^2 + dy^2)$. In such coordinates, the scalar curvature is equal to $\mathrm{R} = -\mathrm{e}^{-2u}\Delta u$, where $\Delta = \partial_x^2 + \partial_y^2$ is the coordinate Laplacian. So we can write the Ricci flow locally in this chart as the LOGARITHMIC FAST DIFFUSION EQUATION

$$\partial_t u = \mathrm{e}^{-2u}\Delta u\,.$$

Ricci flow is motivated by the principle that diffusion equations drive solutions towards equilibria, where the diffusing quantity becomes maximally diffuse. So the hope is that the Ricci flow should diffuse the metric, making it more and more homogeneous. Since the process is smooth and homogeneous spaces are easier to understand than general spaces, this should lead to valuable topological information about the initial manifold.

This philosophy has yielded spectacular advances in some situations (but is far too optimistic in the general case). For instance, Chow (1991) and Hamilton (1988) proved that the Ricci flow of an initial metric on a closed surface always converges, after appropriate time-dependent rescaling, to a metric of constant curvature.

In three and higher dimensions, the Ricci flow on closed manifolds generally encounters curvature singularities before converging 'nicely', except in certain special cases (for instance when the initial metric has positive Ricci curvature; Hamilton 1982). Overcoming singularity formation was the key to Hamilton's programme for proving the Poincaré conjecture (Poincaré 1904) and the more general geometrization conjecture (Thurston 1982). This programme was finally completed by Perelman in 2003.

## 1. Potential theory

We begin with a quick review of basic potential theory. For a more in depth discussion, including a number of details we have left out, see [**2**, Chapter 2].

A function $u \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^n$ is said to be HARMONIC if it satisfies the LAPLACE EQUATION:

$$-\Delta u = 0\,,$$

where

$$\Delta \doteqdot \operatorname{div} \circ \operatorname{grad} = \operatorname{tr} \circ \operatorname{Hess} = \sum_{i=1}^{n} \frac{\partial^2}{(\partial x^i)^2}$$

is the LAPLACIAN. Given any $f \in C^0(\Omega)$, the inhomogeneous Laplace equation

$$-\,\Delta u = f \tag{1.1}$$

for $u \in C^2(\Omega)$ is called POISSON'S EQUATION.

A function $u \in C^2(\Omega)$ is SUBHARMONIC (resp. SUPERHARMONIC) if

$$-\Delta u \leq 0 \ \ (\text{resp. } -\Delta u \geq 0)\,.$$

By the divergence theorem, any subharmonic (resp. superharmonic) function $u \in C^2(B_r(x_0))$ satisfies

$$\begin{aligned} -\int_{\partial B_r(x_0)} \nabla u \cdot N \, d\sigma &= -\int_{B_r(x_0)} \operatorname{div}(\nabla u)\, d\mathcal{L} \\ &= -\int_{B_r(x_0)} \Delta u \, d\mathcal{L} \\ &\leq (\text{resp. } \geq)\, 0\,, \end{aligned} \tag{1.2}$$

where $N$ is the outward unit normal field to $\partial\Omega$, $\nabla u \doteqdot \operatorname{grad} u$, $\mathcal{L}$ denotes the Lebesgue measure on $\mathbb{R}^n$ and $\sigma$ denotes the standard spherical measure on $\partial B_r(x_0)$. In particular, equality holds for a harmonic function.

**Theorem 1.1** (Mean value theorem)**.** *If $u \in C^2(B_r(x_0))$ is subharmonic (resp. superharmonic), then*

$$u(x_0) \leq (\textit{resp. } \geq) \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u \, d\mathcal{L}$$

*and*

$$u(x_0) \leq (\textit{resp. } \geq) \frac{1}{|\partial B_r(x_0)|} \int_{\partial B_r(x_0)} u \, d\sigma\,.$$

*In particular, if $u$ is harmonic, then equality holds in both cases.*

**Proof.** Exercise. (Hint: evaluate the integral in polar coordinates and apply (1.2).) □

The following is a straightforward consequence of the mean value inequalities.

**Theorem 1.2** ((Strong) maximum principle)**.** *Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a subharmonic (resp. superharmonic) function. If $u(x) = \sup_\Omega u$ (resp. $u(x) = \inf_\Omega u$) for some interior $x \in \Omega \setminus \partial\Omega$, then $u$ is constant on the connected component of $\Omega$ containing $x$.*

**Proof.** It suffices to consider the subharmonic case.

Set $M \doteqdot \sup_\Omega u$ and define $\Omega_M \doteqdot \{x \in \Omega \setminus \partial\Omega : u(x) = M\}$. Since $u$ is continuous, $\Omega_M$ is closed. By hypothesis, $\Omega_M$ is non-empty. Given $x \in \Omega_M$, there exists $r > 0$ such that $B_r(x) \subset \Omega$, so that $u$ is subharmonic on $B_r(x)$. Since $\Delta M = 0$, $u - M$ is also subharmonic on $B_r(x)$ and the mean value theorem yields

$$0 = u(x) - M \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} (u(x) - M) d\mathcal{L} \leq 0\,,$$

and hence $u(x) \equiv M$ on $B_r(x)$. This means that $B_r(x) \subset \Omega_M$, and we deduce that $\Omega_M$ is open. The claim follows. $\square$

The following two corollaries are immediate applications of the strong maximum principle.

**Corollary 1.3** ((Weak) maximum principle)**.** *Let $u \in C^2(\Omega)$ be a subharmonic (resp. superharmonic) function. If $\Omega$ is bounded, then*

$$\sup_\Omega u \leq \sup_{\partial\Omega} u \;\; (\textit{resp. } \inf_\Omega u \geq \inf_{\partial\Omega} u)\,.$$

**Corollary 1.4.** *If $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy $\Delta u = \Delta v$ and $u|_{\partial\Omega} \equiv v|_{\partial\Omega}$, then $u \equiv v$.*

A further consequence of the mean value theorem is the Harnack inequality for harmonic functions.

**Theorem 1.5** (Harnack inequality)**.** *Given $\Omega \subset \mathbb{R}^n$ and any connected $\Omega' \Subset \Omega$, there is a constant $C < \infty$ such that any non-negative subharmonic function $u \in C^2(\Omega)$ satisfies*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u\,.$$

**Proof.** Consider any $x \in \Omega$ and $r > 0$ such that $B_{4r}(x) \subset \Omega$. Given any $\underline{x}, \overline{x} \in B_r(x)$, the mean value inequalities give

$$u(\overline{x}) = \frac{1}{|B_r(\overline{x})|} \int_{B_r(\overline{x})} u\, d\mathcal{L} \leq \frac{1}{|B_r(\overline{x})|} \int_{B_{2r}(x)} u\, d\mathcal{L}$$

and

$$u(\underline{x}) = \frac{1}{|B_{3r}(\underline{x})|}\int_{B_{3r}(\underline{x})} u\,d\mathcal{L} \geq \frac{1}{|B_{3r}(\underline{x})|}\int_{B_{2r}(x)} u\,d\mathcal{L}$$

Consequently,

$$\sup_{B_r(x)} u \leq \frac{|B_{3r}(\underline{x})|}{|B_r(\overline{x})|}\inf_{B_r} u = 3^n \inf_{B_r} u\,.$$

Now choose $\overline{x}$ and $\underline{x}$ in $\overline{\Omega}'$ so that $u(\overline{x}) = \max_{\overline{\Omega}'} u$ and $u(\underline{x}) = \min_{\overline{\Omega}'} u$ and choose $r > 0$ so that $\operatorname{dist}(\Omega', \mathbb{R}^n \setminus \Omega) > 4r$. By the Heine–Borel theorem, $\Omega'$ is covered by a finite number of balls of radius $r$. Some subset of these balls certainly covers a path from $\underline{x}$ to $\overline{x}$. Taking $N$ to be the smallest number such that there are $N$ balls with this property (which will thus depend only on $\Omega$ and $\Omega'$), we find, by applying the above inequality $N$ times, that

$$\sup_{\Omega'} u \leq 3^{Nn} \inf_{\Omega'} u\,. \qquad \square$$

Corollary 1.4 establishes, in particular, that harmonic functions (and, more generally, solutions to the Poisson equation) defined in bounded domains are uniquely determined by their boundary values (so long as they are continuous up to the boundary of their domains). *Existence* of solutions with prescribed boundary data is a little more difficult (but nonetheless possible) to establish. In fact, we will only do so in case $\Omega$ is a ball. (Existence of solutions over more general domains can be obtained from the existence result on balls via Perron's method, which we describe in §4.)

First observe that the function

$$\Gamma(x) \doteqdot \begin{cases} \dfrac{|x|^{2-n}}{(2-n)|\partial B_1^n|} & \text{if } n \geq 3 \\[2ex] \dfrac{1}{2\pi}\log|x| & \text{if } n = 2 \end{cases}$$

is harmonic in $\mathbb{R}^n \setminus \{0\}$ (and hence the function $x \mapsto \Gamma(x-y)$ is harmonic in $\mathbb{R}^n \setminus \{y\}$ for any $y \in \mathbb{R}^n$). The function $\Gamma$ is called the FUNDAMENTAL SOLUTION to the Laplace equation.

By carefully applying the divergence theorem, it is possible to derive the remarkable representation formula (called GREEN'S REPRESENTATION FORMULA)

$$\begin{aligned} u(y) = &\int_B \Gamma(x-y)\Delta u(x)\,\mathcal{L}(x) \\ &+ \int_{\partial B}\big(u(x)\nabla_N\Gamma(x-y) - \Gamma(x-y)\nabla_N u(x)\big)d\sigma(x) \end{aligned} \tag{1.3}$$

for any $u \in C^2(B) \cap C^1(\overline{B})$, where $B$ is any ball and $N$ is its outward unit normal. Fixing a centre and taking the radius of $B$ to infinity, we obtain

$$u(y) = \int_{\mathbb{R}^n} \Gamma(x-y)\Delta u(x)\,\mathcal{L}(x) \tag{1.4}$$

for any $u \in C^2(\mathbb{R}^n)$ with compact support. On the other hand, for any harmonic $u \in C^2(B) \cap C^1(\overline{B})$, Green's representation formula yields

$$u(y) = \int_{\partial B} \big(u(x)\nabla_N \Gamma(x-y) - \Gamma(x-y)\nabla_N u(x)\big) d\sigma(x)\,.$$

In particular, this implies that any harmonic $u \in C^2(B)$ is analytic in $B$! Note that the right hand side is just the CONVOLUTION $\Gamma * \Delta u$ of $\Gamma$ and $\Delta u$.

Conversely, if $f \in C^0(\mathbb{R}^n)$ is compactly supported, then the NEWTONIAN POTENTIAL (corresponding to $f$),

$$\gamma_f(y) \doteqdot -\int_{\mathbb{R}^n} \Gamma(x-y) f(x)\,\mathcal{L}(x)\,,$$

is a smooth solution to Poisson's equation (1.1) on $\mathbb{R}^n$, and hence on any domain[4] $\Omega \subset \mathbb{R}^n$. This solution is not unique, however, since adding any harmonic function $h \in C^2(\Omega)$ to $\gamma_f|_\Omega$ yields another solution to Poisson's equation on $\Omega$. On the other hand, given boundary data $\phi \in C^0(\partial\Omega)$, if we are able to find a harmonic function $h_\phi \in C^2(\Omega) \cap C^0(\overline{\Omega})$ taking boundary data $\phi - \gamma_f|_{\partial\Omega}$, then $\gamma_f + h_\phi$ satisfies Poisson's equation with boundary data $\phi$. In fact, by Corollary 1.4, it is the unique solution of class $C^2(\Omega) \cap C^0(\overline{\Omega})$.

Next observe that, by adding a harmonic function $h$ to $\Gamma$, we obtain the more general representation formula

$$\begin{aligned} u(y) = &\int_B G(x,y)\Delta u(x)\,\mathcal{L}(x) \\ &+ \int_{\partial B} \big(u(x)\nabla_N G(x,y) - G(x,y)\nabla_N u(x)\big) d\sigma(x)\,, \end{aligned}$$

where $G(x,y) \doteqdot \Gamma(x-y) + h(x)$. If we can determine a harmonic function $h$ such that $G(\cdot, y)$ vanishes on $\partial B$, then, for a harmonic $u$, this will become

$$u(y) = \int_{\partial B} u(x)\nabla_N G(x,y) d\sigma(x)\,,$$

a representation formula for harmonic functions with prescribed boundary data. The function $G$, if it exists, is unique by Corollary 1.4, and is known as GREEN'S FUNCTION (for the ball $B$).

Constructing a suitable (explicit) harmonic function $h$ for the unit ball is typically achieved by the *method of images*. In short, we find that any

[4]Note that, if $\Omega \subset \mathbb{R}^n$ is bounded and $f \in C^0(\overline{\Omega})$, then $f$ admits a continuous extension to $\mathbb{R}^n$ which is compactly supported in some large ball.

harmonic function $u \in C^2(B_1^n) \cap C^1(\overline{B}_1^n)$ satisfies the POISSON INTEGRAL FORMULA

$$u(y) = \int_{\partial B_1^n} K(x,y)u(x)\,d\sigma(x)\,,$$

where the POISSON KERNEL $K = \nabla_N G$ is defined by

$$K(x,y) \doteqdot \frac{1-|x|^2}{|\partial B_1^n|}\frac{1}{|x-y|^n}\,.$$

In fact, a little more work (an approximation argument) reveals that the Poisson integral formula actually holds for all harmonic $u \in C^2(B_1^n) \cap C^0(\overline{B}_1^n)$. Moreover, with not too much effort, we obtain the converse:

**Theorem 1.6.** *Given any $\phi \in C^0(B_1^n)$ the function $u : \overline{B}_1^n \to \mathbb{R}$ defined by*

$$u(x) \doteqdot \begin{cases} \displaystyle\int_{\partial B_1^n} K(x,y)\phi(x)\,d\sigma(x) & \text{if } x \in B_1^n \\ \phi(x) & \text{if } x \in \partial B_1^n \end{cases}$$

*belongs to $C^2(B_1^n) \cap C^0(\overline{B}_1^n)$ and is harmonic in $B_1^n$.*

Note that, although we only stated the result for the unit ball centred at the origin in $\mathbb{R}^n$, it applies to any ball $B_r(x)$ since we may translate and scale boundary data on $B_r(x)$ to obtain boundary data on $B_1^n$, apply the result to obtain a solution on $B_1^n$, and scale and translate back to obtain the desired solution on $B_r(x)$.

We conclude that, for any ball $B \subset \mathbb{R}^n$, any $f \in C^0(\overline{B})$, and any $\phi \in C^0(\partial B)$, there exists a unique solution $u \in C^2(B) \cap C^0(\overline{B})$ to the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } B \\ \quad u = \phi & \text{on } \partial B\,. \end{cases}$$

Moreover, if $f = 0$, then $u$ is analytic in $B$.

### 1.1. Exercises.

**Exercise 1.1.** Let $u \in C^2(\Omega)$ be a harmonic function defined in $\Omega \subset \mathbb{R}^n$. Convince yourself that the function $u_{a,\lambda,R,v} \in C^2(\Omega_{\lambda,R,v})$ is harmonic for any $a \in \mathbb{R}$, $\lambda > 0$, $R \in O(n)$ and $v \in \mathbb{R}^n$, where

$$u_{a,\lambda,R,v}(x) \doteqdot au(\lambda R(x-v)) \quad \text{and} \quad \Omega_{\lambda,R,v} \doteqdot \{R^{-1}\lambda^{-1}x + v : x \in \Omega\}\,.$$

**Exercise 1.2.** Prove the mean value inequalities (Theorem 1.1).

## 2. The maximum principle

The most important tool in the study of elliptic PDE of second order is the MAXIMUM PRINCIPLE.

**2.1. The weak maximum principle.** The WEAK MAXIMUM PRINCIPLE guarantees that subsolutions to certain linear elliptic PDE necessarily attain their maxima at the boundary of their domain.

Denote by $S^{n\times n}$ the set of symmetric $n\times n$ matrices.

**Theorem 2.1** (The (weak) maximum principle)**.** *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set. Suppose that the coefficients $(a,b,c):\Omega\to S^{n\times n}\times\mathbb{R}^n\times\mathbb{R}$ satisfy*

$$a^{ij}(x)\xi_i\xi_j\geq 0 \;\text{ for all }\; x\in\Omega \;\text{ and all }\; \xi\in\mathbb{R}^n\,, \tag{2.1}$$

$$c(x)\leq 0 \;\text{ for all }\; x\in\Omega\,, \tag{2.2}$$

*and*

$$a^{ij}(x)\xi_i\xi_j+b^k(x)\xi_k+c(x)>0 \;\text{ for all }\; x\in\Omega \;\text{ for some }\; \xi\in\mathbb{R}^n\,. \tag{2.3}$$

*If $u\in C^2(\Omega)\cap C^0(\overline{\Omega})$ is a subsolution to the corresponding equation, that is,*

$$-\left(a^{ij}u_{ij}+b^iu_i+cu\right)\leq 0 \;\text{ in }\; \Omega\,,$$

*then $u\leq\max_{\partial\Omega}u_+$, where $u_+\doteqdot\max\{u,0\}$ is the non-negative part of $u$.*

A linear operator $L=a^{ij}D_iD_j+b^iD_i+c$ (or equation $-Lu=f$) satisfying (2.4) is called ELLIPTIC (or WEAKLY ELLIPTIC). If the inequality (2.4) is strict, we call $L$ STRICTLY (or LOCALLY UNIFORMLY) ELLIPTIC.

**Proof of Theorem 2.1.** Given $\varepsilon>0$, set $u_\varepsilon(x)\doteqdot u(x)+\varepsilon e^{x\cdot\xi}$, where $\xi\in\mathbb{R}^n$ is chosen so that

$$a^{ij}(x)\xi_i\xi_j+b^k(x)\xi_k+c(x)>0 \;\text{ for all }\; x\in\Omega\,.$$

We claim that $u_\varepsilon\leq\max_{\partial\Omega}(u_\varepsilon)_+$ in $\Omega$. Suppose, to the contrary, that

$$u_\varepsilon(x_0)>\max_{\partial\Omega}(u_\varepsilon)_+$$

for some point $x_0\in\Omega$. We may assume that $x_0$ is a local maximum of $u_\varepsilon$. But then, at $x_0$,

$$\begin{aligned}
0\leq&\;-(a^{ij}D_iD_j+b^kD_k)u_\varepsilon\\
=&\;-(a^{ij}D_iD_j+b^kD_k)u-\varepsilon(a^{ij}D_iD_j+b^kD_k)e^{x\cdot\xi}\\
\leq&\; cu-\varepsilon e^{x\cdot\xi}\left(a^{ij}\xi_i\xi_j+b^k\xi_k\right)\\
=&\; cu_\varepsilon-\varepsilon e^{x\cdot\xi}\left(a^{ij}\xi_i\xi_j+b^k\xi_k+c\right)\\
<&\;0\,,
\end{aligned}$$

which is absurd. We conclude that $u_\varepsilon \leq \max_{\partial\Omega}(u_\varepsilon)_+$ in $\Omega$ for all $\varepsilon > 0$. The claim follows since $\varepsilon$ was arbitrary. $\square$

We emphasize that no further conditions (e.g. continuity or regularity) on the coefficients beyond (2.1)–(2.3) are required in Theorem 2.1. Note that (2.1) and (2.3) are implied by the pair of conditions

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \text{ for all } x \in \Omega \text{ and all } \xi \in \mathbb{R}^n\,, \tag{2.4}$$

and

$$\lambda^{-2}|b(x)|^2 - \lambda^{-1}c(x) \leq \Lambda \text{ for all } x \in \Omega \tag{2.5}$$

for some $\lambda > 0$ and $\Lambda < \infty$. An operator $L = a^{ij}D_iD_j + b^iD_i + c$ (or equation $-Lu = f$) satisfying (2.4) is called UNIFORMLY ELLIPTIC.

Since the difference between any two solutions to an affine linear equation satisfies the corresponding linear equation, the maximum principle immediately implies the following uniqueness property for the Dirichlet boundary value problem.

**Corollary 2.2.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Suppose that the coefficients $(a, b, c) : \Omega \to S^{n\times n} \times \mathbb{R}^n \times \mathbb{R}$ satisfy* (2.1)–(2.3). *Given any $f : \Omega \to \mathbb{R}$ and any $\varphi \in C^0(\partial\Omega)$, there exists at most one $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying the Dirichlet problem*

$$\begin{cases} -\left(a^{ij}u_{ij} + b^iu_i + cu\right) = f & \text{in } \Omega \\ \hfill u = \varphi & \text{on } \partial\Omega\,. \end{cases}$$

The maximum principle is also a useful tool in proving *a priori* estimates. We illustrate this with the following estimate, which we will need later when we solve the Dirichlet problem for linear elliptic equations.

**Proposition 2.3.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with* $\operatorname{diam}(\Omega) \leq R$. *Suppose that the coefficients $(a, b, c) : \Omega \to S^{n\times n} \times \mathbb{R}^n \times \mathbb{R}$ satisfy* (2.1)–(2.3). *In particular,*

$$\lambda \doteqdot a^{ij}\xi_i\xi_j + b^i\xi_i + c > 0 \text{ in } \Omega$$

*for some $\xi \in \mathbb{R}^n$. If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies*

$$-\left(a^{ij}u_{ij} + b^iu_i + cu\right) \leq f \text{ in } \Omega\,,$$

*then*

$$\sup_\Omega u \leq \max_{\partial\Omega} u_+ + C\sup_\Omega \frac{f_+}{\lambda}\,,$$

*where* $C \doteqdot \mathrm{e}^{|\xi|R}$.

**Proof.** Consider the function

$$v \doteqdot u - \max_{\partial\Omega} u_+ - \mathrm{e}^{-\underline{d}}\left(\mathrm{e}^{\overline{d}} - \mathrm{e}^{x\cdot\xi}\right)\sup_\Omega \frac{f_+}{\lambda}\,,$$

where $\overline{d} \doteqdot \sup_{x\in\Omega} \xi \cdot x$ and $\underline{d} \doteqdot \inf_{x\in\Omega} \xi \cdot x$. Observe that

$$\begin{aligned} -\left(a^{ij}v_{ij} + b^i v_i + cv\right) &\le f - \left(a^{ij}D_iD_j + b^iD_i + c\right) \mathrm{e}^{x\cdot\xi - \underline{d}} \sup_\Omega \frac{f_+}{\lambda} \\ &= f - \left(a^{ij}\xi_i\xi_j + b^i\xi_i + c\right) \mathrm{e}^{x\cdot\xi - \underline{d}} \sup_\Omega \frac{f_+}{\lambda} \\ &\le 0\,. \end{aligned}$$

Since $v|_{\partial\Omega} \le 0$, the maximum principle (Theorem 2.1) yields $v \le 0$ in $\Omega$, and hence

$$\begin{aligned} \sup_\Omega u &\le \max_{\partial\Omega} u_+ + \sup_\Omega \frac{f_+}{\lambda} \sup_\Omega \left(\mathrm{e}^{\overline{d}-\underline{d}} - \mathrm{e}^{x\cdot\xi-\underline{d}}\right) \\ &\le \max_{\partial\Omega} u_+ + \mathrm{e}^{|\xi|R} \sup_\Omega \frac{f_+}{\lambda}\,. \end{aligned}$$

□

**2.2. The strong maximum principle.** The weak maximum principle guarantees that solutions to certain linear elliptic PDE attain their maximum at the boundary. The *strong maximum principle* guarantees that the maximum cannot also be attained in the interior, except in the extreme case that the solution is constant. It is a consequence of the *Hopf lemma.*

**Theorem 2.4** (Hopf Lemma)**.** *Let $\Omega = B_R(p)$ be the open ball in $\mathbb{R}^n$ of radius $R$ centred at $p \in \mathbb{R}^n$. Suppose that the coefficients $(a, b, c) : B_R(p) \to S^{n\times n} \times \mathbb{R}^n \times \mathbb{R}$ satisfy, for some $\lambda > 0$ and $\Lambda < \infty$,*

$$a^{ij}(x)\xi_i\xi_j \ge \lambda|\xi|^2 \ \textit{ for all } \ x \in B_R(p) \ \textit{ and all } \ \xi \in \mathbb{R}^n \tag{2.6}$$

$$c(x) \le 0 \ \textit{ for all } \ x \in B_R(p)\,, \tag{2.7}$$

*and*

$$\lambda^{-1}\operatorname{tr}(a(x)) \le \Lambda \ \textit{ and } \ \lambda^{-2}|b(x)|^2 - \lambda^{-1}c(x) \le \Lambda \ \textit{ for all } \ x \in B_R(p). \tag{2.8}$$

*Let $u \in C^2(B_R(p)) \cap C^1(\overline{B_R(p)})$ be a subsolution to the corresponding equation:*

$$-\left(a^{ij}u_{ij} + b^iu_i + cu\right) \le 0 \ \textit{ in } \ B_R(p)\,.$$

*If, for some $x_0 \in \partial B_R(p)$, $u(x) < u(x_0) = 0$ for all $x \in B_R(p)$, then*

$$Du|_{x_0} \cdot \nu(x_0) > 0\,,$$

*where $\nu(x_0) = \frac{x_0-p}{|x_0-p|}$ is the outer unit normal to $B_R(p)$ at $x_0$.*

**Proof.** It suffices to consider the case that $B_R(p) = B_1$ is the unit ball centred at the origin. Indeed, if $u$ is defined on $B_R(p)$, then we may consider the function $\tilde{u}$ defined on $B_1$ by

$$\tilde{u}(x) \doteqdot u(p + Rx)\,.$$

This function satisfies the equation

$$-\left(\tilde{a}^{ij}\tilde{u}_{ij}+\tilde{b}^{i}\tilde{u}_{i}+\tilde{c}\tilde{u}\right)\leq 0 \;\text{ in }\; B_1(0)\,,$$

where

$$\tilde{a}^{ij}(x)\doteqdot R^{-2}a^{ij}(p+Rx)\,,\;\; \tilde{b}^{k}(x)\doteqdot R^{-1}b^{k}(p+Rx) \;\text{ and }\; \tilde{c}(x)\doteqdot c(p+Rx)\,.$$

So the hypotheses of the theorem are met by $\tilde{u}$ and the new coefficients, with $\lambda$ replaced by $\tilde{\lambda}=\lambda R^{-2}$. Supposing then that the theorem is proved in case $B=B_1(0)$, we conclude, for a point $p+Rx_0\in\partial B_R(p)$ satisfying $u(p+Rx)<u(p+Rx_0)=0$ for all $x\in B_1(0)$, that

$$0<D\tilde{u}|_{x_0}\cdot\tilde{\nu}(x_0)=RDu|_{p+Rx_0}\cdot\nu(p+Rx_0)\,.$$

So let us continue under the assumption $B_R(p)=B_1$. Given $\varepsilon>0$ and $\mu>0$, define the function

$$u_{\varepsilon,\mu}(x)\doteqdot u(x)+\varepsilon\left(\mathrm{e}^{-\mu|x|^2}-\mathrm{e}^{-\mu}\right)\,.$$

Observe that

$$\begin{aligned}
-\big(a^{ij}&D_iD_j+b^iD_i+c\big)u_{\varepsilon,\mu}\\
&\leq\;-\varepsilon\left(a^{ij}D_iD_j+b^iD_i+c\right)\left(\mathrm{e}^{-\mu|x|^2}-\mathrm{e}^{-\mu}\right)\\
&=\;-\varepsilon\mathrm{e}^{-\mu|x|^2}\left(4\mu^2a^{ij}x_ix_j-2\mu a^{ij}\delta_{ij}-2\mu b^kx_k+c\right)+\varepsilon c\mathrm{e}^{-\mu}\\
&\leq\;-\varepsilon\mathrm{e}^{-\mu|x|^2}\left(3\mu^2\lambda|x|^2-2\mu\,\mathrm{tr}(a)-\lambda^{-1}|b|^2+c\left(1-\mathrm{e}^{-\mu(1-|x|^2)}\right)\right)\\
&\leq\;-\varepsilon\mathrm{e}^{-\mu|x|^2}\left(3\mu^2\lambda|x|^2-2\mu\lambda\Lambda-\lambda\Lambda\right)\,.
\end{aligned}$$

where we made use of the Cauchy–Schwarz inequality to estimate

$$2\mu b^kx_k\leq\mu^2\lambda|x|^2+\lambda^{-1}|b|^2\,.$$

So we may choose $\mu$ sufficiently large that

$$-\big(a^{ij}D_iD_j+b^iD_i+c\big)u_{\varepsilon,\mu}\leq 0 \;\text{ for all }\; x\in B_1\setminus B_{1/2}\,.$$

On the other hand, since $u(x)<0$ in $B_1$, we can find $\delta>0$ such that $u(x)<-\delta$ in $\overline{B}_{1/2}$, and hence choose $\varepsilon$ sufficiently small that

$$\begin{aligned}
u_{\varepsilon,\mu}&=u+\varepsilon\left(\mathrm{e}^{-\mu/4}-\mathrm{e}^{-\mu}\right)\\
&\leq\;-\delta+\varepsilon\left(\mathrm{e}^{-\mu/4}-\mathrm{e}^{-\mu}\right)\\
&<0
\end{aligned}$$

on $\partial B_{1/2}$. Since, by continuity of $u$, $u_{\varepsilon,\mu}\leq 0$ on $\partial B_1$, the maximum principle implies that $u_{\varepsilon,\mu}\leq 0$ in $B_1\setminus B_{1/2}$. Since $u_{\varepsilon,\mu}(x_0)=0$, we conclude that $x_0$

is a maximum of $u_{\varepsilon,\mu}$ in $B_1 \setminus B_{1/2}$ and hence

$$\begin{aligned} 0 \le Du_{\varepsilon,\mu}|_{x_0} \cdot \nu(x_0) = & \left(Du - 2\varepsilon\mu e^{-\mu|x|^2}x\right)\Big|_{x_0} \cdot \nu(x_0) \\ &= Du|_{x_0} \cdot \nu(x_0) - 2\varepsilon\mu e^{-\mu}\,. \end{aligned}$$

The claim follows. □

**Theorem 2.5** (Strong maximum principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Suppose that the coefficients $(a, b, c) : \Omega \to S^{n\times n} \times \mathbb{R}^n \times \mathbb{R}$ satisfy* (2.6)–(2.8)*. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a non-positive subsolution to the corresponding equation:*

$$-\left(a^{ij}u_{ij} + b^i u_i + cu\right) \le 0 \quad in \ \ \Omega\,.$$

*If $u(x_0) = 0$ for some $x_0 \in \Omega$, then $u \equiv 0$ in $\Omega$.*

**Proof.** Set

$$D \doteqdot \{x \in \Omega : u(x) = 0\}\,.$$

By hypothesis, $D$ is non-empty. Since $u$ is continuous, $D$ is relatively closed in $\Omega$. So $\Omega \setminus D$ is open in $\Omega$.

Suppose, contrary to the claim, that $\Omega \setminus D$ is non-empty. Then we can find an open ball $B \subset \Omega \setminus D$ with $\overline{B} \cap D \neq \emptyset$ (take, for example, the largest ball in $\Omega \setminus D$ about a point whose distance to $\partial\Omega$ is greater than its distance to $D$).

Choose some $x_0 \in \overline{B} \cap D$. Since the hypotheses of the Hopf Lemma are satisfied at $x_0$, we conclude that

$$Du|_{x_0} \cdot \nu(x_0) > 0\,.$$

But this is impossible since $x_0 \in D$ is a local maximum of $u$. □

### 2.3. Exercises.

**Exercise 2.1.** Suppose that the coefficients $(a, b, c) : \Omega \to S^{n\times n} \times \mathbb{R}^n \times \mathbb{R}$ satisfy (2.4) and (2.5). Show that they also satisfy satisfy (2.1) and (2.3).

**Exercise 2.2.** Suppose that the coefficients $(a, b, c) : \Omega \to S^{n\times n} \times \mathbb{R}^n \times \mathbb{R}$ satisfy (2.1) and (2.3), and that $c \ge 0$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a positive supersolution to the corresponding linear equation. Show that $\min_\Omega u \ge \min_{\partial\Omega} u$.

**Exercise 2.3.** Prove Corollary 2.2.

**Exercise 2.4.** Suppose that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a subsolution to Poisson's equation on the ball $B_R$ of radius $R$ centred at the origin:

$$-\Delta u \le f \quad \text{in} \ \ B_R\,.$$

Prove that

$$\sup_{B_R} u \leq \sup_{\partial B_R} u_+ + \frac{1}{2n} R^2 \sup_{B_R} f_+ \,.$$

*Hint: Consider the function*

$$v \doteqdot u - \max_{\partial B_R} u_+ - \frac{1}{2n}(R^2 - |x|^2) \sup_{B_R} f_+ \,.$$

**Exercise 2.5** (Bernstein estimates)**.** Let $u : B_R \to \mathbb{R}$ be a harmonic function. Observe that

$$-\Delta |D^k u|^2 = -2|D^{k+1} u|^2 \,.$$

Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a smooth non-negative function satisfying the following conditions

- $\rho|_{\mathbb{R}^n \setminus B_R} \equiv 0$ and $\rho|_{B_{R/2}} \equiv 1$,
- $|D_i \rho|^2 \leq \frac{10\rho}{R^2}$ for each $i$, and
- $|D_i D_j \rho| \leq \frac{10}{R^2}$ for each $i, j$.

Note that such a function can be constructed by, say, mollifying the step function

$$\sigma_{B_{3R/4}}(x) \doteqdot \begin{cases} 0 & \text{if } x \in \mathbb{R}^n \setminus B_{3R/4} \\ 1 & \text{if } x \in B_{3R/4} \,. \end{cases}$$

For each $k \in \mathbb{N}$, set

$$Q_k \doteqdot \sum_{j=0}^{k} a_j R^{2j} \rho^j |D^j u|^2 \,,$$

where, say,

$$a_0 = 1 \quad \text{and} \quad a_j \doteqdot \frac{a_{j-1}}{10nj(3j+2)} \quad \text{for} \quad j \geq 1 \,.$$

(a) Show that

$$-\Delta Q_k \leq 0 \,.$$

*Hint: By the Cauchy–Schwarz inequality (with Peter giving $2j$ to Paul),*

$$-2D^{j+1} u \cdot \left( D^j u \otimes \frac{D\rho}{\rho} \right) \leq 2j \frac{|D\rho|^2}{\rho^2} |D^j u|^2 + \frac{1}{2j} |D^{j+1} u|^2$$

*wherever $\rho \neq 0$.*

(b) Deduce that

$$\max_{B_{R/2}} |D^j u| \leq A_j R^{-j} \,,$$

where

$$A_j \doteqdot \max_{\partial B_R} |u| a_j^{-\frac{1}{2}} \,.$$

(c) Deduce that there are no harmonic functions $u : \mathbb{R}^n \to \mathbb{R}$ of growth

$$|u(x)| \leq o(|x|) \text{ as } |x| \to \infty$$

other than the constant ones. This is known as LIOUVILLE'S THEOREM.

(d) Deduce, furthermore, that every smooth harmonic function $u : \Omega \to \mathbb{R}$ is actually analytic in $\Omega$.

## 3. Schauder's theorem

Schauder's[5] theorem, a fundamental result in the study of elliptic partial differential equations, states, roughly speaking, that solutions to uniformly elliptic affine linear equations are two *Hölder derivatives* more regular than their coefficients. It is considered a *perturbative result*, in that it relies on the fact that (under the conditions of the theorem) the equation closely resembles the Laplace equation at sufficiently small scales. The theorem consists of two results, the *interior estimate*, and the *boundary* (or *global*) *estimate*.

Schauder's theorem is phrased in terms of *Hölder spaces*. These spaces play a crucial role in the development of the theory of nonlinear partial differential equations, so it is worth reviewing them here.

**3.1. Hölder spaces.** Given $\alpha \in (0,1]$, a function $u : \Omega \to \mathbb{R}$ on a bounded open set $\Omega \subset \mathbb{R}^n$ is $\alpha$-HÖLDER CONTINUOUS AT $x \in \Omega$ if

$$\sup_{y\in\Omega\setminus\{x\}} \frac{|u(y)-u(x)|}{|y-x|^\alpha} < \infty\,.$$

A function $u : \Omega \to \mathbb{R}$ is then called $\alpha$-HÖLDER CONTINUOUS if it is $\alpha$-Hölder continuous at all points, and UNIFORMLY $\alpha$-HÖLDER CONTINUOUS if

$$[u]_{C^\alpha(\Omega)} \doteqdot \sup_{(x,y)\in\Omega\times\Omega\setminus\{(z,z):z\in\Omega\}} \frac{|u(y)-u(x)|}{|y-x|^\alpha} < \infty\,.$$

We simply call $u : \Omega \to \mathbb{R}$ (UNIFORMLY) HÖLDER CONTINUOUS (at $x$) if it is (uniformly) $\alpha$-Hölder continuous (at $x$) for some $\alpha \in (0,1]$.

Observe that, for each $\alpha \in (0,1]$, the set $C^\alpha(\Omega)$ of $\alpha$-Hölder continuous functions on $\Omega$ forms a linear space under pointwise addition and scalar multiplication. We equip $C^\alpha(\Omega)$ with the norm[6]

$$|u|_{C^\alpha(\Omega)} \doteqdot |u|_{C^0(\Omega)} + [u]_{C^\alpha(\Omega)}\,.$$

Similarly, we equip the linear space $C^{k,\alpha}(\Omega)$ of functions $u$ on $\Omega$ all of whose partial derivatives of order up to and including $k$ are $\alpha$-Hölder continuous with the norm

$$|u|_{C^\alpha(\Omega)} \doteqdot |u|_{C^k(\Omega)} + \sum_{|\beta|=k} [D^\beta u]_{C^\alpha(\Omega)}\,.$$

[5]Juliusz Schauder was a Polish mathematician of Jewish origin. After the invasion of German troops in Lwów 1941 it was impossible for him to continue his work. Even before the Lwów ghetto was established he wrote to Ludwig Bieberbach pleading for his support. Instead, Bieberbach passed his letter to the Gestapo and Schauder was arrested. In his letters to Swiss mathematicians, he wrote that he had important new results, but no paper to write them down. He was executed by the Gestapo, probably in October 1943.

[6]Note that $[\,\cdot\,]_{C^\alpha(\Omega)}$ is not a norm, since it vanishes on constant functions. Moreover, we allow $|u|_{C^\alpha(\Omega)} = \infty$, though $|u|_{C^\alpha(\Omega')} < \infty$ for every $u \in C^\alpha(\Omega)$ and $\Omega' \Subset \Omega$.

Recall that, for an arbitrary subset $F \subset \mathbb{R}^n$, a function $u : F \to \mathbb{R}$ is continuously differentiable if it extends to a continuously differentiable function on a neighbourhood of $F$. Using this definition, the spaces $C^{k,\alpha}(\overline{\Omega})$ can be defined as for $\Omega$, and in this case form Banach spaces.

The space $C^{k,\alpha}(F, \mathbb{R}^m)$ of functions $u$ from $F \subset \mathbb{R}^n$ into $\mathbb{R}^m$ all of whose partial derivatives of order up to and including $k$ have $\alpha$-Hölder continuous components, and its norm, are defined in the same manner.

The primary reason for the introduction of Hölder norms is that a sequence of functions $u_j$ which is bounded in the $C^k$ topology typically loses a derivative in passing to the limit (along a subsequence) by way of the Arzelà–Ascoli theorem. If the sequence is bounded in $C^{k,\alpha}$, $\alpha > 0$, then the sequence subconverges in $C^{k,\beta}$ for every $\beta < \alpha$. That is, we only need to give up a *fraction* of a derivative. See Exercise 3.5.

**3.2. The interior estimate.** The interior version provides an estimate away from the boundary of the domain.

**Theorem 3.1** (Interior Schauder)**.** *There exists $C = C(n, \lambda, \Lambda, \alpha, \rho, R) < \infty$ with the following property. Let $\Omega \subset \mathbb{R}^n$ be an open set satisfying $\operatorname{diam} \Omega \le R$ and let $a \in C^\alpha(\Omega, S^{n\times n})$, $b \in C^\alpha(\Omega, \mathbb{R}^n)$, $c \in C^\alpha(\Omega)$ be coefficients satisfying*

$$a^{ij} \ge \lambda \delta^{ij} \ \ \textit{in } \Omega\,, \ \ \lambda > 0$$

*and*

$$|a|_{C^\alpha(\Omega, S^{n\times n})} + |b|_{C^\alpha(\Omega,\mathbb{R}^n)} + |c|_{C^\alpha(\Omega)} \le \Lambda\,, \ \ \Lambda < \infty\,.$$

*Suppose that $u \in C^{2,\alpha}(\Omega)$ satisfies*

$$-(a^{ij}u_{ij} + b^i u_i + cu) = f\,.$$

*If $f \in C^\alpha(\Omega)$, then, given any $\Omega' \Subset \Omega$ satisfying $\operatorname{dist}(\Omega', \partial\Omega) \ge \rho$,*

$$|u|_{C^{2,\alpha}(\Omega')} \le C\left(|u|_{C^0(\Omega)} + |f|_{C^\alpha(\Omega)}\right)\,.$$

We first prove a basic Hölder estimate for the Hessian of $C^2$ solutions to the Poisson equation on $\mathbb{R}^n$, from which everything will follow.

**Lemma 3.2.** *There exists $C = C(n, \alpha) < \infty$ such that*

$$[D^2u]_{C^\alpha(\mathbb{R}^n)} \le C[\Delta u]_{C^\alpha(\mathbb{R}^n)}$$

*for every $u \in C^{2,\alpha}(\mathbb{R}^n)$ such that $\Delta u \in C^\alpha(\mathbb{R}^n)$.*

**Proof.** The proof proceeds by *reductio ad absurdum.* So suppose, contrary to the claim, that there exists, for each $\ell \in \mathbb{N}$, some $u_\ell \in C^{2,\alpha}(\mathbb{R}^n)$ such that $\Delta u_\ell \in C^\alpha(\mathbb{R}^n)$ and yet

$$[D^2u_\ell]_{C^\alpha} > \ell[\Delta u_\ell]_{C^\alpha}\,,$$

where $[\,\cdot\,]_{C^\alpha} \doteqdot [\,\cdot\,]_{C^\alpha(\mathbb{R}^n)}$. If we set $v_\ell \doteqdot \lambda_\ell u_\ell$, where $\lambda_\ell \doteqdot [D^2 u_\ell]_{C^\alpha}^{-1}$, then

$$[D^2 v_\ell]_{C^\alpha} = 1 \ \text{ and } \ [\Delta v_\ell]_{C^\alpha} \le \ell^{-1}\,.$$

By the pigeonhole principle, we can find $i, j, k \in \{1, \ldots, n\}$ such that, for infinitely many $\ell$, there exist $x_\ell \in \mathbb{R}^n$ and $h_\ell > 0$ for which

$$\frac{|D_iD_jv_\ell(x_\ell + h_\ell e_k) - D_iD_jv_\ell(x_\ell)|}{h_\ell^\alpha} \ge \frac{1}{2n^3}\,.$$

We now rescale once more: if we set

$$w_\ell(x) \doteqdot h_\ell^{-2-\alpha} v_\ell(x_\ell + h_\ell x) + p_\ell(x),$$

where $p_\ell$ is a quadratic polynomial, then

$$[D^2 w_\ell]_{C^\alpha} \le 1\,, \ \ [\Delta w_\ell]_{C^\alpha} \le \ell^{-1}\,, \ \text{ and } \ |D_iD_jw_\ell(e_k) - D_iD_jw_\ell(0)| \ge \frac{1}{2n^3}\,.$$

We choose $p_\ell$ so that $w_\ell$ vanishes to second order at the origin,

$$w_\ell(0) = 0\,, \ \ Dw_\ell(0) = 0 \ \text{ and } \ D^2w_\ell(0) = 0\,.$$

Applying the Arzelà–Ascoli theorem, we can find a subsequence of the functions $w_\ell$ which converges locally uniformly in $C^2(\mathbb{R}^n)$ to a *harmonic* function $w$ satisfying

$$D^2w(0) = 0 \ \text{ and } \ [D^2 w]_{C^\alpha} \le 1\,, \tag{3.1}$$

and yet

$$|D^2 w(e_k)| \ge \frac{1}{2n^3}\,.$$

But this is absurd: since $w$ is harmonic, (3.1) and the Bernstein estimates (see Exercise 2.5) imply that $D^2 w \equiv 0$. $\square$

We now prove Theorem 3.1 by *freezing the coefficients*, introducing cutoff functions, and applying Lemma 3.2. Note that, by a linear change of coordinates, Lemma 3.2 holds with $\Delta$ replaced by $a^{ij}D_iD_j$ for any positive definite $a \in S^{n\times n}$, with the constant $C$ now depending on $a$ (see Exercise 3.6).

**Proof of Theorem 3.1.** We will only prove the claim for $\Omega = B_2$ and $\Omega' = B_1$. Once this is proved, a simple scaling argument yields the claim for $\Omega = B_{2r}$ and $\Omega' = B_r$ for any $r > 0$. A covering argument then yields the claim for general domains $\Omega$ and $\Omega' \Subset \Omega$.

Fix any $x_0 \in B_1$ and set $a_0 \doteqdot a(x_0)$. In order to apply Lemma 3.2, we multiply $u$ with a cutoff function $\eta$ with support in the ball $B_r$ of radius $r$ about $x_0$, with $r \le 1$. Observe that $v \doteqdot \eta u$ satisfies

$$-a_0^{ij}v_{ij} = Lv + (a^{ij} - a_0^{ij})v_{ij} + b^i v_i + cv \ \text{ in } \ B_r\,,$$

where

$$L \doteqdot -(a^{ij}D_iD_j + b^iD_i + c)\,.$$

We may estimate

$$\begin{aligned}[(a^{ij}-a_0^{ij})v_{ij}]_{C^\alpha(B_r)} &\le [D^2v]_{C^\alpha(B_r)}|a-a_0|_{C^0(B_r)}+[a-a_0]_{C^\alpha(B_r)}|D^2v|_{C^0(B_r)}\\ &\le \Lambda r^\alpha[D^2v]_{C^\alpha(B_r)}+\Lambda|D^2v|_{C^0(B_r)}\,. \qquad (3.2)\end{aligned}$$

Thus, applying Lemma 3.2 (in conjunction with a coordinate transformation and a rescaling),

$$\begin{aligned}[D^2v]_{C^\alpha(B_r)} &= [D^2v]_{C^\alpha(\mathbb{R}^n)}\\ &\le C[a_0^{ij}v_{ij}]_{C^\alpha(\mathbb{R}^n)}\\ &= C[a_0^{ij}v_{ij}]_{C^\alpha(B_r)}\\ &= C[Lv+(a^{ij}-a_0^{ij})v_{ij}+b^iv_i+cv]_{C^\alpha(B_r)}\\ &\le C[Lv]_{C^\alpha(B_r)}+[(a^{ij}-a_0^{ij})v_{ij}]_{C^\alpha(B_r)}+[b^iv_i]_{C^\alpha(B_r)}+[cv]_{C^\alpha(B_r)}\end{aligned}$$

where $C=C(n,\alpha,\lambda,\Lambda)$ (i.e. $C$ depends only on $n,\lambda,\Lambda$ and $\alpha$). Applying (3.2) and choosing $r$ sufficiently small, we may absorb the first term on the right of (3.2). This yields an estimate of the form

$$[D^2v]_{C^\alpha(B_r)}\le C([Lv]_{C^\alpha(B_r)}+|v|_{C^2(B_r)})\,,$$

where $C=C(n,\alpha,\lambda,\Lambda,\rho)$.

Now, if we we arrange that $\eta\equiv 1$ in $B_{\sigma r}$, where $\sigma<1$, then

$$[D^2u]_{C^\alpha(B_{\sigma r})}=[D^2v]_{C^\alpha(B_{\sigma r})}\le[D^2v]_{C^\alpha(B_r)}\,.$$

On the other hand, we may crudely estimate

$$\begin{aligned}[Lv]_{C^\alpha(B_r)} &= [\eta Lu+(a^{ij}\eta_{ij}+b^i\eta_i)u+a^{ij}\eta_iu_j]_{C^\alpha(B_r)}\\ &\le C(|f|_{C^\alpha(B_r)}+|u|_{C^2(B_r)})\,,\end{aligned}$$

where $C=C(n,\lambda,\Lambda,\eta)$, and

$$|v|_{C^2(B_r)}\le C|u|_{C^2(B_r)}\,,$$

where $C=C(n,\eta)$. Thus, by covering $B_1$ with suitable balls of radius $r$ and choosing $\eta$ appropriately, we obtain

$$|u|_{C^{2,\alpha}(B_1)}\le C(|f|_{C^\alpha(B_{3/2})}+|u|_{C^2(B_{3/2})})\,, \qquad (3.3)$$

where $C=C(n,\alpha,\lambda,\Lambda)$.

The final step is to replace the $C^2$-norm on the right hand side by the $C^0$ norm using the following *interpolation inequality* (whose proof we omit):

$$|D^\ell w|_{C^0(B_1)}\le\varepsilon[D^kw]_{C^\alpha(B_1)}+C_\varepsilon|w|_{C^0(B_1)} \qquad (3.4)$$

for $1\le\ell\le k$, where $C_\varepsilon=C_\varepsilon(n,\alpha,k,\varepsilon)$. Though this does not look good enough, since the norms on the right are over the larger ball $B_{3/2}$, it can actually still be achieved by exploiting the fact that (3.3) holds for *all* solutions to equations satisfying the hypotheses.

Fix any $y \in B_2$, set $\sigma = \frac{1}{3}\operatorname{dist}(y, \partial B_2)$, and consider the function $\tilde{u} : B_2 \to \mathbb{R}$ defined by

$$\tilde{u}(x) \doteqdot u(y + \sigma x)\,.$$

Observe that

$$\tilde{L}\tilde{u}(x) = \sigma^2 Lu(y + \sigma x)\,,$$

where

$$\tilde{L}(x) \doteqdot a^{ij}(y + \sigma x)D_i D_j + \sigma b^i(y + \sigma x)D_i + \sigma^2 c(y + \sigma x)\,.$$

Since $y + \sigma B_2 \subset B_2$ and $\sigma \leq 1$, the estimate (3.3) yields

$$\begin{aligned} |\tilde{u}|_{C^{2,\alpha}(B_1)} &\leq C([\tilde{L}\tilde{u}]_{C^\alpha(B_2)} + |\tilde{u}|_{C^2(B_2)}) \\ &\leq C([Lu]_{C^\alpha(B_2)} + |u|_{C^0(B_2)} + \sigma^2|D^2u|_{C^0(B_{2\sigma}(y))})\,, \end{aligned} \tag{3.5}$$

where $C = C(n, \alpha, \lambda, \Lambda)$, and we used the standard interpolation inequality to estimate

$$2\sigma|Du|^2_{C^0(B_2)} \leq C(n)\big(|u|^2_{C^0(B_2)} + \sigma^2|D^2u|^2_{C^0(B_2)}\big)\,.$$

Applying the interpolation inequality (3.4), we may estimate

$$\begin{aligned} \frac{1}{9}\operatorname{dist}(y, \partial B_2)^2|D^2u(y)| &\leq \sigma^2|D^2u|_{C^0(B_\sigma(y))} \\ &= |D^2\tilde{u}|_{C^0(B_1)} \\ &\leq \varepsilon|\tilde{u}|_{C^{2,\alpha}(B_1)} + C_\varepsilon|\tilde{u}|_{C^0(B_1)}\,. \end{aligned}$$

Recalling (3.5) and choosing $\varepsilon = \frac{1}{18C}$, we obtain

$$\frac{1}{9}\operatorname{dist}(y, \partial B_2)^2|D^2u(y)| \leq C\left(|Lu|^2_{C^\alpha(B_2)} + |u|_{C^0(B_2)}\right) + \frac{1}{18}Q\,,$$

where

$$Q(x) \doteqdot \sup_{x \in B_2} \operatorname{dist}(x, \partial B_2)^2|D^2u(x)|^2\,.$$

Since $y$ was arbitrary, we conclude that

$$|D^2u|_{C^0(B_{3/2})} \leq 4Q \leq 8C\left(|f|^2_{C^\alpha(B_2)} + |u|_{C^0(B_2)}\right)\,.$$

This completes the proof. □

**3.3. The boundary estimate.** The boundary estimate provides an estimate in the neighbourhood of any boundary point (with additional dependence on the boundary data). Together, the interior and boundary estimates yield a global estimate.

An open set $\Omega \subset \mathbb{R}^n$ is said to be OF CLASS $C^{k,\alpha}$, if each point $p$ of its boundary $\partial\Omega$ admits a neighbourhood $U$ in $\mathbb{R}^n$ and an injective map $\phi : U \to \mathbb{R}^n$ onto $V \subset \mathbb{R}^n$ such that

$$\phi(\Omega \cap U) \subset \mathbb{R}^n_+\,,\quad \phi(\partial\Omega \cap U) \subset \partial\mathbb{R}^n_+\,,\quad \phi \in C^{k,\alpha}(U) \text{ and } \phi^{-1} \in C^{k,\alpha}(V)\,,$$

where $\mathbb{R}^n_+$ is the UPPER HALF-SPACE $\mathbb{R}^{n-1} \times (0,\infty)$ in $\mathbb{R}^n$. The pair $(\phi, U)$ is called a CHART for $\Omega$. If $k \geq 1$, then, by the implicit function theorem, $\partial\Omega$ may also be represented as a graph over its tangent plane (see Exercise 3.7).

**Theorem 3.3** (Global Schauder)**.** *There exists $C = C(n, \lambda, \Lambda, \alpha, \Omega) < \infty$ with the following property. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with boundary of class $C^{2,\alpha}$ and let $a \in C^\alpha(\Omega, S^{n\times n})$, $b \in C^\alpha(\Omega, \mathbb{R}^n)$, $c \in C^\alpha(\Omega)$ be coefficients satisfying*

$$a^{ij} \geq \lambda \delta^{ij} \ \ in \ \Omega\,, \ \ \lambda > 0$$

*and*

$$|a|_{C^\alpha(\Omega, S^{n\times n})} + |b|_{C^\alpha(\Omega,\mathbb{R}^n)} + |c|_{C^\alpha(\Omega)} \leq \Lambda\,, \ \ \Lambda < \infty\,.$$

*Suppose that $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ satisfies*

$$\begin{cases} -(a^{ij}u_{ij} + b^i u_i + cu) = f & in \ \ \Omega \\ \hfill u = \varphi & on \ \ \partial\Omega \end{cases}$$

*If $f \in C^\alpha(\overline{\Omega})$ and $\varphi \in C^{2,\alpha}(\overline{\Omega})$, then*

$$|u|_{C^{2,\alpha}(\Omega)} \leq C\left(|u|_{C^0(\Omega)} + |f|_{C^\alpha(\Omega)} + |\varphi|_{C^{2,\alpha}(\partial\Omega)}\right).$$

The key estimate in this case is a Hölder estimate for the Hessian of $C^2$ solutions to Poisson's equation in the *halfspace* $\mathbb{R}^n_+ \doteqdot \mathbb{R}^{n-1} \times [0,\infty)$.

**Lemma 3.4.** *There exists $C = C(n, \alpha) < \infty$ such that*

$$[D^2u]_{C^\alpha(\mathbb{R}^n_+)} \leq C\left([\Delta u]_{C^\alpha(\mathbb{R}^n_+)} + [D^2u|_{\partial\mathbb{R}^n_+}]_{C^\alpha(\partial\mathbb{R}^n_+)}\right)$$

*for every $u \in C^{2,\alpha}(\mathbb{R}^n_+)$ such that $\Delta u \in C^\alpha(\mathbb{R}^n_+)$ and $D^2u|_{\partial\mathbb{R}^n_+} \in C^\alpha(\partial\mathbb{R}^n_+)$.*

**Proof.** It suffices to prove the claim in the case $D^2u|_{\partial\mathbb{R}^n_+} \equiv 0$. Indeed, in the general case, we set, for each $(x, x_n) \in \mathbb{R}^{n-1} \times [0,\infty)$,

$$\varphi(x, x_n) \doteqdot u|_{\partial\mathbb{R}^n_+}(x)$$

Observe that the function $v \doteqdot u - \varphi$ vanishes on $\partial\mathbb{R}^n_+$. So, assuming the theorem holds in this case, we find that

$$\begin{aligned}
[D^2u]_{C^\alpha(\mathbb{R}^n_+)} &= [D^2v + D^2\varphi]_{C^\alpha(\mathbb{R}^n_+)} \\
&\leq [D^2v]_{C^\alpha(\mathbb{R}^n_+)} + [D^2\varphi]_{C^\alpha(\mathbb{R}^n_+)} \\
&\leq C[\Delta v]_{C^\alpha(\mathbb{R}^n_+)} + [D^2\varphi]_{C^\alpha(\mathbb{R}^n_+)} \\
&= C[\Delta u - \Delta\varphi]_{C^\alpha(\mathbb{R}^n_+)} + [D^2\varphi]_{C^\alpha(\mathbb{R}^n_+)} \\
&\leq C'\left([\Delta u]_{C^\alpha(\mathbb{R}^n_+)} + [D^2\varphi]_{C^\alpha(\mathbb{R}^n_+)}\right) \\
&= C'\left([\Delta u]_{C^\alpha(\mathbb{R}^n_+)} + [D^2\varphi]_{C^\alpha(\partial\mathbb{R}^n_+)}\right),
\end{aligned}$$

where $C' = C'(n, \alpha)$.

So we may assume that $D^2u|_{\partial\mathbb{R}^n_+} \equiv 0$. The proof in this case proceeds, as in Lemma 3.4, by *reductio ad absurdum*. So suppose, contrary to the claim, that there exists, for each $\ell \in \mathbb{N}$, some $u_\ell \in C^2(\mathbb{R}^n_+)$ such that $\Delta u_\ell \in C^\alpha(\mathbb{R}^n_+)$ and $D^2u_\ell|_{\partial\mathbb{R}^n_+} \equiv 0$, and yet

$$[D^2u_\ell]_{C^\alpha} > \ell[\Delta u_\ell]_{C^\alpha}\,,$$

where $[\,\cdot\,]_{C^\alpha} \doteqdot [\,\cdot\,]_{C^\alpha(\mathbb{R}^n_+)}$. If we set $v_\ell \doteqdot \lambda_\ell u_\ell$, where $\lambda_k \doteqdot |D^2u_\ell|^{-1}_{C^\alpha}$, then

$$[D^2v_\ell]_{C^\alpha} = 1 \;\text{ and }\; [\Delta v_\ell]_{C^\alpha} \leq \ell^{-1}\,.$$

By the pigeonhole principle, we can find $i, j, k \in \{1, \ldots, n\}$ such that, for infinitely many $\ell$, there exist $x_\ell \in \mathbb{R}^n$ and $h_\ell > 0$ for which

$$\frac{|D_iD_jv_\ell(x_\ell + h_\ell e_k) - D_iD_jv_\ell(x_\ell)|}{h_\ell^\alpha} \geq \frac{1}{2n^3}\,.$$

After passing to a subsequence, we may arrange that either

$$h_\ell^{-1}\operatorname{dist}(x_\ell, \partial\mathbb{R}^n_+) \to \infty \;\text{ as }\; \ell \to \infty\,,$$

or

$$\sup_\ell h_\ell^{-1}\operatorname{dist}(x_\ell, \partial\mathbb{R}^n_+) < \infty\,.$$

In the first case, we may proceed exactly as in the proof of Lemma 3.2: by rescaling and translating appropriately, and adding a suitable quadratic, we obtain a sequence of functions $w_\ell$ with $|D^2v_\ell(0)| \geq \frac{1}{2n^3}$ which converges in the $C^2$ topology on compact subsets of $\mathbb{R}^n$ to the zero function, an impossibility.

If, instead, $\gamma \doteqdot \sup_\ell h_\ell^{-1}\operatorname{dist}(x_\ell, \partial\mathbb{R}^n_+) < \infty$, then we can find, for all sufficiently large $\ell$, some $z_\ell \in \partial\mathbb{R}^n_+$ such that $\operatorname{dist}(x_\ell, z_\ell) \leq 2\gamma h_\ell$. Consider, for each $\ell$, the function

$$w_\ell(x) \doteqdot h_\ell^{-2-\alpha}v_\ell(z_\ell + h_\ell x) + p_\ell(x),$$

where $p_\ell$ is a quadratic polynomial. Note that $w_\ell$ is still defined on the halfspace $\mathbb{R}^n_+$ and that the point $y_\ell \doteqdot h_\ell^{-1}(x_\ell - z_\ell)$ lies, for all $\ell$, in a fixed compact set (the ball $\overline{B}_{2\gamma} \cap \mathbb{R}^n_+$). Moreover,

$$[D^2w_\ell]_{C^\alpha} \leq 1, \;\; [\Delta w_\ell]_{C^\alpha} \leq \ell^{-1}, \;\text{ and }\; |D_iD_jw_\ell(y_\ell + e_k) - D_iD_jw_\ell(y_\ell)| \geq \frac{1}{2n^3}.$$

We again choose $p_\ell$ so that $w_\ell$ vanishes to second order at the origin. Applying the Arzelà–Ascoli theorem, we can find $y \in \overline{B}_{2\gamma} \cap \mathbb{R}^n_+$ and a subsequence of the functions $w_\ell$ which converges locally uniformly in $C^2(\mathbb{R}^n_+)$ to a *harmonic* limit $w$ which satisfies

$$D^2w(0) = 0 \;\text{ and }\; [D^2w]_{C^\alpha} \leq 1\,, \tag{3.6}$$

and yet

$$|D^2w(y+e_k) - D^2w(y)| \geq \frac{1}{2n^3}\,.$$

But this is absurd: since $w$ vanishes along $\partial\mathbb{R}^n_+$, it extends by reflection across $\partial\mathbb{R}^n_+$ to a harmonic function on $\mathbb{R}^n$, so the Bernstein estimates and (3.6) imply that $D^2w \equiv 0$. □

Freezing the coefficients, introducing cutoff functions, "straightening the boundary" using charts and applying the interpolation inequality as in the proof of Theorem 3.1 leads to an estimate for linear equations on bounded domains in a neighbourhood of the boundary (which now depends on the boundary condition and the boundary charts). Combined with the interior estimate, this proves Theorem 3.3. We omit the details.

### 3.4. Exercises.

**Exercise 3.1.** Provide an example of

(a) a function $u : [0,1] \to \mathbb{R}$ which is in $C^\alpha([0,1])$ but not in $C^\beta([0,1])$ for any $\beta > \alpha$.

(b) a function $u : [0,1] \to \mathbb{R}$ which is in $C^{1,1}([0,1])$ but not in $C^2([0,1])$.

**Exercise 3.2.** Suppose that $f : [0,1] \to \mathbb{R}$ is continuously differentiable. Show that $f$ is $\alpha$-Hölder continuous for all $\alpha \in (0,1]$.

**Exercise 3.3.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Show, for any choice of $k \in \mathbb{N}$ and $\alpha \in (0,1]$, that $C^{k,\alpha}(\overline{\Omega})$ is a Banach space.

**Exercise 3.4.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Given $u, v \in C^\alpha(\Omega)$, $\alpha \in (0,1]$, show that

$$[uv]_{C^\alpha(\Omega)} \leq \sup_\Omega |u|[v]_{C^\alpha(\Omega)} + \sup_\Omega |v|[u]_{C^\alpha(\Omega)}\,.$$

**Exercise 3.5.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Given $\alpha, \beta \in (0,1]$ with $\beta \leq \alpha$, show that

- $C^\alpha(\overline{\Omega}) \subset C^\beta(\overline{\Omega})$.
- the inclusion map $\iota : C^\alpha(\overline{\Omega}) \to C^\beta(\overline{\Omega})$ is continuous.
- the embedding $\iota : C^\alpha(\overline{\Omega}) \to C^\beta(\overline{\Omega})$ is compact[7] if $\beta < \alpha$. *Hint: by the Arzelà–Ascoli theorem, a bounded sequence $\{u_j\}_{j\in\mathbb{N}} \subset C^\alpha(\overline{\Omega})$ admits a subsequence which converges in the* uniform *topology to some limit $u$. Show that $v_j \doteqdot u_j - u$ converges to zero in the $\beta$-Hölder topology.*

[7] That is, bounded subsets of $C^\alpha(\overline{\Omega})$ are relatively compact in $C^\beta(\overline{\Omega})$.

**Exercise 3.6.** Given a positive definite $a \in S_+^{n\times n}$, suppose that $u \in C^2(\mathbb{R}^n)$ satisfies

$$-a^{ij}u_{ij} = 0\,.$$

Find a linear change of coordinates $A \in GL(\mathbb{R}^n)$ such that $x \mapsto u(Ax)$ is harmonic.

**Exercise 3.7.** Suppose that $\Omega \underset{\text{open}}{\subset} \mathbb{R}^n$ is of class $C^{k,\alpha}$ for some $k \geq 1$. The TANGENT SPACE (or TANGENT PLANE) $T_x\partial\Omega$ to $\partial\Omega$ at $x \in \partial\Omega$ is the affine subspace of $\mathbb{R}^n$ defined by

$$T_x\partial\Omega \doteqdot \{x + D_v\phi^{-1} : v \in \partial\mathbb{R}^n_+\}\,,$$

where $\phi : U \to \mathbb{R}^n$ is some chart for $\Omega$ with $x \in U$. The OUTWARD UNIT NORMAL vector $\nu(x)$ to $\Omega$ at $x$ is the unit normal vector to $T_x\partial\Omega$ with the property that $x + \varepsilon\nu(x) \in \mathbb{R}^n \setminus \Omega$ for all $\varepsilon$ sufficiently small.

(a) Show that $T_x\partial\Omega$ does not depend on the choice of chart about $x$.

(b) Show that, for each $x \in \partial\Omega$, there is a function $u \in C^{k,\alpha}(T_x\partial\Omega)$ and a neighbourhood $U$ of $x$ in $\mathbb{R}^n$ such that

$$\Omega \cap U = \{y + r\nu(x) : y \in T_x\partial\Omega \text{ and } r < u(y)\} \cap U$$

and

$$\partial\Omega \cap U = \{y + u(y)\nu(x) : y \in T_x\partial\Omega\} \cap U\,.$$

*Hint: You will need the impilict function theorem.*

More generally, a subset $M \subset \mathbb{R}^n$ is called an EMBEDDED SUBMANIFOLD (of dimension $\ell < n$ and class $C^{k,\alpha}$) if, for each $x \in M$, there is a neighbourhood $U$ of $x$ in $\mathbb{R}^n$ and a diffeomorphism $\phi : U \to V \subset \mathbb{R}^n$ of class $C^{k,\alpha}$ such that $\phi(U \cap M) = V \cap (\mathbb{R}^\ell \times \{0\})$. If $k \geq 1$, then $M$ admits a tangent space and unit normal at each point, and may be represented as the graph of a $C^{k,\alpha}$ function over each tangent space.

## 4. Solubility in Hölder spaces of the Dirichlet problem for linear elliptic equations

Estimates of the kind proven by Schauder are referred to as *a priori* estimates, as they provide uniform regularity of *all* solutions to a given equation prior to any knowledge of the *existence* of a solution. Somewhat counterintuitively, *a priori* estimates play a crucial role in proving the existence of solutions as well. Using the Schauder theorem, we will prove (using multiple approaches) the following theorem.

**Theorem 4.1** (Solving the Dirichlet problem in Hölder spaces[8])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $C^{2,\alpha}$ and suppose that the coefficients $(a, b, c) : \overline{\Omega} \to S^{n\times n} \times \mathbb{R}^n \times \mathbb{R}$ are $\alpha$-Hölder continuous, $a$ is positive definite, and $c \leq 0$. If $f \in C^\alpha(\overline{\Omega})$ and $\phi \in C^{2,\alpha}(\overline{\Omega})$, then the Dirichlet problem*

$$\begin{cases} -(a^{ij}u_{ij} + b^i u_i + cu) = f & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

*admits a unique solution in $C^{2,\alpha}(\overline{\Omega})$.*

Theorem 4.1 says that the spaces $C^{2,\alpha}(\overline{\Omega})$ and $C^\alpha(\overline{\Omega}) \times C^{2,\alpha}(\partial\Omega)$ are isomorphic, and the linear map

$$u \mapsto \big( - (a^{ij}u_{ij} + b^i u_i + cu), u|_{\partial\Omega}\big)$$

is an isomorphism.

Note that we have already proved uniqueness (in the larger space $C^2(\Omega) \cap C^0(\overline{\Omega})$) as a consequence of the maximum principle (Corollary 2.2).

Moreover, in order to prove existence, it suffices to consider the case $\phi \equiv 0$. This is because

$$f' \doteqdot f - \big(a^{ij}\phi_{ij} + b^i\phi_i + c\phi\big) \in C^\alpha(\overline{\Omega})$$

and adding $\phi$ to any solution $v \in C^{2,\alpha}(\Omega)$ to the problem

$$\begin{cases} a^{ij}v_{ij} + b^i v_i + cv = f' & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

yields a solution $u \in C^{2,\alpha}(\overline{\Omega})$ to the problem (4.9).

Finally, by hypothesis, we can find $\lambda > 0$ such that

$$a^{ij} \geq \lambda\delta^{ij} \ \text{ in } \Omega$$

and $\Lambda < \infty$ such that

$$|a|_{C^\alpha(\Omega)} + |b|_{C^\alpha(\Omega)} + |c|_{C^\alpha(\Omega)} \leq \Lambda\,.$$

[8] In fact, we shall see that the Dirichlet problem (4.9) can also be (uniquely) solved in the larger space $C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ if the coefficients $(a, b, c)$ and the inhomogeneity $f$ are merely bounded and $\alpha$-Hölder continuous in $\Omega$, and $\phi \in C^0(\partial\Omega)$.

**4.1. Approximation by smooth equations.** Our first approach exploits the existence of smooth solutions to linear equations with smooth coefficients over smooth domains, which we take for granted. (It may be proved independently using the Sobolev theory — the Fredholm alternative, the maximum principle for weak solutions, and Sobolev embeddings; see [**1**]).

We note that any bounded domain of class $C^{k,\alpha}$ may be approximated by smooth domains $\Omega_j \subset \Omega$ such that, for any $\beta < \alpha$, $\Omega_j \to \Omega$ in $C^{k,\beta}$ in the sense that there exist $\varepsilon_j \to 0$ and $\delta > 0$ such that

$$\{x \in \Omega : \operatorname{dist}(\partial\Omega, x) > \varepsilon_j\} \subset \Omega_j$$

for all $j$ and, for each $x \in \partial\Omega$, we can find charts $(\phi, U)$ for $\Omega$ and $(\phi_j, U_j)$ for $\Omega_j$ such that $B_\delta(x) \subset U \cap U_j$ and $|\phi_j^{-1} - \phi^{-1}|_{C^{2,\beta}(B_\delta(x))} \to 0$ uniformly in $x$. We can also arrange that $|\phi_j^{-1}|_{B_\delta(x)}$ is bounded independently of $j$ and $x$. This may be proved by mollifying the distance-to-the-boundary function (which is of class $C^{2,\alpha}$ on a sufficiently small neighbourhood of $\partial\Omega$). We will study the distance-to-the-boundary function further in §7.

**Proof of Theorem 4.1. (assuming smooth solubility over smooth domains.)** Approximate $\Omega$ by a sequence of smooth domains $\Omega_k$ as above. Denote by $\eta_\varepsilon$ the standard mollifier with $\operatorname{spt}\eta_\varepsilon \subset B_\varepsilon$. For each $k \in \mathbb{N}$, set

$$a_k^{ij} \doteqdot \eta_{1/k} * a^{ij}\,, \quad b_k^i \doteqdot \eta_{1/k} * b^i\,, \quad c_k \doteqdot \eta_{1/k} * c \;\text{ and }\; f_k \doteqdot \eta_{1/k} * f\,.$$

These coefficients are all smoothly defined on $\Omega$ and converge locally uniformly on compact subsets of $\Omega$ to the original coefficients. They satisfy

$$a_k^{ij} \geq \lambda_k \delta^{ij} \;\text{ and }\; c_k \leq 0 \;\text{ in }\; \Omega_k$$

and

$$|a_k|_{C^\alpha(\Omega_k)} + |b_k|_{C^\alpha(\Omega_k)} + |c_k|_{C^\alpha(\Omega_k)} \leq \Lambda_k\,,$$

where $\lambda_k \to \lambda$ and $\Lambda_k \to \Lambda$ as $k \to \infty$.

By the Sobolev theory, there exists a solution $u^k \in C^\infty(\Omega_k)$ to the Dirichlet problem

$$\begin{cases} -(a_k^{ij} u_{ij}^k + b_k^i u_i^k + c_k u^k) = f_k \;\text{ in }\; \Omega_k \\ \hfill u^k = 0 \;\text{ on }\; \partial\Omega_k\,. \end{cases}$$

By the Schauder estimates,

$$|u^k|_{C^{2,\alpha}(\Omega_k)} \leq C|f_k|_{C^\alpha(\Omega_k)}\,,$$

where, for $k$ sufficiently large, $C$ depends only on $n$, $\lambda$, $\Lambda$, $\alpha$ and $\Omega$. It now follows from the Arzelà–Ascoli theorem that a subsequence of these solutions converges locally uniformly in the $C^{2,\beta}$ topology, for any $\beta < \alpha$, to a solution $u \in C^{2,\alpha}(\Omega)$ to the original problem. $\square$

**4.2. The method of continuity.** Another approach to establishing existence of solutions to PDE, which will prove particularly fruitful later in the context of nonlinear equations, is THE METHOD OF CONTINUITY. In this approach, one embeds the problem in a continuous family of problems which connect the problem in question to a solved problem, and then shows that the family of parameters corresponding to solved problems is the entire interval. Typically, this is done by showing that this family is both open and closed; *a priori* estimates are employed in deducing closedness.

In the case at hand, we make use of Banach's fixed-point theorem and the solubility of Poisson's equation.

**Lemma 4.2.** *Let $B$ be a Banach space and $V$ be a normed linear space and let $L_0$ and $L_1$ be bounded linear operators from $B$ into $V$. For each $t \in [0,1]$, consider the bounded linear map*

$$L_t \doteqdot tL_0 + (1-t)L_1 \,.$$

*Suppose that*

$$\min_{t\in[0,1]} \min_{x\in B\setminus\{0\}} \frac{|L_t x|_V}{|x|_B} > 0 \,. \tag{4.2}$$

*If $L_0$ is surjective, then $L_1$ is surjective.*

**Proof.** By (4.2), $L_t$ is injective for each $t \in [0,1]$. Suppose that $L_s$ is surjective for some $s \in [0,1]$. Then $L_s$ is invertible and, by (4.2),

$$|L_s^{-1}| \le C \doteqdot \frac{1}{\min_{t\in[0,1]} \min_{x\in B\setminus\{0\}} \frac{|L_t x|_V}{|x|_B}} \,.$$

We claim that $t$ is invertible for $t$ sufficiently close to $s$. Observe that the equation

$$L_t x = y$$

is equivalent to the equation

$$x = L_s^{-1} y - (t-s) L_s^{-1}(L_1 - L_0) x \,.$$

By the contraction mapping theorem, this equation is soluble for

$$|t-s| < \delta \doteqdot \frac{1}{C(|L_0| + |L_1|)}$$

since this ensures that the mapping

$$x \mapsto Tx \doteqdot L_s^{-1} y - (t-s) L_s^{-1}(L_1 - L_0) x \tag{4.3}$$

is a contraction mapping.

Since $\delta > 0$, we may cover $[0,1]$ by a finite number of intervals of length $\delta$. Since $L_0$ is surjective, the claim follows by induction. $\square$

The Schauder estimate and Lemma 4.2 now yield solutions to linear elliptic PDE in Hölder spaces, assuming only the solubility in Hölder spaces of Poisson's equation. The latter result, known as KELLOGG'S THEOREM, may be established as in §4.1 (using the global Schauder estimate) once solubility in $C^\infty$ for smooth data has been established[9].

**Proof of Theorem 4.1. (assuming the solubility in Hölder spaces of Poisson's equation).** Consider the family of problems

$$\begin{cases} L_t u = f \;\text{ in }\; \Omega \\ \quad u = 0 \;\text{ on }\; \partial\Omega \end{cases} \tag{4.4}$$

for $t \in [0,1]$, where

$$L_t \doteqdot t(a^{ij}D_iD_j + b^iD_i + c) + (1-t)\Delta\,.$$

Observe that the coeficients $(a_t, b_t, c_t) \doteqdot t(a,b,c) + (1-t)(\delta, 0, 0)$ satisfy

$$a_t^{ij} \geq \lambda_t \delta^{ij} \;\text{ in }\; \Omega$$

and

$$|a_t|_{C^\alpha(\Omega)} + |b_t|_{C^\alpha(\Omega)} + |c_t|_{C^\alpha(\Omega)} \leq \Lambda_t\,,$$

where $\lambda_t \doteqdot \min\{1, \lambda\}$ and $\Lambda_t \doteqdot \max\{1, \Lambda\}$.

Observe that $L_t$ maps $C_0^{2,\alpha}(\overline{\Omega}) \doteqdot \{u \in C^{2,\alpha}(\overline{\Omega}) : u|_{\partial\Omega} \equiv 0\}$ into $C^\alpha(\overline{\Omega})$ and, as a linear map between these spaces, is bounded. Invertability of $L_t$ is equivalent to the solubility of the Dirichlet problem (4.4) in $C^{2,\alpha}(\overline{\Omega})$ for any $f \in C^\alpha(\overline{\Omega})$.

We need to verify (4.2) in Lemma 4.2. So fix some $u \in C_0^{2,\alpha}(\overline{\Omega})$. Since $u|_{\partial\Omega} \equiv 0$, Proposition 2.3 yields

$$|u| \leq C \sup_\Omega |L_t u| \leq C|L_t u|_{C^\alpha(\Omega)}\,,$$

where $C$ depends only on $n$, $\lambda$, $\Lambda$, and the diameter of $\Omega$. Schauder's estimate (Theorem 3.3) then yields

$$|u|_{C^{2,\alpha}(\Omega)} \leq C|L_t u|_{C^\alpha(\Omega)}\,.$$

Since $u \in C_0^{2,\alpha}(\overline{\Omega})$ was arbitrary, we conclude that

$$\min_{t\in[0,1]} \min_{u \in C_0^{2,\alpha}(\overline{\Omega})\setminus\{0\}} \frac{|L_t u|_{C^\alpha(\Omega)}}{|u|_{C^{2,\alpha}(\Omega)}} > 0\,.$$

Since $L_0$ is invertible, Lemma 4.2 implies that $L_1$ is invertible. □

[9]The latter may be established by various methods. Traditionally, solutions were obtained using potential theory. Another approach is via the $L^2$ theory for equations of divergence form. We will provide a proof in the following subsection using Perron's method, which reduces the general case to the case $\Omega = B_1$. We take this case for granted since it is typically covered in undergraduate PDE courses (by deriving an explicit representation formula for solutions).

**4.3. Perron's method.** Finally, we present PERRON'S METHOD, which reduces solubility of an equation in a general domain $\Omega$ to its solubility in small balls. The idea here is to exploit the maximum principle, which implies that (for suitable equations) any subsolution taking the same boundary values as a solution $u$ necessarily lies below $u$. If there exists at least one subsolution, then the "largest" subsolution exists. If this happens to be smooth, then it must actually be a solution.

4.3.1. *Barrier subsolutions.* Consider, then, the linear elliptic operator

$$L \doteqdot a^{ij} D_i D_j + b^i D_i + c\,.$$

We assume throughout this section that the operator $L$ satisfies the hypotheses of Theorem 4.1. In particular, $L$ adheres to the maximum principle and the strong maximum principle. We shall, in addition, assume a *local* solubility condition for Dirichlet problems corresponding to $L$ (to be made precise below). This condition may be verified independently for Poisson's equation (and hence, by the method of continuity, for all $L$ satisfying our hypotheses).

We say that a function $v : \Omega \to \mathbb{R}$ satisfies $-Lv \leq f$ (or that $v$ is a subsolution to the equation $-Lu = f$) IN THE BARRIER SENSE if, for every $\Omega' \Subset \Omega$ and $\varphi \in C^2(\Omega')$ with $-L\varphi = f$ in $\Omega'$,

$$v \leq \varphi \text{ on } \partial\Omega' \implies v \leq \varphi \text{ in } \Omega\,.$$

Since $L$ adheres to the maximum principle, any classical subsolution $v \in C^2(\Omega)$ is a subsolution in the barrier sense.

If $f$ is continuous and we are able to solve the Dirichlet problem for the equation $-Lu = f$ in $C^2$ for sufficiently small balls about any point of $\Omega$, then the converse is also true; that is, every barrier subsolution $v \in C^2(\Omega)$ is a classical subsolution. To see this, suppose, to the contrary, that $-Lv(x_0) > f(x_0)$ at some point $x_0 \in \Omega$ for some barrier subsolution $v$. Since $f$ and $Lv$ are continuous, we can find $r > 0$ such that $-Lv > f$ on $B_r(x_0)$ (in the classical sense). If $r$ is sufficiently small, then, by hypothesis, we may solve the boundary value problem

$$\begin{cases} -L\varphi = f \text{ in } B_r(x_0) \\ \quad\ \varphi = v \text{ on } \partial B_r(x_0) \end{cases}$$

in $C^2(B_r(x_0)) \cap C^0(\overline{B}_r(x_0))$. So the maximum principle implies that $v \geq \varphi$ in $B_r(x_0)$. On the other hand, $v \leq \varphi$ since $v$ is a subsolution, so we conclude that $v \equiv \varphi$. But then $-Lv \equiv f$, in contradiction with our assumption.

In fact, we will assume a slightly stronger local solubility condition[10]. Namely, for fixed $\alpha \in (0,1)$, bounded domain $\Omega \subset \mathbb{R}^n$, and right hand side

[10]The following local solubility condition will suffice to obtain global solutions with $C^{2,\alpha}$ interior regularity. To obtain $C^{2,\alpha}$ regularity up to the boundary for the global solutions, we will need to make a corresponding local solubility assumption. This is the subject of §4.3.4.

$f \in C^\alpha(\Omega)$, we assume that we can find, for every $x \in \Omega$, some $r > 0$ and some $\delta \in (0,1)$ such that $B_r(x) \Subset \Omega$ and the Dirichlet problem

$$\begin{cases} -Lu = f \ \text{ in } \ B_{\delta r}(y) \\ \quad\ u = \phi \ \text{ on } \ \partial B_{\delta r}(y) \end{cases} \tag{4.5}$$

is soluble in $C^{2,\alpha}(B_{\delta r}(y)) \cap C^0(\overline{B}_{\delta r}(y))$ over every $B_{\delta r}(y) \subset B_r(x)$ for every $\phi \in C^0(\partial B_{\delta r})$.

We may define barrier supersolutions analogously, and these will satisfy corresponding supersolution properties.

Next observe that barrier sub- and supersolutions admit a comparison principle.

**Proposition 4.3.** *Suppose that $u, v \in C^0(\overline{\Omega})$ satisfy, respectively, $-Lu \leq f$ and $-Lv \geq f$ in the barrier sense. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\Omega$.*

**Proof.** Set $M \doteqdot \sup_\Omega(u - v)$ and suppose, contrary to the claim, that $M > 0$. Since $u - v \leq 0$ on $\partial\Omega$, we can find $x_0 \in \Omega$ and $r > 0$ such that $u(x_0) - v(x_0) = M$ and $u - v \not\equiv M$ on $\partial B_r(x_0)$. In fact, we can also arrange that $r$ is as small as we like, so that we may find solutions $\overline{u}, \overline{v} \in C^2(B_r(x_0)) \cap C^0(\overline{B}_r(x_0))$ to the Dirichlet problem for $L$ in $B_r(x_0)$ with respective boundary data $u$ and $v$ on $\partial B_r(x_0)$. Note that $u \leq \overline{u}$ and $v \geq \overline{v}$ in $B_r(x_0)$. So the maximum principle implies that

$$M = (u - v)(x_0) \leq (\overline{u} - \overline{v})(x_0) \leq \max_{\partial B_r(x_0)} (\overline{u} - \overline{v}) = \max_{\partial B_r(x_0)} (u - v) \leq M\,,$$

and hence $(\overline{u} - \overline{v})(x_0) = \max_{\overline{B}_r(x_0)}(\overline{u} - \overline{v})$. That is, $\overline{u} - \overline{v}$ attains its maximum at an interior point. But this violates the strong maximum principle since $\overline{u} - \overline{v}$ is not constant. □

**Corollary 4.4.** *If $u, v \in C^0(\overline{\Omega})$ are both barrier subsolutions to the equation $-L\varphi = 0$, then the function $\max\{u, v\}$ is also a subsolution.*

**Proof.** Exercise. □

Consider now the Dirichlet problem

$$\begin{cases} -Lu = f \ \text{ in } \ \Omega \\ \quad\ u = \phi \ \text{ on } \ \partial\Omega\,. \end{cases} \tag{4.6}$$

Define a function $u : \Omega \to \mathbb{R}$ by

$$u(x) \doteqdot \sup\{v(x) : v \in C^0(\overline{\Omega})\,,\ -Lv \leq f \ \text{ in } \Omega\,,\ \text{and } v \leq \phi \text{ on } \partial\Omega\}\,.$$

We shall call a function $v \in C^0(\overline{\Omega})$ satisfying $-Lv \leq f$ in $\Omega$ (in the barrier sense) and $u \leq \phi$ on $\partial\Omega$ a (BARRIER) SUBSOLUTION TO THE DIRICHLET PROBLEM (4.6).

Observe that $u$ takes finite values. Indeed, choosing $\xi$ so that

$$\lambda \doteqdot a^{ij}\xi_i\xi_j + b^i\xi_i + c > 0$$

and setting $\overline{d} \doteqdot \sup_{x\in\Omega}(\xi \cdot x)$ and $\underline{d} \doteqdot \inf_{x\in\Omega} \xi \cdot x$, we find that the function $\underline{u} \in C^0(\overline{\Omega})$ defined by

$$\underline{u}(x) \doteqdot -\sup_{\partial\Omega}(-\phi)_+ - \mathrm{e}^{-\underline{d}}\left(\mathrm{e}^{\overline{d}} - \mathrm{e}^{x\cdot\xi}\right)\sup_{\Omega}\frac{(-f)_+}{\lambda}$$

satisfies $-L\underline{u} \le f$ in $\Omega$ and $\underline{u} \le \phi$ on $\partial\Omega$. Thus (by definition of $u$) $u \ge \underline{u}$ in $\Omega$ and hence

$$\inf_{\Omega} u \ge -\sup_{\partial\Omega}(-\phi)_+ - \mathrm{e}^{|\xi|R}\sup_{\Omega}\frac{(-f)_+}{\lambda}\,, \tag{4.7}$$

where $R \ge \operatorname{diam}\Omega$. On the other hand, by Proposition 4.3 (cf. Proposition 2.3),

$$\sup_{\Omega} u \le \sup_{\partial\Omega}\phi_+ + \mathrm{e}^{|\xi|R}\sup_{\Omega}\frac{f_+}{\lambda}\,.$$

We conclude that

$$\sup_{\Omega}|u| \le \sup_{\partial\Omega}|\phi| + C\sup_{\Omega}|f|\,, \tag{4.8}$$

where $C$ depends only on $n$, $R$, and the coefficients of $L$.

4.3.2. *Smoothness of the Perron (sub)solution.* We claim that $u \in C^{2,\alpha}(\Omega)$ and $-Lu = f$. To see this, fix some $x \in \Omega$ and let $u_j \in C^0(\overline{\Omega})$ form a sequence of subsolutions to (4.6) such that $u_j(x) \to u(x)$. Given $r \le 1$ sufficiently small, we can, by hypothesis, find a function $U_j \in C^0(\overline{\Omega}) \cap C^{2,\alpha}(B_r(x))$ satisfying $U_j \equiv u_j$ on $\Omega \setminus B_r(x)$ and

$$\begin{cases} -LU_j = f & \text{in } B_r(x) \\ \quad U_j = u_j & \text{on } \partial B_r(x)\,. \end{cases}$$

The interior Schauder estimate (Theorem 3.1), Proposition 2.3 and the estimate (4.8) yield

$$|U_j|_{C^{2,\alpha}(B_{r/2}(x))} \le C\left(\sup_{\partial\Omega}|\phi| + |f|_{C^\alpha(\Omega)}\right)\,,$$

where $C$ depends only on $n$, $\alpha$, and the coefficients of $L$. So the Arzelà–Ascoli theorem implies that a subsequence of the functions $U_j$ converges in $C^2(B_{r/2}(x))$ to some limit $U \in C^{2,\alpha}(B_{r/2}(x))$ satisfying $-LU = f$ in $B_{r/2}(x)$. Moreover, since $u_j(x) \to u(x)$ and (by Proposition 4.3) $u_j \le U_j$, we find that $U(x) = u(x)$. Moreover, by definition of $u$, $U \le u$ in $B_{r/2}$.

We claim that $U \equiv u$ in $B_{\delta r}(x)$ for some $\delta > 0$. Suppose, to the contrary, that for any $\delta > 0$ we can find $y \in B_{\delta r}(x)$ such that $U(y) < u(y)$. By the definition of $u$, we can find a subsolution $v \in C^0(\overline{\Omega})$ to (4.6) such that

$U(y) < v(y) \leq u(y)$. Set $v_j \doteqdot \max\{U_j, v\}$. For $\delta$ sufficiently small, we can find a function $V_j \in C^0(\overline{\Omega})$ satisfying $V_j \equiv v_j$ on $\Omega \setminus B_{4\delta r}(y)$ and

$$\begin{cases} -LV_j = f \text{ in } B_{4\delta r}(y) \\ \quad V_j = v_j \text{ on } \partial B_{4\delta r}(y)\,. \end{cases}$$

Since $U_j \leq V_j$ on $\partial B_{4\delta r}$, we have $U_j \leq V_j$ in $B_{4\delta r}(y)$. Since $V_j$ is a subsolution to the Dirichlet problem (4.6), we also have $V_j \leq u$ by the definition of $u$. Since $v$ is a subsolution and $v \leq V_j$ on $\partial B_{4\delta r}$, we also have $v \leq V_j$ in $B_{4\delta r}$. In particular, $\lim_{j\to\infty} V_j(x) = u(x)$, and $U(y) < \lim_{j\to\infty} V_j(y) \leq u(y)$.

Since $V_j \equiv v_j \geq U_j$ on $\Omega \setminus B_{4\delta r}(y)$, the interior Schauder estimate, Propositions 2.3 and 4.3 and the estimate (4.8) yield, for $j$ sufficiently large,

$$|V_j|_{C^{2,\alpha}(B_{2\delta r}(y))} \leq C\left(\sup_{\partial\Omega}|\phi| + |f|_{C^\alpha(\Omega)}\right),$$

where $C$ depends on $n$, $\alpha$, and the coefficients of $L$. So the Arzelà–Ascoli theorem implies that a subsequence of the functions $V_j$ converges in $C^2(B_{2\delta r}(y))$ to some limit $V \in C^{2,\alpha}(B_{2\delta r}(x))$ satisfying $-LV = f$ and $U \leq V \leq u$ in $B_{2\delta r}(x)$, and $U(y) < V(y)$. Since $-LU = -LV = f$ in $B_{2\delta r}(y)$, $U \leq V$ and $U(x) = V(x)$ at $x \in B_{\delta r}(y)$, the strong maximum principle implies that $U \equiv V$ in $B_{2\delta r}(y)$, in contradiction with the inequality $U(y) < V(y)$.

We conclude that $U \equiv u$ in $B_{\delta r}(x)$. But then $u \in C^{2,\alpha}(B_{\delta r})$ and $-Lu = f$ in $B_{\delta r}$. The claim follows since $x$ was arbitrary.

4.3.3. *Continuity of the Perron solution up to the boundary.* So we have found a solution $u \in C^2(\Omega)$ to the equation $-Lu = f$. However, we have not yet shown that $u$ attains the boundary values $\phi$. This will be possible so long as $\Omega$ is of class $C^2$. In fact, it suffices for $\Omega$ to satisfy the EXTERIOR BALL CONDITION, which means that for each point $x_0 \in \partial\Omega$ we can find a ball $B \subset \mathbb{R}^n \setminus \overline{\Omega}$ such that $\overline{B} \cap \overline{\Omega} = \{x_0\}$.

So fix $x_0 \in \partial\Omega$. We need to show that $u(x) \to \phi(x)$ as $x \to x_0$. To do this, we shall construct upper and lower barriers for $u$ near $x_0$ which take values arbitrarily close to $\phi(x_0)$ at $x_0$. First observe that we can find a function $w : \Omega \to \mathbb{R}$ such that $-Lw \geq 1$ in $\Omega$, $w > 0$ on $\partial\Omega \setminus \{x_0\}$, and $w(x_0) = 0$. Indeed, given a ball $B_r(p)$ in $\mathbb{R}^n \setminus \Omega$ such that $\overline{B}_r(p) \cap \overline{\Omega} = \{x_0\}$, we can set

$$w(x) = w_{\mu,\sigma} \doteqdot \mu\left(r^{-\sigma} - |x-p|^{-\sigma}\right)$$

for suitable $\mu > 0$ and $\sigma > 0$. To see this, observe that

$$-Lw_{\mu,\sigma} \geq \frac{\mu}{|x-p|^{\sigma+2}}\left(\sigma(\sigma+2)a^{ij}\frac{(x-p)_i}{|x-p|}\frac{(x-p)_j}{|x-p|} - \sigma\left(\operatorname{tr} a + b^i(x-p)_i\right)\right).$$

The claim follows from the uniform positivity of $a$ and the bounds for $a$ and $b$.

Now fix any sequence of numbers $\varepsilon_i \searrow 0$ and, for each $i$, choose $\delta_i > 0$ such that $|\phi(x) - \phi(x_0)| < \varepsilon_i$ for $x \in B_{\delta_i}(x_0) \cap \partial\Omega$. We may further choose $k_i > 0$ so that

$$k_i w \geq \sup_{\partial\Omega} |\phi| + C \sup_{\Omega} |f| + |\phi(x_0)| \text{ on } \partial B_{\delta_i}(x_0) \cap \Omega\,,$$

where $C$ is the constant in the estimate (4.8). If we also ensure that $k_i \geq \sup_\Omega |f + c\phi(x_0)|$, then we find that the functions

$$\underline{w}_i \doteqdot \phi(x_0) - k_i w - \varepsilon_i \text{ and } \overline{w}_i \doteqdot \phi(x_0) + k_i w + \varepsilon_i\,,$$

satisfy

$$\begin{cases} -L\underline{w}_i \leq 0 \text{ in } B_{\delta_i}(x_0) \cap \Omega \\ \quad \underline{w}_i \leq \phi \text{ on } B_{\delta_i}(x_0) \cap \partial\Omega \\ \quad \underline{w}_i \leq \underline{M} \doteqdot -\sup_{\partial\Omega} |\phi| - C \sup_{\Omega} |f| \text{ on } \partial B_{\delta_i}(x_0) \cap \Omega \end{cases}$$

and

$$\begin{cases} -L\overline{w}_i \geq 0 \text{ in } B_{\delta_i}(x_0) \cap \Omega \\ \quad \overline{w}_i \geq \phi \text{ on } B_{\delta_i}(x_0) \cap \partial\Omega \\ \quad \overline{w}_i \geq \overline{M} \doteqdot \sup_{\partial\Omega} |\phi| + C \sup_{\Omega} |f| \text{ on } \partial B_{\delta_i}(x_0) \cap \Omega\,. \end{cases}$$

Finally, we set

$$\underline{u}_i \doteqdot \begin{cases} \max\{\underline{u}_i, \underline{m}\} \text{ in } B_{\delta_i}(x_0) \cap \Omega \\ \qquad\qquad \underline{m} \text{ in } \Omega \setminus \overline{B}_{\delta_i}(x_0) \end{cases}$$

and

$$\overline{u}_i \doteqdot \begin{cases} \min\{\overline{u}_i, \overline{m}\} \text{ in } B_{\delta_i}(x_0) \cap \Omega \\ \qquad\qquad \overline{m} \text{ in } \Omega \setminus \overline{B}_{\delta_i}(x_0)\,, \end{cases}$$

where $\underline{m} \geq \underline{M}$, resp. $\overline{m} \leq \overline{M}$, is a subsolution, resp. supersolution, to the Dirichlet problem (4.6). For example, we could take

$$\overline{m} \doteqdot \sup_{\partial\Omega} \phi_+ + \mathrm{e}^{-\underline{d}} \left(\mathrm{e}^{\overline{d}} - \mathrm{e}^{x\cdot\xi}\right) \sup_{\Omega} \frac{f_+}{\lambda}$$

and

$$\underline{m} \doteqdot -\inf_{\partial\Omega} (-\phi)_+ - \mathrm{e}^{-\underline{d}} \left(\mathrm{e}^{\overline{d}} - \mathrm{e}^{x\cdot\xi}\right) \sup_{\Omega} \frac{(-f)_+}{\lambda}\,.$$

By Corollary 4.4, $\underline{u}_i$ and $\overline{u}_i$ are, respectively, sub- and supersolutions to the Dirichlet problem (4.6), so we find that $\underline{u}_i \leq u \leq \overline{u}_i$. Taking $i \to \infty$ then yields $u(x_0) = \phi(x_0)$. Since $x_0$ was arbitrary, we conclude that $u|_{\partial\Omega} = \phi$.

It remains to prove that $u$ is of class $C^{2,\alpha}$ up to the boundary of $\Omega$ (assuming $\Omega$ is of class $C^{2,\alpha}$, $\phi \in C^{2,\alpha}(\overline{\Omega})$ and $f \in C^{\alpha}(\overline{\Omega})$). But let us first note that we have proved the following result.

**Theorem 4.5.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain which satisfies the exterior ball condition (e.g. $\partial\Omega$ is of class $C^2$) and suppose that the coefficients $(a, b, c) : \Omega \to S^{n\times n} \times \mathbb{R}^n \times \mathbb{R}$ are bounded and $\alpha$-Hölder continuous, $a$ is positive definite, and $c \leq 0$. If $f$ is bounded and $\alpha$-Hölder continuous and $\phi \in C^0(\partial\Omega)$, then the Dirichlet problem*

$$\begin{cases} -(a^{ij}u_{ij} + b^i u_i + cu) = f & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases} \tag{4.9}$$

*admits a unique solution in $C^{2,\alpha}(\Omega) \cap C^0(\Omega)$.*

4.3.4. *Smoothness of the Perron solution up to the boundary.* Finally, we prove (under the full hypotheses of Theorem 4.1) that $u \in C^{2,\alpha}(\overline{\Omega})$, so long as the local solutions (to the problems (4.5)) are of class $C^{2,\alpha}$ up to the boundary.

Fix some $x_0 \in \partial\Omega$. Since $\Omega$ is of class $C^{2,\alpha}$, we can find a neighbourhood (in $\mathbb{R}^n$) $U$ of $x_0$ an open set $V \subset \mathbb{R}^n$ containing $\overline{B}_1$, a $C^{2,\alpha}$ diffeomorphism $\Phi : U \to V$, and some $\rho > 0$ such that $B_1 \subset \Phi(\Omega)$ and $B_\rho(\tilde{x}_0) \cap \partial B_1 \subset \Phi(\partial\Omega \cap U) \subset \partial B_1$, where $\tilde{x}_0 \doteqdot \Phi(x_0)$.

Set $(\tilde{a}, \tilde{b}, \tilde{c}) \doteqdot (a, b, c) \circ \Phi^{-1}|_{\overline{B}_1}$, $\tilde{f} \doteqdot f \circ \Phi^{-1}|_{\overline{B}_1}$, and $\tilde{u} \doteqdot u \circ \Phi^{-1}|_{\overline{B}_1}$, and let $\tilde{\phi} : B_{3/2} \to \mathbb{R}$ be the radial extension of $\tilde{u}$; that is, $\tilde{\phi}(x) \doteqdot |x|\tilde{u}(x/|x|)$ for $x \neq 0$ and $\tilde{\phi}(0) \doteqdot 0$. For each $k \in \mathbb{N}$, consider $\tilde{\phi}_k \doteqdot \tilde{\phi} * \eta_{1/k}$, where $\eta_\varepsilon$ is the standard mollifier. Note that $\tilde{\phi}_k \to \phi$ in the uniform topology and $|\tilde{\phi}_k|_{C^{2,\alpha}(B_\rho(\tilde{x}_0))} \leq 2|\tilde{\phi}|_{C^{2,\alpha}(B_\rho(\tilde{x}_0))}$ for $k$ sufficiently large.

By hypothesis, we can find, for each $k \in \mathbb{N}$, a solution $\tilde{u}_k \in C^{2,\alpha}(\overline{\Omega})$ to the Dirichlet problem

$$\begin{cases} -\tilde{L}\tilde{u}_k = \tilde{f} & \text{in } B_1 \\ \tilde{u}_k = \tilde{\phi}_k & \text{on } \partial B_1 \,, \end{cases}$$

where $\tilde{L} \doteqdot \tilde{a}^{ij}D_iD_j + \tilde{b}^k D_k + \tilde{c}$. It follows from the maximum principle, the interior Schauder estimate, and the boundary Schauder estimate, in conjunction with the Arzelà–Ascoli theorem, that, after passing to a subsequence, the solutions $\tilde{u}_k$ converge to some limit $\tilde{u}_\infty \in C^0(\overline{B}_1) \cap C^2(B_1) \cap C^{2,\alpha}(\overline{B}_1 \cap B_\rho(\tilde{x}_0))$ uniformly in $\overline{B}_1$, in $C^2$ on compact subsets of $B_1$ and in $C^2(\overline{B}_1 \cap B_\rho(\tilde{x}_0))$. But since $\tilde{u}_\infty$ takes the same boundary values as $\tilde{u}$, we must have $\tilde{u}_\infty \equiv \tilde{u}$. The claim follows.

**4.4. Epilogue.** The previous sections provide a number of different paths to Theorem 4.1. For example, one could exploit the full force of the Sobolev theory to obtain $C^\infty$ solutions to our general class of linear equations, and then proceed by approximation using the Schauder theory (as in §4.1). Alternatively, one could avoid the Sobolev theory altogether, instead using potential theory (or $L^2$ theory) to solve Poisson's equation in $C^\infty$ (and hence in $C^{2,\alpha}$ by approximation) and applying the method of continuity. In fact, Perron's method reduces the problem to the solution of Poisson's equation in $C^\infty$ over the unit ball.

In case the condition $c \le 0$ is not met, existence or uniqueness of solutions may fail. However, it is still possible to state a *Fredholm alternative* (see [**2**, Theorem 6.15]).

**4.5. Exercises.**

**Exercise 4.1.** Prove Corollary 4.4.

**Exercise 4.2.** Suppose that the coefficients $(a, b, c) : \Omega \to S^{n\times n} \times \mathbb{R}^n \times \mathbb{R}$ and the function $f : \Omega \to \mathbb{R}$ are $\alpha$-Hölder continuous. Suppose that $u \in C^2(\Omega)$ satisfies $-Lu = f$ in $\Omega$. Show that $u \in C^{2,\alpha}(B)$ for every ball $B \Subset \Omega$.

**Exercise 4.3.** Suppose that the coefficients $(a, b, c) : \Omega \to S^{n\times n} \times \mathbb{R}^n \times \mathbb{R}$ and the function $f : \Omega \to \mathbb{R}$ are of class $C^{k,\alpha}$ (resp. $C^\infty$) and the boundary datum $\phi$ is of class $C^{k+2,\alpha}$ (resp. $C^\infty$). Suppose that $u \in C^2(\Omega)$ satisfies $-Lu = f$ in $\Omega$. Show that $u \in C^{k+2,\alpha}(B)$ (resp. $C^\infty(B)$) for every ball $B \Subset \Omega$. *Hint 1: If $u$ is of class $C^{j+2}$ for some $j < k$, then the equation $-Lu = f$ may be differentiated $j$ times. Hint 2: Given a function $v : \Omega \to \mathbb{R}$, a unit coordinate direction $e_\ell$, $x \in B$, and sufficiently small $h$, consider the difference quotient*

$$\delta_\ell^h v(x) \doteqdot \frac{v(x + he_\ell) - v(x)}{h}\,.$$

A similar argument provides higher boundary regularity, assuming higher boundary regularity of the data. See [**2**, §6.4].

## 5. Quasilinear equations — an introduction

*Hilbert's 19th problem*[11] asks whether minimizers $u : \Omega \to \mathbb{R}$ of *elliptic energy functionals*

$$E(u) \doteqdot \int_\Omega F(x, u(x), Du(x))dx \tag{5.1}$$

are necessarily smooth. Here, $\Omega \subset \mathbb{R}^n$ is any bounded open set and we require that $F$ be smooth in all arguments $(x, z, p) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ and strictly convex with respect to the third (this is the "ellipticity" requirement). In order to ensure smoothness up to $\partial\Omega$, $\partial\Omega$ should be smooth too, but this is not a requirement for interior regularity.

Minimization problems of the form (5.1) are ubiquitous in physics. Indeed, the Lagrangian school of thought (*Fermat's principle*) postulates that physical laws must emerge from an underlying energy minimization principle. Elliptic minimization problems also arise frequently in geometry.

Observe that any smooth stationary point $u$ of (5.1) (with respect to smooth, compactly supported perturbations) satisfies the equation[12]

$$a^{ij}(\cdot, u, Du)u_{ij} + b(\cdot, u, Du) = 0\,, \tag{5.2}$$

where

$$a^{ij}(\cdot, u, Du) \doteqdot \frac{\partial^2 F}{\partial p_i \partial p_j}(\cdot, u, Du)$$

and

$$b(\cdot, u, Du) \doteqdot \frac{\partial^2 F}{\partial p_i \partial z}(\cdot, u, Du)u_i + \frac{\partial^2 F}{\partial p_i \partial x^i}(\cdot, u, Du) - \frac{\partial F}{\partial z}(\cdot, u, Du)\,.$$

Indeed, given any $\eta \in C^\infty(\Omega)$ with $\operatorname{spt}\eta \Subset \Omega$,

$$\begin{aligned} 0 = \frac{d}{ds}\Big|_{s=0} E(u + s\eta) &= \frac{d}{ds}\Big|_{s=0} \int_\Omega F(x, (u+s\eta)(x), D(u+s\eta)(x))dx \\ &= \int_\Omega \left(\frac{\partial F}{\partial p_i}(\cdot, u, Du)\eta_i + \frac{\partial F}{\partial z}(\cdot, u, Du)\eta\right) dx \qquad (5.3) \\ &= -\int_\Omega \left(D_i \frac{\partial F}{\partial p_i}(\cdot, u, Du) - \frac{\partial F}{\partial z}(\cdot, u, Du)\right)\eta\, dx \\ &= -\int_\Omega \left(\frac{\partial^2 F}{\partial p_i \partial p_j}(\cdot, u, Du)u_{ij} + \frac{\partial^2 F}{\partial p_i \partial z}(\cdot, u, Du)u_i \right. \\ &\qquad \left. + \frac{\partial^2 F}{\partial p_i \partial x^i}(\cdot, u, Du) - \frac{\partial F}{\partial z}(\cdot, u, Du)\right)\eta\, dx\,. \end{aligned}$$

[11] In fact, Hilbert's formulation demands that $u$ be analytic (assuming, of course, that $F$ is analytic).

[12] Equation (5.2) is called the Euler–Lagrange equation corresponding to the energy $E$.

The equation (5.2) is not necessarily a linear equation, since the coefficients $a$ and $b$ may depend on $x$, $u$ and $Du$ in a nonlinear way. On the other hand, the second derivatives do appear in a linear way, so the equation is referred to as QUASILINEAR.

A function $u \in W^{1,1}(\Omega)$ is said to satisfy (5.2) weakly (IN THE SENSE OF DISTRIBUTIONS) if (5.3) holds. "Direct methods" in the calculus of variations were already known to provide the existence of minimizers[13] $u \in W^{1,2}(\Omega)$ of the functional $E$ (with respect to appropriate boundary conditions). So Hilbert's 19th problem is reduced to proving that solutions to equations of the form (5.2) having one weak derivative in $L^2$ are necessarily smooth. Observe that the strict convexity of the density $F$ ensures that (5.2) is STRICTLY ELLIPTIC; that is,

$$a^{ij} > 0 \text{ in } \Omega$$

in the sense of symmetric bilinear forms. However, since we only know that $u \in W^{1,2}(\Omega)$, the coefficients of (5.2) need not be continuous nor even bounded.

The *existence* of solutions to partial differential equations of the form (5.2) (with appropriately prescribed boundary conditions) is the content of *Hilbert's 20th problem.*

A key observation is that any derivative of a solution to (5.2) satisfies a *divergence form* linear equation. Indeed, by testing (5.3) against $D_s\eta$ and integrating by parts, we find that any weak derivative, $v = D_s u \in L^2(\Omega)$, of a solution $u \in W^{1,2}(\Omega)$ of (5.2) satisfies (weakly)

$$D_i\left(a^{ij}v_j + b^i v\right) + c^i v_i + dv = D_i f^i + g\,, \tag{5.4}$$

where

$$\begin{aligned}
a^{ij}(x) &\doteqdot \frac{\partial^2 F}{\partial p_i \partial p_j}(x, u(x), Du(x))\,,\\
b^{i}(x) &\doteqdot \frac{\partial^2 F}{\partial p_i \partial z}(x, u(x), Du(x))\,,\\
c^{i}(x) &\doteqdot -\frac{\partial^2 F}{\partial z \partial p_i}(x, u(x), Du(x))\,,\\
d(x) &\doteqdot -\frac{\partial^2 F}{\partial z^2}(x, u(x), Du(x))\,,\\
f^{i}(x) &\doteqdot -\frac{\partial^2 F}{\partial p_i \partial x^s}(x, u(x), Du(x))\,,
\end{aligned}$$

[13]So long as $E$ happens to be suitably *coercive*. Note that, by Hölder's inequality, $W^{1,2}(\Omega) \subset W^{1,1}(\Omega)$.

and

$$g(x) \doteqdot \frac{\partial^2 F}{\partial z \partial x^s}(x, u(x), Du(x))\,.$$

Building on the work of Bernstein and Schauder, Hilbert's 19th and 20th problems were resolved, independently, by Ennio de Giorgi and John F. Nash Jr in 1957[14].

The essential breakthrough was an estimate for the Hölder continuity of solutions to (5.4) (and hence of $v = Du$, where $u$ is a stationary point of (5.1)). This result has far-reaching consequences and is considered one of the most significant mathematical breakthroughs of the 20th century[15]. It is of a different nature than any of the regularity results which had existed previously — these results can all be seen as arising from perturbations of the Laplace equation (they are "perturbative results"); in Schauder-type estimates, for example, one always exploits the fact that, when zooming in on a solution at a point, the operator is closer and closer to the Laplacian. In the de Giorgi–Nash theorem, this is no longer the case: the uniform ellipticity is preserved by scaling, but the equation does not become "better", nor any closer to the Laplace equation.

We will prove the de Giorgi–Nash theorem in §6. We observe here that it provides an affirmative resolution to the 19th problem since Schauder had already proved that solutions with Hölder continuous first derivatives are smooth. Indeed, let $u \in W^{1,2}(\Omega)$ be a minimizer of (5.1). Since any derivative of $u$ satisfies (5.4), the de Giorgi–Nash estimate implies that $Du$ is uniformly Hölder continuous on any domain compactly contained in $\Omega$. But then $u$ is bounded and the coefficients of (5.2) are uniformly Hölder continuous on any such domain, so the Schauder estimate[16] implies that $D^2u$ is uniformly Hölder continuous on any domain compactly contained in $\Omega$. Since differentiation of (5.2) $k$-times yields a linear equation for $D^k u$ with the same leading coefficients $a^{ij}$ and remaining coefficients depending smoothly on $D^\ell u$ for $\ell$ at most $k+1$, arguing inductively we conclude that $D^k u$ is uniformly Hölder continuous on any domain compactly contained in $\Omega$ for all $k \geq 0$. In particular, $u$ is smooth in $\Omega$. Bernstein's method then shows that $u$ is analytic if $F$ is.

Moreover, as we shall see, by applying the method of continuity and the implicit function theorem, the a priori estimates of de Giorgi–Nash and Schauder also yield a satisfactory solution to Hilbert's 20th problem.

[14] A different proof was later given by Jürgen Moser. The two dimensional case had previously been solved by Morrey.

[15] It is oft said that had only one of the pair, de Giorgi or Nash, reached the solution, then he would surely have been awarded the Fields medal for the discovery. But such is the nature of awards.

[16] In fact, a "divergence form" counterpart of the estimate stated in Theorem 3.1.

Finally, let us note that, by proving a version of the de Giorgi–Nash estimate for linear equations in *non-divergence* form, we shall also be able to establish the (unique) existence of smooth solutions to many *fully nonlinear* elliptic equations (that is, equations which are nonlinear in the second derivatives). This is the content of the celebrated KRYLOV–SAFANOV THEORY, which will be taken up in Sections 10–11.

**5.1. Appendix: Sobolev spaces.** Recall that a function $u : \Omega \to \mathbb{R}$, $\Omega \underset{\text{open}}{\subset} \mathbb{R}^n$, is WEAKLY DIFFERENTIABLE if, for each $i = 1, \dots \mathbb{N}$, there exists a function $u_i \in L^1_{\text{loc}}(\Omega)$ such that

$$\int_\Omega u_i \eta = -\int_\Omega u \eta_i \text{ for all } \eta \in C^\infty_0(\Omega)\,.$$

The family of linear maps $x \mapsto Du|_x : \mathbb{R}^n \to \mathbb{R}$ defined by $Du|_x(v) \doteqdot D_v u_x \doteqdot v^i u_i(x)$ is called the WEAK DERIVATIVE of $u$. Since $C^\infty_0(\Omega)$ is dense in $L^1_{\text{loc}}(\Omega)$, the weak derivative of a function, if it exists, is unique up to pointwise almost everywhere equivalence. So the weak derivative of a continuously differentiable function agrees (up to pointwise everywhere equivalence) with the pointwise derivative.

The SOBOLEV SPACE $W^{k,p}(\Omega)$ consists of the functions $u \in L^p(\Omega)$ which admit $k$ weak derivatives, each of which lies in $L^p(\Omega)$. It is equipped with the norm $|u|_{W^{k,p}(\Omega)}$ defined by

$$|u|^p_{W^{k,p}(\Omega)} \doteqdot \sum_{j=0}^k |D^j u|^p_{L^p(\Omega)}\,,$$

where $D^j u$ has the obvious interpretation as a multilinear map[17]. The resulting normed linear space is complete, and hence $W^{k,p}(\Omega)$ is a Banach space.

A fundamental property of the Sobolev spaces is the fact that the smooth functions form a dense subspace. That is, every Sobolev function $u \in W^{k,p}(\Omega)$ may be approximated in the $W^{k,p}(\Omega)$ topology by smooth functions.

We refer the reader to the book of Evans [**1**] for further development of the theory of Sobolev spaces and weak solutions to partial differential equations.

**5.2. Exercises.**

**Exercise 5.1.** Show that the Laplace equation,

$$\Delta u = 0\,,$$

[17]We shall always use the Hilbert–Schmidt norm for multilinear maps.

is the Euler–Lagrange equation for the Dirichlet energy,

$$E(u) \doteqdot \int_\Omega |\nabla u|^2 dx\,,$$

where

$$\Delta u \doteqdot \operatorname{div}(\nabla u)\,.$$

**Exercise 5.2.** Show that the minimal surface equation,

$$H_{\operatorname{graph} u} = 0\,,$$

where

$$H_{\operatorname{graph} u} \doteqdot \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\,,$$

is the Euler–Lagrange equation for the area functional,

$$E(u) \doteqdot \int_\Omega \sqrt{1+|\nabla u|^2}\,dx\,.$$

**Exercise 5.3.** Given $\kappa \in \mathbb{R}$ and $\beta \in C^0(\partial\Omega)$, show that critical points of the energy

$$E(u) \doteqdot \underbrace{\int_\Omega \sqrt{1+|\nabla u|^2}\,dx}_{\text{surface energy}} + \underbrace{\frac{\kappa}{2}\int_\Omega u^2\,dx}_{\text{gravitational potential}} + \underbrace{\int_{\partial\Omega} \beta u\,dx}_{\text{wetting energy}}$$

with respect to perturbations $\eta \in C^\infty(\overline{\Omega})$ satisfy the capillary surface problem

$$\begin{cases} H_{\operatorname{graph} u} = \kappa u \ \text{ in } \ \Omega \\ \langle \nu\,,\,\gamma\rangle = \beta \ \text{ on } \ \partial\Omega\,, \end{cases}$$

where $\nu(x)$ is the upward unit normal to $\operatorname{graph} u$ at $(x, u(x))$ and $\gamma$ is the outward unit normal to $\Omega$.

## 6. The Harnack inequality of de Giorgi, Nash and Moser — Hölder continuity of solutions to linear elliptic equations of divergence form

Our goal is to prove an *a priori* Hölder estimate for weak solutions to linear elliptic equations of divergence form. That is, equations of the form

$$- \operatorname{div}\big(A(\cdot, u, Du)\big) = B(\cdot, u, Du) \ \text{ in } \ \Omega\,, \tag{6.1}$$

where $A : \Omega \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $B : \Omega \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are of the form

$$A^i(x, z, p) = a^{ij}(x)p_j + b^i(x)z + f^i(x) \ \text{ and } \ B(x, z, p) = c^i(x)p_i + d(x)z + g(x)\,.$$

We recall that, assuming $A$ and $B$ are measurable, a function $u : \Omega \to \mathbb{R}$ is a WEAK SOLUTION to (6.1) if $u \in W^{1,2}(\Omega)$ and

$$\int_\Omega A(\cdot, u, Du) \cdot D\eta = \int_\Omega B(\cdot, u, Du)\eta$$

for every $\eta \in W^{1,2}_0(\Omega)$. By the divergence theorem, classical solutions are weak solutions.

Our motivation, as outlined in §5, is to prove an *a priori* Hölder estimate for the first derivative of solutions to quasilinear equations.

The Hölder estimate (Theorem 6.5 below) is obtained from the Harnack inequality of de Giorgi and Nash (Theorem 6.3 below), which, in turn, results from the combination of two estimates: a so-called *mean value inequality* for subsolutions and a so-called *weak Harnack inequality* for supersolutions.

The mean value inequality provides an estimate for the supremum of a non-negative *subsolution* to (6.1) in terms of its $L^p$-norm, $p > 1$, and the equation data, while the weak Harnack inequality provides an estimate for the $L^p$-norm, $1 \leq p < \frac{n}{n-2}$, of a non-negative *supersolution* to (6.1) in terms of its infimum and the equation data. Combining the two, we are able to estimate the supremum of a solution to (6.1) in terms of its infimum and the equation data; i.e. a *Harnack inequality*. We follow the argument of Moser and John–Nirenberg.

Before proving these inequalities, let us illustrate the central idea in the special case where $b, c, d, f$ and $g$ are all zero. The idea is to obtain integral estimates for solutions using the divergence theorem and exploit the fact that, for a measurable function[18] $u : \Omega \to \mathbb{R}$,

$$\lim_{p\to\infty}\left(\fint_\Omega |u|^p\right)^{\frac{1}{p}} = \operatorname*{ess\,sup}_\Omega |u| \quad\text{and}\quad \lim_{p\to-\infty}\left(\fint_\Omega |u|^p\right)^{\frac{1}{p}} = \operatorname*{ess\,inf}_\Omega |u|\,.$$

Suppose that $u \in W^{1,2}(B_1)$ and consider the test function $\zeta \doteqdot u^\beta\eta^2$, where $\beta \neq 0$ and $\eta$ is a smooth function with support in $B_1$. If $\zeta \in W^{1,2}_0(B_1)$, then, assuming $a \in L^\infty(B_1(y), S^{n\times n})$,

$$\int_{B_1} a^{ij}D_iuD_j\zeta = \beta\int_{B_1} u^{\beta-1}\eta^2a^{ij}D_iuD_ju + 2\int_{B_1} u^\beta\eta a^{ij}D_iuD_j\eta\,.$$

In case $u$ is a subsolution, we consider only $\beta > 0$, and in case $u$ is a supersolution, we consider only $\beta < 0$, so that

$$|\beta|\int_{B_1} u^{\beta-1}\eta^2a^{ij}D_iuD_ju \le 2\left|\int_{B_1} u^\beta\eta a^{ij}D_iuD_j\eta\right|\,.$$

Assuming $a \ge \lambda\delta$ and $|a| \le \Lambda$, the Cauchy–Schwarz inequality yields

$$\begin{aligned}|\beta|\lambda\int_{B_1} u^{\beta-1}|Du|^2\eta^2 &\le 2\Lambda\int_{B_1} u^\beta\eta|Du||D\eta|\\ &\le \frac{|\beta|\lambda}{2}\int_{B_1} u^{\beta-1}|Du|^2\eta^2 + \frac{2\Lambda^2}{|\beta|\lambda}\int_{B_1} u^{\beta+1}|D\eta|^2\end{aligned}$$

and hence

$$\int_{B_1} u^{\beta-1}|Du|^2\eta^2 \le \frac{4\Lambda^2}{\beta^2\lambda^2}\int_{B_1} u^{\beta+1}|D\eta|^2\,.$$

Assuming $\beta \neq -1$, set $2\gamma \doteqdot \beta+1$ and $w \doteqdot u^\gamma$, so that

$$\int_{B_1} |Dw|^2\eta^2 \le \frac{4\Lambda^2\gamma^2}{\beta^2\lambda^2}\int_{B_1} w^2|D\eta|^2$$

and hence

$$\int_{B_1} |D(w\eta)|^2 \le \left(\frac{\Lambda^2(\beta+1)^2}{\lambda^2\beta^2}+1\right)\int_{B_1} w^2|D\eta|^2\,.$$

We now apply the Sobolev inequality to obtain

$$\left(\int_{B_1} |w\eta|^{2\kappa}\right)^{\frac{1}{2\kappa}} \le C\left(\int_{B_1} w^2|D\eta|^2\right)^{\frac{1}{2}}\,,$$

[18] Recall that the ESSENTIAL SUPREMUM of a function $u$ is the least *essential upper bound* for $u$, where a value $a \in \mathbb{R}$ is said to be an ESSENTIAL UPPER BOUND if the set of points whose value exceeds $a$ has measure zero. The ESSENTIAL INFIMUM is defined similarly. The $L^\infty$ norm $|u|_{L^\infty(\Omega)}$ of a measurable function $u : \Omega \to \mathbb{R}$ is the essential supremum over $\Omega$ of $|u|$. Since every essentially bounded measurable function agrees almost everywhere with some bounded measurable function, every $u \in L^\infty(\Omega)$ has a representative.

where $\kappa = \frac{n}{n-2}$ when $n \geq 3$ or any fixed number in $(0, \infty)$ when $n = 2$, and, assuming $\beta$ is bounded away from zero, $C$ depends only on $n$ and $\Lambda/\lambda$. Choosing $\eta$ so that $\eta = 1$ on $B_{3/4}$ and $|D\eta| \leq 10$, we find that

$$\left(\int_{B_{3/4}} |u|^{2\kappa\gamma}\right)^{\frac{1}{2\kappa}} \leq C \left(\int_{B_1} |u|^{2\gamma}\right)^{\frac{1}{2}} .$$

So we have bootstrapped an $L^2$ estimate for $u^\gamma$ in $B_1$ to an $L^{2\kappa}$ estimate in the smaller ball $B_{3/4}$. Note that $\kappa > 1$. The idea now is to iterate this until we arrive at the desired estimates (in $B_{1/2}$).

## 6.1. The mean value inequality.

**Theorem 6.1.** *There exists $C = C(n, p, q, \Lambda/\lambda, \nu) < \infty$ with the following property. Suppose that the coefficients $a \in L^\infty(\Omega, S^{n\times n})$, $b, c, f \in L^q(\Omega, \mathbb{R}^n)$ and $d, g \in L^{\frac{q}{2}}(\Omega)$, $q > n$, satisfy*

$$a^{ij} \geq \lambda\delta^{ij} \ \text{ in } \Omega \ \text{ and } |a|_{L^\infty(\Omega,S^{n\times n})} \leq \Lambda\,, \quad \lambda > 0\,,\ \Lambda < \infty \tag{6.2}$$

*and*

$$\frac{R^{2-\frac{2n}{q}}}{\lambda^2}\left(|b|^2_{L^q(\Omega,\mathbb{R}^n)} + |c|^2_{L^q(\Omega,\mathbb{R}^n)}\right) + \frac{R^{2-\frac{2n}{q}}}{\lambda}|d|_{L^{\frac{q}{2}}(\Omega)} \leq \nu\,, \quad \nu < \infty\,. \tag{6.3}$$

*Let $u \in W^{1,2}(\Omega)$ be a subsolution to (6.1). If $p > 1$ and $B_{2R}(y) \subset \Omega$, then $u \in L^\infty(B_R(y))$ and*

$$\sup_{B_R(y)} u \leq C\left(R^{-\frac{n}{p}}|u_+|_{L^p(B_{2R}(y))} + \frac{R^{1-\frac{n}{q}}}{\lambda}|f|_{L^q(B_{2R}(y),\mathbb{R}^n)} + \frac{R^{2-\frac{2n}{q}}}{\lambda}|g|_{L^{\frac{q}{2}}(B_{2R}(y))}\right).$$

**Proof.** We may assume, without loss of generality, that $R = 1/2$ and $u \geq 0$. We consider only the case that $f$ and $g$ are zero. The general case is proved by replacing

$$u \mapsto u + \lambda^{-1}\left(|f|_{L^q(B_1(y),\mathbb{R}^n)} + |g|_{L^{\frac{q}{2}}(B_1(y))}\right)$$

in the proof.

We proceed as outlined above. Given $\beta > 0$ and a smooth function $\eta$ with support compactly contained in $B_1$, consider the function $\zeta \doteqdot G(u)\eta^2$, where $G(z) \doteqdot z^\beta$ if $\beta \leq 1$ and

$$G(z) \doteqdot \begin{cases} z^\beta & \text{if } z \leq N \\ N^\beta + \beta N^{\beta-1}(z - N) & \text{if } z > N \end{cases}$$

if $\beta > 1$ (this modification is needed to ensure that $\zeta \in W^{1,2}_0(B_1)$). If $u \in L^{2\gamma}(B_s)$, where $2\gamma \doteqdot \beta + 1$ and $s \leq 1$, then we may proceed as above, applying the Hölder inequality to control the terms involving $b, c$ and $d$, choosing $\eta$ appropriately and taking $N \to \infty$, to obtain the estimate

$$\left(\int_{B_r} |u|^{2\kappa\gamma}\right)^{\frac{1}{2\kappa\gamma}} \leq \left(\frac{C\gamma}{s-r}\right)^{\frac{1}{\gamma}} \left(\int_{B_s} |u|^{2\gamma}\right)^{\frac{1}{2\gamma}}$$

for any $r < s$, where $C = C(n, \Lambda/\lambda)$.

We now iterate this inequality to obtain an $L^\infty$ estimate: set $2\gamma_j \doteqdot \kappa^j p$ and define $r_j$ by $r_0 \doteqdot 1$ and $r_{j+1} \doteqdot \frac{1}{2}(\frac{1}{2} + r_j)$. Then

$$\left(\int_{B_{r_j}} u^{p\kappa^j}\right)^{\frac{1}{p\kappa^j}} \leq C_j^{\frac{2}{p}} \left(\int_{B_1} u^p\right)^{\frac{1}{p}} ,$$

where

$$C_j \doteqdot \prod_{i=0}^{j} (Cp)^{\kappa^{-i}} 2^{(i+1)\kappa^{-i}} \kappa^{i\kappa^{-i}} = (Cp)^{\sum_{i=0}^{j} \kappa^{-i}} 2^{\sum_{i=0}^{j} (i+1)\kappa^{-i}} \kappa^{\sum_{i=0}^{j} i\kappa^{-i}} .$$

Since $\kappa > 1$, $\sum_{i=0}^{\infty} \kappa^{-i} < \infty$ and $\sum_{i=0}^{\infty} i\kappa^{-i} < \infty$, taking $j \to \infty$, we conclude that

$$\sup_{B_{1/2}} u \leq C \left(\int_{B_1} |u|^{2p}\right)^{\frac{1}{2p}} . \qquad \square$$

### 6.2. The weak Harnack inequality.

**Theorem 6.2.** *There exists $C = C(n, p, q, \Lambda/\lambda, \nu) < \infty$ with the following property. Suppose that the coefficients $a \in L^\infty(\Omega, S^{n\times n})$, $b, c, f \in L^q(\Omega, \mathbb{R}^n)$ and $d, g \in L^{\frac{q}{2}}(\Omega)$, $q > n$, satisfy* (6.2) *and* (6.3)*. Let $u \in W^{1,2}(\Omega)$ be a non-negative supersolution to* (6.1)*. If $1 \leq p < \frac{n}{n-2}$ and $B_{4R}(y) \subset \Omega$, then*

$$R^{-\frac{n}{p}} |u|_{L^p(B_{2R}(y))} \leq C \left( \inf_{B_R(y)} u + \frac{R^{1-\frac{n}{q}}}{\lambda} |f|_{L^q(B_{2R}(y), \mathbb{R}^n)} + \frac{R^{2-\frac{2n}{q}}}{\lambda} |g|_{L^{\frac{q}{2}}(B_{2R}(y))} \right).$$

**Proof.** We may assume, without loss of generality, that $R = 1$. We consider only the case that $f$ and $g$ are zero. The general case is proved by replacing

$$u \mapsto u + \lambda^{-1} \left( |f|_{L^q(B_1(y))} + |g|_{L^{\frac{q}{2}}(B_1(y))} \right)$$

in the proof.

Proceeding as above, we obtain the estimate

$$\left(\int_{B_r} |u|^{2\kappa\gamma}\right)^{\frac{1}{2\kappa\gamma}} \leq \left(\frac{C\gamma}{s-r}\right)^{\frac{1}{\gamma}} \left(\int_{B_s} |u|^{2\gamma}\right)^{\frac{1}{2\gamma}} \tag{6.4}$$

if $\gamma > 0$, or

$$\left( \int_{B_s} |u|^{2\gamma} \right)^{\frac{1}{2\gamma}} \leq \left( \frac{C|\gamma|}{s-r} \right)^{\frac{1}{|\gamma|}} \left( \int_{B_r} |u|^{2\kappa\gamma} \right)^{\frac{1}{2\kappa\gamma}} \tag{6.5}$$

if $\gamma < 0$, where $C = C(n, \Lambda/\lambda)$.

Fix $p_0 > 0$. Iterating (6.5) with $2\gamma_j = -\kappa^j p_0$ yields

$$\left( \int_{B_{3/2}} u^{-p_0} \right)^{-\frac{1}{p_0}} \leq C \inf_{B_{1/2}} u \,,$$

where $C = C(n, p_0, \Lambda/\lambda)$.

On the other hand, iterating (6.4) with $2\gamma_j = \kappa^{-j} p$ yields, for any $0 < p < \kappa$,

$$\left( \int_{B_1} u^p \right)^{\frac{1}{p}} \leq C \left( \int_{B_{3/2}} u^{\kappa^{-j} p} \right)^{\frac{1}{\kappa^{-j} p}}$$

for all $j = 1, 2, \ldots$, where $C = C(n, p, \Lambda/\lambda)$. If $p > p_0$, then taking $j$ sufficiently large and applying the Hölder inequality yields

$$\left( \int_{B_1} u^p \right)^{\frac{1}{p}} \leq C \left( \int_{B_{3/2}} u^{p_0} \right)^{\frac{1}{p_0}} .$$

The theorem now follows from the fact that

$$\left( \int_{B_{3/2}} u^{p_0} \right) \left( \int_{B_{3/2}} u^{-p_0} \right) \leq C$$

for some $p_0 \in (0,1)$ and $C = C(n)$. This is a consequence of the John–Nirenberg inequality; however, we shall not prove it here (the details can be found in [**2**, §8.6]). □

**6.3. The Harnack inequality.** Combining the mean value and weak Harnack inequalities yields the following Harnack inequality.

**Theorem 6.3.** *There exists $C = C(n, p, q, \Lambda/\lambda, \nu, r, R) < \infty$ with the following property. Let $\Omega \subset \mathbb{R}^n$ be a connected open set with* $\operatorname{diam} \Omega \leq R$. *Suppose that the coefficients $a \in L^\infty(\Omega, S^{n\times n})$, $b, c, f \in L^q(\Omega, \mathbb{R}^n)$ and $d, g \in L^{\frac{q}{2}}(\Omega)$, $q > n$, satisfy* (6.2) *and* (6.3). *If $u \in W^{1,2}(\Omega)$ is a non-negative solution to* (6.1)*, then $u \in L^\infty_{\mathrm{loc}}(\Omega)$ and, for every $\Omega' \Subset \Omega$ with* $\operatorname{dist}(\Omega', \Omega) \geq r$,

$$\sup_{\Omega'} u \leq C \left( \inf_{\Omega'} u + \lambda^{-1} \left[ |f|_{L^q(\Omega, \mathbb{R}^n)} + |g|_{L^{\frac{q}{2}}(\Omega)} \right] \right) .$$

**Proof.** Fix some $p \in (1, \frac{n}{n-2})$. Given any ball $B_{4R}(y) \Subset \Omega$, Theorems 6.1 and 6.2 yield

$$\sup_{B_R(y)} u \leq C\left(\inf_{B_R(y)} u + \frac{R^{1-\frac{n}{q}}}{\lambda}|f|_{L^q(B_{2R}(y),\mathbb{R}^n)} + \frac{R^{2-\frac{2n}{q}}}{\lambda}|g|_{L^{\frac{q}{2}}(B_{2R}(y))}\right), \quad (6.6)$$

where $C = C(n, p, q, \Lambda/\lambda, \nu)$. We may cover $\Omega'$ by a finite collection of open balls $B_j = B_{R_j(y_j)}$, $j = 0, 1, \ldots, N$, such that $B_{4R_j}(y_j) \Subset \Omega$ and $\cup_{j=0}^N B_j$ is connected, where $R_j \leq \operatorname{diam}\Omega$ and $N = N(\operatorname{dist}(\Omega', \partial\Omega))$.

Given any pair of points $x, y \in \Omega'$ we can find a path $\gamma : [0, 1] \to \cup_{j=0}^n B_j$ joining $x = \gamma(0)$ and $y = \gamma(1)$. After possibly relabelling the balls, we may arrange that $\gamma(0) \in B_0$, $\gamma(1) \in B_M$, $M \leq N$, and divide $[0, 1]$ into intervals $[t_{j-1}, t_j]$, $0 = t_0 < t_1 < \cdots < t_M = 1$ such that that $\gamma(t_j) \in B_{j-1} \cap B_j$ for $1 \leq j \leq M - 1$.

After possibly perturbing $x$, $y$ and $t_j$ slightly (to account for the fact that $u$ is only defined up to sets of zero measure), (6.6) implies that

$$\begin{aligned} u(\gamma(t_j)) &\leq \sup_{B_j(y)} u \\ &\leq C\left(\inf_{B_j(y)} u + \frac{R_j^{1-\frac{n}{q}}}{\lambda}|f|_{L^q(\Omega,\mathbb{R}^n)} + \frac{R_j^{2-\frac{2n}{q}}}{\lambda}|g|_{L^{\frac{q}{2}}(\Omega)}\right) \\ &\leq C\left(u(\gamma(t_{j+1})) + \frac{R_j^{1-\frac{n}{q}}}{\lambda}|f|_{L^q(\Omega,\mathbb{R}^n)} + \frac{R_j^{2-\frac{2n}{q}}}{\lambda}|g|_{L^{\frac{q}{2}}(\Omega)}\right) \end{aligned}$$

and hence

$$u(x) \leq C'\left(u(y) + \lambda^{-1}\left[|f|_{L^q(\Omega,\mathbb{R}^n)} + |g|_{L^{\frac{q}{2}}(\Omega)}\right]\right),$$

where $C = C(n, p, q, \Lambda/\lambda, \nu, r, R)$. Taking the essential supremum over $x \in \Omega'$ and the essential infimum over $y \in \Omega'$ yields the claim. $\square$

**6.4. Hölder continuity of solutions to linear elliptic equations of divergence form.** The key application of the de Giorgi–Nash–Moser theory is a Hölder estimate for solutions to linear elliptic equations of divergence form.

We will need the following lemma.

**Lemma 6.4.** *Given $\gamma, \tau < 1$ and $\mu \in (0, 1)$, there exist $C = C(\gamma, \tau) < \infty$ and $\alpha = \alpha(\gamma, \tau, \mu) \in (0, 1)$ with the following property. Let $\omega : (0, R_0] \to \mathbb{R}$ be a non-decreasing function satisfying, for all $R \in (0, R_0]$,*

$$\omega(\tau R) \leq \gamma\omega(R) + \sigma(R)$$

*for some non-decreasing function $\sigma$. For any $R \in (0, R_0]$,*

$$\omega(R) \leq C\left(\left(\frac{R}{R_0}\right)^{\alpha} \omega(R_0) + \sigma(R^{\mu} R_0^{1-\mu})\right).$$

**Proof.** Fix some $0 < R_1 < R_0$. Iterating the hypothesis yields

$$\begin{aligned}\omega(\tau^m R) &\leq \gamma^m \omega(R) + \sum_{i=1}^{m-1} \sigma(R) \\ &\leq \gamma^m \omega(R) + \frac{\sigma(R)}{1-\gamma}\end{aligned}$$

for all $m \in \mathbb{N}$ and $R \leq R_0$. Fix $R_1 \leq R_0$. Given $R \in (0, R_1]$, we can choose $m$ so that $\tau^m R_1 \leq R \leq \tau^{m-1} R_1$, so that, by the monotonicity of $\omega$ and $\sigma$,

$$\begin{aligned}\omega(R) &\leq \omega(\tau^{m-1} R_1) \\ &\leq \gamma^{m-1} \omega(R_1) + \frac{\sigma(R_1)}{1-\gamma} \\ &\leq \gamma^{m-1} \omega(R_0) + \frac{\sigma(R_1)}{1-\gamma}.\end{aligned}$$

Since $\tau^m \leq R/R_1$,

$$\gamma^m \leq \left(\frac{R}{R_1}\right)^{\frac{\log \gamma}{\log \tau}},$$

and hence

$$\omega(R) \leq \frac{1}{\gamma}\left(\frac{R}{R_1}\right)^{\frac{\log \gamma}{\log \tau}} \omega(R_0) + \frac{\sigma(R_1)}{1-\gamma}.$$

If we take $R_1 \doteqdot R_0^{1-\mu} R^{\mu}$, then

$$\omega(R) \leq \frac{1}{\gamma}\left(\frac{R}{R_0}\right)^{(1-\mu)\frac{\log \gamma}{\log \tau}} \omega(R_0) + \frac{\sigma(R_0^{1-\mu} R^{\mu})}{1-\gamma}.$$

The claim follows. □

**Theorem 6.5** (de Giorgi–Nash)**.** *There exist $C(n, \Lambda/\lambda, \nu, q, R, \rho) < \infty$ and $\alpha(n, \Lambda/\lambda, \nu R, \rho) \in (0, 1)$ with the following property. Let $\Omega \subset \mathbb{R}^n$ be a connected open set with* $\operatorname{diam} \Omega \leq R$*. Suppose that the coefficients $a \in L^{\infty}(\Omega, S^{n\times n})$, $b, c, f \in L^q(\Omega, \mathbb{R}^n)$ and $d, g \in L^{\frac{q}{2}}(\Omega)$, $q > n$, satisfy* (6.2) *and* (6.3)*. If $u \in W^{1,2}(\Omega)$ is a solution to* (6.1)*, then $u \in C^{\alpha}(\Omega)$ and*

$$|u|_{C^{\alpha}(\Omega')} \leq C\left(|u|_{L^2(\Omega)} + \lambda^{-1}\left[|f|_{L^q(\Omega, \mathbb{R}^n)} + |g|_{L^{\frac{q}{2}}(\Omega)}\right]\right)$$

*for every $\Omega' \Subset \Omega$ with* $\operatorname{dist}(\Omega', \partial\Omega) \geq \rho$*.*

**Proof.** Suppose that $B_{R_0} \doteqdot B_{R_0}(x) \subset \Omega$. Given $r \leq R_0$, define

$$M_r \doteqdot \sup_{B_r} u \quad \text{and} \quad m \doteqdot \inf_{B_r} u$$

and set

$$M \doteqdot \sup_{B_{R_0}} |u| \,.$$

Applying the Harnack inequality[19] (Theorem 6.3) with $p = 1$ to $M_{4R} - u$ and $u - m_{4R}$ on the domain on $B_{4R}$, where $R \leq R_0/4$, yields

$$M_{4R} - m_R \leq C(M_{4R} - M_R + k(R))$$

and

$$M_R - m_{4R} \leq C(m_R - m_{4R} + k(R))\,,$$

where

$$k(R) \doteqdot \frac{R^{1-\frac{n}{q}}}{\lambda}\left(|f|_{L^q(B_{R_0},\mathbb{R}^n)} + M|b|_{L^q(B_{R_0},\mathbb{R}^n)}\right) + \frac{R^{2-\frac{2n}{q}}}{\lambda}\left(|g|_{L^{\frac{q}{2}}(B_{R_0})} + M|d|_{L^{\frac{q}{2}}(B_{R_0})}\right)\,.$$

Adding the two inequalities yields

$$(M_{4R} - m_{4R}) + (M_R - m_R) \leq C\left(M_{4R} - m_{4R} - (M_R - m_R) + 2k(R)\right)\,.$$

Thus, if we define

$$\omega(R) \doteqdot \operatorname{osc}_{B_R} u \doteqdot \sup_{x,y\in B_R}(u(x) - u(y))\,,$$

then we find that

$$\omega(R) \leq \gamma\omega(4R) + 2k(R)\,,$$

where $\gamma = \gamma(n, \lambda/\Lambda, \nu R_0, q)$.

Taking $\sigma(R) \doteqdot 2k(R/4)$ for $R \leq R_0$, Lemma 6.4 now yields

$$\omega(R) \leq C\left(\left(\frac{R}{R_0}\right)^{\alpha}\omega(R_0) + \sigma(R^{\mu}R_0^{1-\mu})\right)$$

for $R \leq R_0$, where $\alpha = \alpha(n, \frac{\lambda}{\Lambda}, \nu R_0, q, \mu) \in (0,1)$ and $C = C(n, \frac{\lambda}{\Lambda}, \nu R_0, q) < \infty$. Estimating

$$\sigma(R^{\mu}R_0^{1-\mu}) \leq 2\left(\frac{R}{R_0}\right)^{\mu(1-n/q)} k(R_0)$$

and choosing $\mu$ so that $\mu(1 - n/q) = \alpha$, we obtain

$$\begin{aligned}\omega(R) &\leq C\left(\frac{R}{R_0}\right)^{\alpha}\left(\omega(R_0) + k(R_0)\right)\\ &\leq C\left(\frac{R}{R_0}\right)^{\alpha}\left(\sup_{B_{R_0}}|u| + k(R_0)\right)\,,\end{aligned}$$

where $C = C(n, \lambda/\Lambda, \nu R_0, q) < \infty$ and $\alpha = \alpha(n, \lambda/\Lambda, \nu R_0, q)$.

[19] In fact, we could also invoke the weak Harnack inequality here.

Applying the mean value inequality (Theorem 6.1), we conclude that

$$\omega(R) \leq C \left(\frac{R}{R_0}\right)^{\alpha} \left(|u|_{L^2(B_{R_0})} + k(R_0)\right) . \tag{6.7}$$

The claim follows from a chaining argument in the usual way, since this estimate holds on any ball $B_{R_0}(x)$ in $\Omega$. $\square$

**6.5. Estimates up to the boundary.** The mean value and weak Harnack inequalities both admit boundary versions. These may be stated, respectively, as follows.

**Theorem 6.6.** *There exists $C = C(n, p, q, \Lambda/\lambda, \nu) < \infty$ with the following property. Suppose that the coefficients $a \in L^\infty(\Omega, S^{n\times n})$, $b, c, f \in L^q(\Omega, \mathbb{R}^n)$ and $d, g \in L^{\frac{q}{2}}(\Omega)$, $q > n$, satisfy (6.2) and (6.3). Let $u \in W^{1,2}(\Omega)$ be a subsolution to (6.1). If $p > 1$ and $\sup_{\partial\Omega\cap B_{2R}(y)} u_+ \leq M$, then $u \in L^\infty(B_R(y) \cap \Omega)$ and*

$$\sup_{B_R(y)} u_M \leq C \left( R^{-\frac{n}{p}} |u_M|_{L^p(B_{2R}(y))} + \frac{R^{1-\frac{n}{q}}}{\lambda} |f|_{L^q(B_{2R}(y),\mathbb{R}^n)} + \frac{R^{2-\frac{2n}{q}}}{\lambda} |g|_{L^{\frac{q}{2}}(B_{2R}(y))} \right) ,$$

*where*

$$u_M \doteqdot \begin{cases} \max\{u(x), M\} & \text{if } x \in \Omega \\ M & \text{if } x \notin \Omega \, . \end{cases}$$

**Theorem 6.7.** *There exists $C = C(n, p, q, \Lambda/\lambda, \nu) < \infty$ with the following property. Suppose that the coefficients $a \in L^\infty(\Omega, S^{n\times n})$, $b, c, f \in L^q(\Omega, \mathbb{R}^n)$ and $d, g \in L^{\frac{q}{2}}(\Omega)$, $q > n$, satisfy (6.2) and (6.3). Let $u \in W^{1,2}(\Omega)$ be a non-negative supersolution to (6.1). If $1 \leq p < \frac{n}{n-2}$ and $\inf_{\partial\Omega\cap B_{4R}(y)} u \geq m$, then*

$$R^{-\frac{n}{p}} |u_m|_{L^p(B_{2R}(y))} \leq C \left( \inf_{B_R(y)} u_m + \frac{R^{1-\frac{n}{q}}}{\lambda} |f|_{L^q(B_{2R}(y),\mathbb{R}^n)} + \frac{R^{2-\frac{2n}{q}}}{\lambda} |g|_{L^{\frac{q}{2}}(B_{2R}(y))} \right) ,$$

*where*

$$u_m \doteqdot \begin{cases} \min\{u(x), m\} & \text{if } x \in \Omega \\ m & \text{if } x \notin \Omega \, . \end{cases}$$

The inequalities $\sup_{\partial\Omega\cap B_{2R}(y)} u_+ \leq M$ and $\inf_{\partial\Omega\cap B_{4R}(y)} u \geq m$ in Theorems 6.6 and 6.7 are interpreted in the following sense: a function $u \in W^{1,2}(\Omega)$ is said to satisfy $u \leq 0$ in a subset $T$ of $\overline{\Omega}$ if $u_+$ is the limit of a sequence of functions in $C_0^1(\overline{\Omega} \setminus T)$.

Note that the only difference with respect to the interior versions is the replacement of $u$ by $u_M$ and $u_m$. Indeed, Theorems 6.6 and 6.7 are proved as in the proofs of Theorems 6.1 and 6.7, with $u$ replaced by $u_M$ or $u_m$, respectively, and the cutoff function modified accordingly (see [**2**, §8.10]).

Combining Theorems 6.6 and 6.7 yields a global Hölder estimate, so long as $\Omega$ satisfies the EXTERIOR CONE CONDITION. This means that each point $x \in \partial\Omega$ admits a neighbourhood $U$ and a right circular cone $C_x$ with vertex $x$ such that $C \cap U \cap \overline{\Omega} = \{x\}$. We denote by $\vartheta(C_x)$ the half-opening angle of $C_x$. Note that the exterior cone condition is weaker than the exterior sphere condition.

**Theorem 6.8.** *There exist constants* $C(n, \Lambda/\lambda, \nu, q, \alpha_0, R, \vartheta_0) < \infty$ *and* $\alpha(n, \Lambda/\lambda, \nu R, \alpha_0, \vartheta_0) \in (0,1)$ *with the following property. Let* $\Omega \subset \mathbb{R}^n$ *be a connected open set satisfying the exterior cone condition. Suppose that* $\operatorname{diam}(\Omega) \leq R$ *and* $\min_{x\in\partial\Omega} \vartheta(C_x) \geq \vartheta_0 > 0$*, and that the coefficients* $a \in L^\infty(\Omega, S^{n\times n})$*,* $b, c, f \in L^q(\Omega, \mathbb{R}^n)$ *and* $d, g \in L^{\frac{q}{2}}(\Omega)$*,* $q > n$*, satisfy* (6.2) *and* (6.3)*. If* $u \in W^{1,2}(\Omega)$ *is a solution to* (6.1) *and*

$$\operatorname{osc}_{\partial\Omega\cap B_r(x_0)} u \leq K r^{\alpha_0} \quad \textit{for all } x_0 \in \partial\Omega \ \textit{ and } r > 0\,,$$

*then* $u \in C^\alpha(\overline{\Omega})$ *and*

$$|u|_{C^\alpha(\Omega)} \leq C\left(\sup_\Omega |u| + K + \lambda^{-1}\left[|f|_{L^q(\Omega,\mathbb{R}^n)} + |g|_{L^{\frac{q}{2}}(\Omega)}\right]\right).$$

**Proof.** See [**2**, Theorems 8.27 and 8.29]. □

## 7. Equations of mean curvature type

As we shall see, the Hölder estimate of de Giorgi and Nash reduces the solution of (suitable) quasilinear elliptic boundary value problems to the establishment of *a priori* estimates in $C^1$. We will illustrate this in the context of equations of *mean curvature type*, which play an important role in many areas of geometry, materials science, mathematical physics and topology.

**7.1. Graphical hypersurfaces.** We will need to recall some basic differential geometry of graphical hypersurfaces. So suppose that $M = \operatorname{graph} u \subset \mathbb{R}^{n+1}$ is the graph of a $C^2$ function $u : \Omega \to \mathbb{R}$, for some open set $\Omega \subset \mathbb{R}^n$. We may parametrize $\operatorname{graph} u$ using the map $X : \Omega \to \mathbb{R}^{n+1}$ defined by

$$X(x) \doteqdot x + u(x)e_{n+1}\,.$$

This map is an EMBEDDING — it is a homeomorphism onto its image and its derivative $DX : \mathbb{R}^n \to \mathbb{R}^{n+1}$ is non-degenerate. So the tangent space $T_pM$ to $M$ at a point $p = x + u(x)e_{n+1}$ is spanned by the COORDINATE TANGENT VECTORS $\partial_i X|_x$, which are given by

$$\partial_i X \doteqdot \frac{\partial X}{\partial x^i} = e_i + u_i e_{n+1}\,.$$

From this, we see that the downwards pointing unit normal field is given by

$$\nu = \frac{(Du, -1)}{\sqrt{1 + |Du|^2}}$$

and the INDUCED METRIC TENSOR (a.k.a. the FIRST FUNDAMENTAL FORM) is given by $g = g_{ij} dX^i \otimes dX^j$, where

$$g_{ij} = \langle \partial_i X\,,\, \partial_j X\rangle = \delta_{ij} + u_i u_j\,.$$

The cometric is then given by $g^{ij}\partial_i X \otimes \partial_j X$, where

$$g^{ij} = \delta_{ij} - \frac{u_i u_j}{1 + |Du|^2}\,.$$

In particular, the GRADIENT vector field $\operatorname{grad} f$ of a differentiable function $f : \operatorname{graph} u \to M$ is given by

$$\operatorname{grad} f = g^{ij}\partial_i f \partial_j X\,.$$

The DIFFERENTIAL covector field $\nabla f$ is defined by its action on tangent vectors via $v \mapsto \nabla_v f = v^i \partial_i f$. Since the two are related to each other by the metric,

$$g(v, \operatorname{grad} f) = \nabla_v f\,,$$

we will often denote the gradient by $\nabla f$ as well.

Next, we recall that the induced COVARIANT DERIVATIVE $\nabla$ and SECOND FUNDAMENTAL FORM $A$ of $\operatorname{graph} u$ are, respectively, the tangential and normal components of the Hessian of $X$. In graphical coordinates, they are determined by $\nabla_{\partial_i X}(\partial_j X) = \Gamma_{ij}{}^k\partial_k = g^{k\ell}\Gamma_{ij\ell}\partial_k$ and $A = A_{ij}dx^i \otimes dx^j$, where

$$\Gamma_{ijk} = g(\nabla_{\partial_i X}(\partial_j X), \partial_k) = \left\langle \frac{\partial^2 X}{\partial x^i \partial x^j}\,,\, \partial_k X \right\rangle = u_{ij}u_k$$

and

$$A_{ij} = -\left\langle \frac{\partial^2 X}{\partial x^i \partial x^j}\,,\, \nu \right\rangle = \frac{u_{ij}}{\sqrt{1+|Du|^2}}\,.$$

They are therefore related to each other by the WEINGARTEN EQUATION

$$\frac{\partial^2 X}{\partial x^i \partial x^j} = \nabla_{\partial_i X}(\partial_j X) - A_{ij}\nu\,.$$

Modulo identification of the parallel hyperplanes $T_p \operatorname{graph} u$ and $T_{\nu(p)}S^n$, the SHAPE OPERATOR $A \doteqdot D\nu : T\operatorname{graph} u \to TS^n$ coincides with the WEINGARTEN TENSOR $A : T\operatorname{graph} u \to T\operatorname{graph} u$, which is the automorphism of $T\operatorname{graph} u$ related to the second fundamental form by the metric isomorphism $T\operatorname{graph} u \cong T^*\operatorname{graph} u$ (it is given by $A_i{}^j dX^i \otimes \partial_j X$, where $A_i{}^j = g^{jk}A_{ik}$). This justifies the use of the same symbol to denote all three tensors.

The MEAN CURVATURE $H$ is the trace of the Weingarten tensor. Thus,

$$\begin{aligned} H &= g^{ij}A_{ij} \\ &= \sum_{i,j}\left(\delta_{ij} - \frac{u_i u_j}{1+|Du|^2}\right)\frac{u_{ij}}{\sqrt{1+|Du|^2}} \\ &= \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right). \end{aligned} \tag{7.1}$$

The mean curvature also arises as the first variation of area. The AREA of a compact set $K \subset \operatorname{graph} u$ is given by

$$\operatorname{area}(K) \doteqdot \mu(K) = \int_K d\mu\,,$$

where the INDUCED MEASURE[20] $\mu$ is given by

$$d\mu(x + u(x)e_{n+1}) = \sqrt{1+|Du(x)|^2}\,d\mathcal{L}(x)\,,$$

[20] This coincides with the ambient $n$-dimensional Hausdorff measure.

where $\mathcal{L}$ denotes the Lebesgue measure on $\mathbb{R}^n$. In particular, if $u_\varepsilon = u + \varepsilon\eta$ is a smooth perturbation of $u$ with $\operatorname{spt}\eta \Subset \Omega$, then

$$\begin{aligned}
\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \operatorname{area}(\operatorname{graph} u_\varepsilon) &= \int_\Omega \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \sqrt{1+|Du_\varepsilon|^2}\, d\mathcal{L} \\
&= \int_\Omega \frac{Du \cdot D\eta}{\sqrt{1+|Du|^2}}\, d\mathcal{L} \\
&= -\int_\Omega \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)\eta\, d\mathcal{L} \\
&= -\int_\Omega H\eta\, d\mathcal{L}\,.
\end{aligned}$$

We note that the induced measure $\mu$ coincides on measurable subsets of $\operatorname{graph} u$ with the $n$-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$.

**7.2. Equations of mean curvature type.** We will consider equations of the form

$$-H_{\operatorname{graph} u} = \psi(\cdot, u, Du) \ \text{ in } \ \Omega\,, \tag{7.2}$$

where $H|_{\operatorname{graph} u}$ is the mean curvature of the graph of $u$. By (7.1), equation (7.2) is quasilinear and strictly elliptic. It is only uniformly elliptic if $\sup_\Omega |Du| < \infty$, however.

Let us record some important examples.

**Examples 7.1.**

(1) The (graphical) MINIMAL SURFACE EQUATION asks for a surface with zero mean curvature. So in this example

$$\psi(x, z, p) = 0\,.$$

(2) (Graphical) CAPILLARY SURFACES satisfy (7.2) with

$$\psi(x, z, p) = -\kappa z\,,$$

where $\kappa$ is a positive constant.

(3) The (graphical) PRESCRIBED MEAN CURVATURE EQUATION asks for a surface whose mean curvature is a prescribed function of points in ambient three-space. So in this example

$$\psi(x, z, p) = -f(x, z)$$

for some function $f : \mathbb{R}^{n+1} \to \mathbb{R}$.

(4) The right hand side of the (graphical) TRANSLATOR EQUATION is given by

$$\psi(x, z, p) = -\frac{1}{\sqrt{1+|p|^2}}\,.$$

> Solutions to the translator equation correspond to hypersurfaces which evolve by translation with constant velocity $\vec{v} = e_{n+1}$ under mean curvature flow. Such solutions arise naturally in the analysis of *singularities* and *ancient solutions* of the flow.

Our goal will be to solve the Dirichlet problem

$$\begin{cases} -H_{\mathrm{graph}\, u} = \psi(\cdot, u, Du) \text{ in } \Omega \\ u = \phi \text{ on } \partial\Omega\,. \end{cases} \tag{7.3}$$

We shall assume that $\Omega$ is bounded and of class $C^{2,\alpha}$, $\phi \in C^{2,\alpha}(\overline{\Omega})$ and $(x, z, p) \mapsto \psi(x, z, p)$ is of class $C^2$ and nondecreasing in $z$. In contrast to the linear setting, however, these conditions will not be sufficient to guarantee the existence of a solution to the problem (7.3). Indeed, if we do find a solution $u$ to (7.3), then the divergence theorem implies that

$$\begin{aligned} \int_\Omega \psi(\cdot, u, Du) &= -\int_\Omega \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) \\ &= \int_{\partial\Omega} \frac{Du \cdot N}{\sqrt{1+|Du|^2}} \\ &\le |\partial\Omega|\,, \end{aligned} \tag{7.4}$$

where $N$ is the outward unit normal field to $\partial\Omega$ and $|\partial\Omega|$ is its $n$-dimensional Hausdorff measure. So the relationship between $\Omega$ and $\psi$ cannot be arbitrary (see Exercise 7.3, Examples 7.2 and 7.3, and [**2**, §14.4]).

Similarly, given any $\eta \in C^\infty(\Omega)$ with $\operatorname{spt} \eta \Subset \Omega$, $\psi(\cdot, u, Du)$ must satisfy

$$\int_\Omega \psi(\cdot, u, Du)\, \eta \le \int_\Omega |D\eta|\ . \tag{7.5}$$

**Example 7.2** (The Grim Reaper)**.** Consider the one-dimensional translator equation

$$\frac{u_{xx}}{1+u_x^2} = 1\,.$$

This is a second order ODE which is readily solved directly. Its solutions are the two-parameter family of translates of the Grim Reaper $u : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$, which is defined by

$$u(x) \doteqdot -\log\cos x\,.$$

Note that $u(x) \to \infty$ as $x \to \pm\frac{\pi}{2}$. In particular, this means that the Dirichlet problem

$$\begin{cases} \dfrac{u_{xx}}{1+u_x^2} = 1 \text{ in } (a, b) \\ (u(a), u(b)) = (A, B) \end{cases}$$

is soluble only if $b - a < \pi$.

**7.3. The height estimate.** The maximum principle implies the following basic observation.

**Proposition 7.1** (Comparison principle)**.** *Suppose that* $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *satisfy*

$$-H|_{\operatorname{graph} u} - \psi(\cdot, u, Du) \leq -H|_{\operatorname{graph} v} - \psi(\cdot, v, Dv)\,,$$

*where* $\psi \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ *is non-increasing in* $z \in \mathbb{R}$*. If* $u \leq v$ *on* $\partial\Omega$ *and either* $\sup_\Omega |Du| < \infty$ *or* $\sup_\Omega |Dv| < \infty$*, then* $u \leq v$ *in* $\Omega$*.*

**Proof.** Set $u_\vartheta \doteqdot \vartheta u + (1-\vartheta) v$ and

$$a(p) \doteqdot \frac{1}{\sqrt{1+|p|^2}}\left(I - \frac{p \otimes p}{1+|p|^2}\right).$$

Observe that

$$\begin{aligned} 0 &\leq a(Du)\cdot D^2 u + \psi(\cdot, u, Du) - a(Dv)\cdot D^2 v - \psi(\cdot, v, Dv) \\ &= \int_0^1 \frac{d}{d\vartheta}\big(a(Du_\vartheta)\cdot D^2 u_\vartheta + \psi(\cdot, u_\vartheta, Du_\vartheta)\big)\, d\vartheta \\ &= a^{ij} w_{ij} + b^k w_k + cw\,, \end{aligned}$$

where $w \doteqdot u - v$ and the coefficients $a : \Omega \to S^{n\times n}$, $b : \Omega \to \mathbb{R}^n$ and $c : \Omega \to \mathbb{R}$ are defined by

$$\begin{aligned} a^{ij}(x) &\doteqdot \int_0^1 a(Du_\vartheta(x))^{ij} d\vartheta\,, \\ b^k(x) &\doteqdot \int_0^1 a_{p_k}(Du_\vartheta(x))^{ij}(u_\vartheta)_{ij} d\vartheta - \int_0^1 \psi_{p_k}(x, u_\vartheta(x), Du_\vartheta(x))\, d\vartheta\,, \end{aligned}$$

and

$$c(x) \doteqdot \int_0^1 \psi_z(x, u_\vartheta(x), Du_\vartheta(x))\, d\vartheta\,.$$

The claim now follows from the maximum principle since $a$ is uniformly positive definite (due to the boundedness of either $|Du|$ or $|Dv|$) and $c$ is non-positive (due to the monotonicity of $\psi$). $\square$

This reduces the establishment of an *a priori* estimate in $C^0$ for solutions to (7.3) to the construction of sub- and supersolutions. We say that a function $v$ is a SUBSOLUTION TO THE DIRICHLET PROBLEM (7.3) if

$$\begin{cases} -H_{\operatorname{graph} v} \leq \psi(\cdot, v, Dv) & \text{in } \Omega \\ \qquad\qquad v \leq \phi & \text{on } \partial\Omega\,. \end{cases}$$

Supersolutions are defined analogously.

**Corollary 7.2.** *Suppose that $\psi_z \le 0$ and that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies the Dirichlet problem* (7.3). *If* (7.3) *admits a supersolution $\overline{u}$, then*

$$\sup_\Omega u \le \sup_\Omega \overline{u}$$

*If* (7.3) *admits a subsolution $\underline{u}$, then*

$$\inf_\Omega u \ge \inf_\Omega \underline{u}\,.$$

Sub- and supersolutions arise naturally in certain situations, but they may not always be available. Let us present an alternative argument, which holds under a slightly stronger constraint than the necessary condition (7.5).

**Proposition 7.3.** *Suppose that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies the Dirichlet problem* (7.3) *with $\psi$ non-increasing in $z$. If there exists $\varepsilon > 0$ such that*

$$\int_\Omega \psi(\cdot, \eta, D\eta)\,\eta \le (1-\varepsilon)\int_\Omega |D\eta| \tag{7.6}$$

*for all non-negative $\eta \in W^{1,2}(\Omega)$ with $\operatorname{spt}\eta \Subset \Omega$, then*

$$\sup_\Omega u \le \sup_{\partial\Omega} \phi_+ + C(n, \varepsilon, |\Omega|)\,.$$

*If there exists $\varepsilon > 0$ such that* (7.6) *holds for all non-positive $\eta \in W^{1,2}(\Omega)$ with $\operatorname{spt}\eta \Subset \Omega$, then*

$$\inf_\Omega u \ge \inf_{\partial\Omega} \phi_- - C(n, \varepsilon, |\Omega|)\,.$$

**Proof.** Given any $k \ge k_0 \doteqdot \max_{\partial\Omega}\phi_+$, define[21]

$$u_k \doteqdot (u-k)_+ \;\text{ and }\; A_k \doteqdot \{x \in \Omega : u(x) > k\}\,.$$

Observe that $u_k|_{\partial\Omega} = 0$ and $u_k \in C^{0,1}(\overline{\Omega}) = W^{1,\infty}(\Omega) \subset W^{1,2}(\Omega)$. Note also that $|A_k|$ is monotonically decreasing in $k$.

[21] We will avoid the use of subscripts to denote derivatives for the remainder of the proof.

Since $u_k \in W^{1,2}(\Omega)$, the divergence theorem and the hypotheses for $\psi$ yield

$$\begin{aligned}\int_{A_k} \frac{|Du_k|^2}{\sqrt{1+|Du_k|^2}} &= \int_\Omega \frac{Du \cdot Du_k}{\sqrt{1+|Du|^2}}\\ &= -\int_\Omega \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) u_k\\ &= \int_\Omega \psi(\cdot, u, Du) u_k\\ &\leq \int_\Omega \psi(\cdot, u_k, Du_k) u_k\\ &\leq (1-\varepsilon) \int_\Omega |Du_k|\\ &= (1-\varepsilon) \int_{A_k} |Du_k|\,.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{A_k} |Du_k| &\leq \int_{A_k} \sqrt{1+|Du_k|^2}\\ &= \int_{A_k} \frac{1+|Du_k|^2}{\sqrt{1+|Du_k|^2}}\\ &= \int_{A_k} \frac{1}{\sqrt{1+|Du_k|^2}} + \int_{A_k} \frac{|Du_k|^2}{\sqrt{1+|Du_k|^2}}\\ &\leq |A_k| + (1-\varepsilon) \int_{A_k} |Du_k|\,,\end{aligned}$$

and hence

$$\int_{A_k} |Du_k| \leq \varepsilon^{-1} |A_k|\,.$$

We now employ the Sobolev inequality to estimate

$$\left(\int_{A_k} u_k^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq C \int |Du_k| \leq C\varepsilon^{-1} |A_k|\,,$$

where $C = C(n)$ is the Sobolev constant.

Hölder's inequality now yields

$$\begin{aligned}\int_{A_k} u_k &\leq |A_k|^{\frac{1}{n}} \left(\int_{A_k} u_k^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}\\ &\leq C\varepsilon^{-1} |A_k|^{1+\frac{1}{n}}\,.\end{aligned}$$

On the other hand, given any $h > k \geq k_0$,

$$(h-k)|A_h| = \int_{A_h} (h-k) \leq \int_{A_h} (u-k) = \int_{A_h} u_k \leq \int_{A_k} u_k \,.$$

We conclude that

$$(h-k)|A_h| \leq C\varepsilon^{-1}|A_k|^{1+\frac{1}{n}} \,. \tag{7.7}$$

By iterating this inequality, we can deduce that $A_k = 0$ for $k$ sufficiently large.

Given $k \geq k_0$, set $\varphi(k) \doteqdot |A_k|$. For each $r = 1, 2, \ldots$, set

$$k_r \doteqdot k_0 + d - \frac{d}{2^r} \,,$$

where $d > 0$ will be determined. Note that

$$k_{r+1} - k_r = \frac{d}{2^{r+1}} \;\text{ and }\; k_r \to k_0 + d \;\text{ as }\; t \to \infty \,.$$

Since $k_{r+1} > k_r \geq k_0$, (7.7) yields

$$\varphi(k_{r+1}) \leq C\varepsilon^{-1}\frac{2^{r+1}}{d}\varphi(k_r)^{1+\frac{1}{n}} \,. \tag{7.8}$$

We claim that, upon choosing

$$d \doteqdot 2^{n+1}C\varepsilon^{-1}\varphi(k_0)^{\frac{1}{n}} \,,$$

we may estimate, for all $r$,

$$\varphi(k_r) \leq 2^{-nr}\varphi(k_0) \,. \tag{7.9}$$

This is proved by induction. Indeed, the base case $r = 0$ is clear, so suppose that (7.9) holds for some $r \geq 0$. Then (7.8) and our choice of $d$ yield

$$\begin{aligned}
\varphi(k_{r+1}) &\leq C\varepsilon^{-1}\frac{2^{r+1}}{d}\varphi(k_r)^{1+\frac{1}{n}} \\
&\leq C\varepsilon^{-1}\frac{2^{r+1}}{d} \cdot 2^{-(n+1)r}\varphi(k_0)^{1+\frac{1}{n}} \\
&= C\varepsilon^{-1}\frac{2^{n+1}}{d} \cdot 2^{-n(r+1)}\varphi(k_0)^{1+\frac{1}{n}} \\
&= 2^{-n(r+1)}\varphi(k_0) \,.
\end{aligned}$$

This proves (7.9). Taking $r \to \infty$, we conclude that $|A_{k_0+d}| = 0$, which means that

$$\sup_\Omega u \leq k_0 + d \,.$$

The first claim follows.

The second claim follows from the first, since the function $v \doteqdot -u$ satisfies

$$-H_{\operatorname{graph} v} = -\psi(\cdot, -v, -Dv) \,. \qquad \square$$

The argument employed in Proposition 7.3 is known as STAMPACCHIA ITERATION, after Guido Stampacchia.

We conclude by noting that, although the condition (7.6) is very natural (in view of (7.5)), it is not easy to check in general. Since $\psi_z \leq 0$, it suffices to check the condition with $\psi(\cdot, \eta, D\eta)$ replaced by $\psi(\cdot, 0, D\eta)$. In the important special case of prescribed mean curvature equations (where $\psi = \psi(x, z)$), it then suffices to check that

$$\int \psi(\cdot, 0)\eta \leq (1 - \varepsilon) \int |D\eta|$$

for all $\eta \in W^{1,2}(\Omega)$.

**7.4. The gradient estimate.** Next, we obtain an *a priori* estimate for the gradient of solutions to (7.3). We first estimate the modulus of the gradient at any interior point by its values at the boundary, using the maximum principle.

7.4.1. *Interior gradient estimate.* Consider the function $v : \operatorname{graph} u \to \mathbb{R}$ defined by

$$v \doteqdot - \langle \nu\, ,\, e_{n+1} \rangle\ .$$

Note that

$$v(x + u(x)e_{n+1}) = \frac{1}{\sqrt{1 + |Du(x)|^2}}\,.$$

We will estimate $v$ from below (and hence $|Du|$ from above) by a constant depending on $n$, $\min_{\partial \operatorname{graph} u} v$, and $\sup_\Omega |u|$ using the maximum principle.

We want to compute $\Delta v$, where

$$\Delta \doteqdot \operatorname{div} \operatorname{grad} = g^{ij}\nabla_i\nabla_j = g^{ij}(\partial_i\partial_j - \Gamma_{ij}{}^k\partial_k)$$

is the induced LAPLACE–BELTRAMI OPERATOR, so let us assume that $u \in C^3(\Omega) \cap C^1(\overline{\Omega})$ (we will weaken this to $C^2(\Omega) \cap C^1(\overline{\Omega})$ later, by an approximation argument). Observe that

$$\begin{aligned} \partial_i v &= \langle \partial_i \nu\, ,\, e_{n+1} \rangle \\ &= A_i{}^j \langle \partial_j X\, ,\, e_{n+1} \rangle \end{aligned}$$

and hence

$$\operatorname{grad} v = A(e_{n+1}^\top)\,,$$

where grad is the gradient operator on $\operatorname{graph} u$ and $\cdot^\top : \mathbb{R}^n \to T \operatorname{graph} u$ is the tangential projection map. It follows that

$$-\,\Delta v = |A|^2 v + \nabla_{e_{n+1}^\top} H\,, \tag{7.10}$$

where $|A|^2 = g^{ik}g^{jl}A_{ij}A_{kl}$ is the squared norm of the second fundamental form.[22] We can already deduce our estimate in the most important cases: the minimal, constant mean curvature and translator equations. Indeed, if $H$ is constant, then the second term on the right vanishes. If $H \equiv v$, then it is a gradient term. Since $v > 0$, we conclude from (7.10) and the maximum principle (see Exercise 7.9) that it cannot realize an interior minimum, and hence

$$\min_{\operatorname{graph} u} v \geq \min_{\partial \operatorname{graph} u} v \,.$$

That is,

$$\max_{\Omega} |Du| \leq \max_{\partial\Omega} |Du| \,.$$

For general $\psi$, we must assume, in addition to the monotonicity condition $\psi_z \leq 0$, that

$$\sup_{\Omega\times\{0\}\times\mathbb{R}^n} \big(|\psi_x| + |p|^2|\psi_p|\big) < \infty \,, \tag{7.11}$$

where $\psi_x$ is the derivative of $\psi$ with respect to the first factor and $\psi_p$ is its derivative with respect to the third factor. Note that (7.11) is easily verified for the minimal, constant mean curvature, capillary surface and translator equations, and also all prescribed mean curvature equations satisfying the very mild condition $\psi(\cdot, 0) \in C^1(\overline{\Omega})$.

We need to estimate the term $\nabla_{e_{n+1}^\top} H$. Observe that

$$\begin{aligned}
\nabla_{e_{n+1}^\top} H &= \langle \operatorname{grad} H \,,\, e_{n+1}\rangle \\
&= \big\langle g^{ij}\partial_i H \partial_j X \,,\, e_{n+1}\big\rangle \\
&= g^{ij}\partial_i H u_j \\
&= -\, g^{ij}(\psi_{x^i} + \psi_z u_i + \psi_{p_k} u_{ki})u_j \,,
\end{aligned}$$

where we are implicitly evaluating the derivatives of $\psi$ at $(\cdot, u, Du)$. By hypothesis,

$$-\psi_z g^{ij} u_i u_j \geq 0 \,,$$

[22]One may wonder why we have chosen to apply the Laplacian to $v$, and not some other elliptic operator. The reason is not (just) that the Laplace–Beltrami operator $\Delta$ is the most natural second order elliptic operator arising from a given metric, but rather that the operator $-(\Delta + |A|^2)$ arises as the *linearization* of the mean curvature. See Exercise 7.8.

so this term is not harmful. Note also that the final term can be rewritten as a gradient term. Indeed,

$$\begin{aligned} v_k &= g^{ij} A_{ik} \langle \partial_j X \,,\, e_{n+1} \rangle \\ &= g^{ij} A_{ik} u_j \\ &= \frac{g^{ij} u_{ik} u_j}{\sqrt{1+|Du|^2}} \\ &= v g^{ij} u_{ik} u_j \,, \end{aligned}$$

so that

$$\psi_{p_k} g^{ij} u_{ik} u_j = \frac{\psi_{p_k} v_k}{v} \,.$$

Rewriting

$$\begin{aligned} g^{ij} u_j &= \left( \delta_{ij} - \frac{u_i u_j}{1+|Du|^2} \right) u_j \\ &= u_i \left( 1 - \frac{|Du|^2}{1+|Du|^2} \right) \\ &= \frac{u_i}{1+|Du|^2} \\ &= v^2 u_i \,, \end{aligned}$$

we may estimate

$$\begin{aligned} g^{ij} \psi_{x^i} u_j = v^2 \psi_{x^i} u_i & \\ &\le v^2 |\psi_x| |Du| \\ &\le v |\psi_x| . \end{aligned}$$

Thus,

$$-\Delta v \ge \big( |A|^2 - |\psi_x| \big)\, v - \frac{\psi_{p_k} v_k}{v} \,.$$

To deal with the negative reaction term, we will estimate $\mathrm{e}^{\lambda h} v$ from below for some $\lambda > 0$, where the HEIGHT FUNCTION $h : \operatorname{graph} u \to \mathbb{R}$ is defined by

$$h(p) = \langle p \,,\, e_{n+1} \rangle \,.$$

Note that $h(x + u(x) e_{n+1}) = u(x)$.

Observe that

$$\begin{aligned} \operatorname{grad} h &= e_{n+1}^{\top} \\ &= e_{n+1} + v\nu \end{aligned}$$

and

$$-\Delta h = -Hv \,.$$

Thus,

$$\nabla(\mathrm{e}^{\lambda h}v) = \mathrm{e}^{\lambda h}\nabla v + \lambda \mathrm{e}^{\lambda h}v\nabla h$$

and hence

$$\begin{aligned}
-\Delta(\mathrm{e}^{\lambda h}v) = &\, -\mathrm{e}^{\lambda h}\Delta v - v\Delta \mathrm{e}^{\lambda h} - 2\left\langle \nabla \mathrm{e}^{\lambda h}\,,\, \nabla v\right\rangle \\
= &\, -\mathrm{e}^{\lambda h}\Delta v - \lambda v\mathrm{e}^{\lambda h}\Delta h - \lambda^2\mathrm{e}^{\lambda h}v|\nabla h|^2 - 2\lambda\mathrm{e}^{\lambda h}\left\langle \nabla h\,,\, \nabla v\right\rangle \\
\geq &\, \mathrm{e}^{\lambda h}v\left(|A|^2 - |\psi_x| - \frac{\psi_{p_k}v_k}{v^2}\right) - \lambda\mathrm{e}^{\lambda h}v^2 H - \lambda^2 v\mathrm{e}^{\lambda h}|\nabla h|^2 \\
&\, - 2\lambda\mathrm{e}^{\lambda h}\left\langle \nabla h\,,\, \nabla v\right\rangle\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
-\frac{\Delta(\mathrm{e}^{\lambda h}v)}{\mathrm{e}^{\lambda h}v} + \left\langle 2\lambda\nabla h - \frac{g^{k\ell}\psi_{p_k}\partial_{x^\ell}}{v}\,,\, \frac{\nabla(\mathrm{e}^{\lambda h}v)}{\mathrm{e}^{\lambda h}v}\right\rangle& \\
\geq |A|^2 - |\psi_x| + \lambda\frac{\psi_{p_k}h_k}{v} - \lambda vH + \lambda^2|\nabla h|^2\,.&
\end{aligned}$$

Observe that

$$|\nabla h|^2 = 1 - v^2$$

and

$$|A|^2 \geq \tfrac{1}{n}H^2\,.$$

Thus, estimating

$$\lambda vH \leq \tfrac{1}{n}H^2 + \tfrac{n}{4}\lambda^2 v^2$$

and, since $h_k(x+u(x)e_{n+1}) = u_k(x)$ and $v^{-1} = \sqrt{1+|Du|^2} \leq |Du|$,

$$\frac{\psi_{p_k}h_k}{v} \geq -|Du|^2|\psi_p|\,,$$

we arrive at

$$\begin{aligned}
-\frac{\Delta(\mathrm{e}^{\lambda h}v)}{\mathrm{e}^{\lambda h}v} + \left\langle 2\lambda\nabla h - \frac{g^{k\ell}\psi_{p_k}\partial_{x^\ell}}{v}\,,\, \frac{\nabla(\mathrm{e}^{\lambda h}v)}{\mathrm{e}^{\lambda h}v}\right\rangle& \\
\geq -\sup_{\Omega\times\mathbb{R}\times\mathbb{R}^n}\big(|\psi_x| + |p|^2|\psi_p|\big) + \lambda^2\left(1 - (1+\tfrac{n}{4})v^2\right)\,.&
\end{aligned} \tag{7.12}$$

If we set

$$\lambda^2 \doteqdot 2\sup_{\Omega\times\mathbb{R}\times\mathbb{R}^n}\big(|\psi_x| + |p|^2|\psi_p|\big)\,,$$

then $(1+\frac{n}{4})v^2 \geq \frac{1}{2}$ at any interior minimum of $\mathrm{e}^{\lambda h}v$, and hence

$$\mathrm{e}^{\lambda h}v \geq \min\left\{\min_{\partial\,\mathrm{graph}\,u}\big(\mathrm{e}^{\lambda h}v\big), \sqrt{\tfrac{2}{n+4}}\,\mathrm{e}^{\lambda h}\right\}\,.$$

We conclude that

$$\sup_{\Omega}|Du| \leq C\,, \tag{7.13}$$

where $C = C\left(n, \sup_{\Omega\times\mathbb{R}\times\mathbb{R}^n}\big(|\psi_x| + |p|^2|\psi_p|\big), \sup_\Omega|u|, \max_{\partial\Omega}|Du|\right)$.

We now show that the estimate (7.13) holds under the weaker regularity condition $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Indeed, since we may approximate such $u$ in the $C^2(\Omega) \cap C^1(\overline{\Omega})$ topology by smooth functions, the inequality (7.12) still holds, albeit in the integral form (note that integration over graph $u$ obeys the divergence theorem; see Exercise 7.1)

$$\int \left\langle \nabla(\mathrm{e}^{\lambda h} v)\,,\, \nabla\eta \right\rangle + \int \nabla_V(\mathrm{e}^{\lambda h} v)\eta \geq -\lambda^2(1+\tfrac{n}{4}) \int \left(\tfrac{2}{n+4} - v^2\right) \mathrm{e}^{\lambda h} v\eta$$

for all non-negative $\eta \in W^{1,2}(\operatorname{graph} u)$ with $\operatorname{spt}\eta \Subset \operatorname{graph} u$, where

$$V \doteqdot 2\lambda\nabla h - \frac{g^{k\ell}\psi_{p_k}\partial_{x^\ell}}{v}\,, \quad \lambda^2 \doteqdot 2 \sup_{\Omega\times\{0\}\times\mathbb{R}^n} \left(|\psi_x| + |p|^2|\psi_p|\right)\,,$$

and the integrals are taken with respect to the induced measure $\mu$.

If we set $\eta \doteqdot (m - \mathrm{e}^{\lambda h} v)_+$ , where

$$m \doteqdot \sqrt{\frac{2}{n+4}} \min_{\operatorname{graph} u} \mathrm{e}^{\lambda h}\,,$$

then, since $v \leq \sqrt{\frac{2}{n+4}}$ wherever $\mathrm{e}^{\lambda h} v \leq m$, we obtain

$$\int |\nabla\eta|^2 + \cancel{\frac{1}{2}\int \nabla_V \eta^2} \leq \lambda^2(1+\tfrac{n}{4}) \int \left(\tfrac{2}{n+4} - v^2\right) \mathrm{e}^{\lambda h} v\eta \leq 0\,.$$

It follows that $\eta$ is constant and $(\frac{2}{n+4} - v^2)\eta \equiv 0$, from which we may conclude that $\eta \equiv 0$. That is, $\mathrm{e}^{\lambda h} v \geq m$.

In summary, we have proved the following.

**Proposition 7.4.** *There exists $C = C(n, K, L, M) < \infty$ with the following property. Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$. Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a solution to*

$$-H|_{\operatorname{graph} u} = \psi(\cdot, u, Du)\,,$$

*where $\psi \in C^1(\Omega \times \mathbb{R}^n \times \mathbb{R}^n)$ satisfies*

$$\psi_z \leq 0 \;\; \textit{and} \;\; \sup_{\Omega\times\{0\}\times\mathbb{R}^n} \left(|\psi_x| + |p|^2|\psi_p|\right) \leq L\,.$$

*If $\sup_\Omega |u| \leq K$ and $\max_{\partial\Omega} |Du| \leq M$, then*

$$|Du| \leq C. \tag{7.14}$$

7.4.2. *Boundary gradient estimate.* If the domain $\Omega$ and the boundary condition $\phi$ are of class $C^1$, then we may estimate the gradient of any solution $u$ to (7.3) in directions tangent to the boundary by those of the boundary values $\phi$. Indeed, if $\tau$ is a unit tangent vector to $\partial\Omega$ at $x \in \partial\Omega$, then (by definition of tangent vectors) we can find a curve $\gamma : (-s_0, s_0) \to \partial\Omega$ such that $\gamma'(0) = \tau$. Since $u$ coincides with $\phi$ on $\partial\Omega$, we find that

$$0 \equiv \frac{d}{ds}\bigg|_{s=0} (u - \phi) \circ \gamma = D_\tau u(x) - D_\tau \phi(x) \,.$$

So it remains to estimate $Du$ in directions normal to $\partial\Omega$. This is straightforward if the problem (7.3) admits upper and lower barriers $\overline{u}$, $\underline{u} \in C^1(\overline{\Omega})$ which take the boundary values $\phi$. Indeed, since $\underline{u} \le u \le \overline{u}$ in $\Omega$, we find that

$$\begin{aligned} 0 \le & \lim_{s \searrow 0} \frac{\overline{u}(x + s\nu(x)) - u(x + s\nu(x))}{s} \\ = & \lim_{s \searrow 0} \frac{\overline{u}(x + s\nu(x)) - u(x) + u(x) - u(x + s\nu(x))}{s} \\ = & \, D_{\nu(x)}\overline{u} - D_{\nu(x)} u \end{aligned}$$

and

$$\begin{aligned} 0 \ge & \lim_{s \searrow 0} \frac{\underline{u}(x + s\nu(x)) - u(x + s\nu(x))}{s} \\ = & \lim_{s \searrow 0} \frac{\underline{u}(x + s\nu(x)) - u(x) + u(x) - u(x + s\nu(x))}{s} \\ = & \, D_{\nu(x)}\underline{u} - D_{\nu(x)} u \,, \end{aligned}$$

where $\nu(x)$ is the inward pointing unit normal to $\partial\Omega$ at $x$. Thus,

$$D_{\nu(x)}\underline{u} \le D_{\nu(x)} u \le D_{\nu(x)}\overline{u} \,.$$

We conclude that

$$|Du(x)|^2 \le |\nabla\phi(x)|^2 + \max\{|D_{\nu(x)}\underline{u}|^2, |D_{\nu(x)}\overline{u}|^2\} \,,$$

where $\nabla$ is the induced gradient operator on $\partial\Omega$.

Note that we only need to construct barriers in a neighbourhood the boundary.

7.4.3. *Construction of barriers near the boundary.* We first note that it is not always possible to construct barriers — there is a necessary condition arising from the geometry of $\partial\Omega$.

**Example 7.3.** The function $u \in C^\infty(B_1)$ defined by

$$u(x) = -\sqrt{1 - |x|^2}$$

satisfies the Dirichlet problem

$$\begin{cases} H_{\mathrm{graph}\, u} = n & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1 \,. \end{cases}$$

But

$$Du(x) \cdot \frac{x}{|x|} \to \infty$$

as $x \to \partial B_1$.

To pinpoint what goes wrong in the above example, consider a graphical hypersurface $M = \operatorname{graph} u$ which is smooth up to its boundary, so that there is a smoothly embedded hypersurface $\tilde{M}$ such that $M \Subset \tilde{M}$. If the boundary $\partial M$ is smooth but $|Du(x)|$ becomes unbounded as $x \to x_0$, then the tangent plane to $\tilde{M}$ at $x_0 + u(x_0)e_{n+1}$ is vertical. This means that it is also the tangent plane to the bounding cylinder $\partial\Omega \times \mathbb{R}$. We may represent $\tilde{M}$ and $\partial\Omega \times \mathbb{R}$ in a neighbourhood $U$ of $x_0 + u(x_0)e_{n+1}$ as graphs of smooth functions $v : L \to \mathbb{R}$ and $w : L \to \mathbb{R}$ over the mutual tangent plane $L$. Note that $v(0) = w(0) = 0$ and $Dv(0) = Dw(0)$. The latter implies that

$$A_{\tilde{M}}|_{x_0+u(x_0)e_{n+1}} = \pm D^2 v(0) \;\text{ and }\; A_{\partial\Omega\times\mathbb{R}}|_{x_0+u(x_0)e_{n+1}} = D^2 w(0)\,,$$

where the positive sign is taken if the downward pointing normal to $\operatorname{graph} u$ coincides with the outward pointing normal to $\partial\Omega \times \mathbb{R}$ at $x_0$, and the negative sign is taken if these normals have opposite orientation at $x_0$. Since $v > w$ at points of $M$, and "half" of the boundary tangent directions at $x_0 + u(x_0)e_{n+1}$ point into $M$, Taylor's theorem implies that

$$D^2 v(0) \geq D^2 w(0)$$

and hence

$$H_{\tilde{M}}(x_0 + u(x_0)e_{n+1}) \geq \pm H_{\partial\Omega\times\mathbb{R}}(x_0 + u(x_0)e_{n+1})\,,$$

where the signs correspond to the orientation of $\operatorname{graph} u$ with respect to $\partial\Omega \times \mathbb{R}$ as before. Since $H_{\partial\Omega\times\mathbb{R}}(x + u(x)e_{n+1}) = H_{\partial\Omega}(x)$ for any $x \in \partial\Omega$, where $H_{\partial\Omega}$ is the mean curvature of $\partial\Omega$ with respect to its outward pointing unit normal field, this behaviour is impossible if

$$H_{\partial\Omega}(x) > \limsup_{|p|\to\infty} |\psi(x, \phi(x), p)| \tag{7.15}$$

for all $x \in \partial\Omega$, where $\phi \doteqdot u|_{\partial\Omega}$. The condition 7.15 on $\partial\Omega$, $\phi$ and $\psi$ will actually be sufficient to construct barriers.

Our barriers will be constructed as functions of the distance-to-the-boundary function. So fix a bounded open subset $\Omega \subset \mathbb{R}^n$ and denote by $d : \mathbb{R}^n \to \mathbb{R}$ the distance to $\partial\Omega$,

$$d(x) \doteqdot \min_{p\in\partial\Omega} |x - p|\,.$$

We claim that $d$ is Lipschitz (with Lipschitz constant 1). To see this, fix $x, y \in \mathbb{R}^n$. Since $\partial\Omega$ is compact (and non-empty) there must exist at least one point $q \in \partial\Omega$ such that $d(y) = |y - q|$. So the triangle inequality yields

$$\begin{aligned} d(x) - d(y) &= \min_{p\in\partial\Omega} |x-p| - |y-q| \\ &\le |x-q| - |y-q| \\ &\le |x-y| \,. \end{aligned}$$

The claim follows since $x$ and $y$ were arbitrary.

Now assume that $\partial\Omega$ is of class $C^1$, so that it admits a tangent space $T_x\partial\Omega$ and an outer unit normal vector $\nu(x)$ at each $x \in \partial\Omega$ (both of which depend continuously on $x$). Moreover, for each $x_0 \in \partial\Omega$, we can find a neighbourhood $U$ (in $\mathbb{R}^n$) and a $C^1$ function $u : T_x\partial\Omega \to \mathbb{R}$ such that

$$\partial\Omega \cap U = \{x + u(x)\nu(x_0) : x \in T_{x_0}\partial\Omega\} \cap U \,.$$

That is, $\partial\Omega$ coincides in $U$ with the image of the embedding $X : T_{x_0}\partial\Omega \to \mathbb{R}^{n+1}$ defined by

$$X(x) \doteqdot x + u(x)\nu(x_0) \,.$$

We may identify $T_{x_0}\partial\Omega$ with $\mathbb{R}^{n-1}$ by choosing an orthonormal basis $\{e_i\}_{i=1}^{n-1}$. We thus obtain coordinates $\{x^i\}_{i=1}^{n-1}$ for $\partial\Omega \cap U$ via $x^i(y^j e_i + u(y^j e_j)) \doteqdot y^i$. If $\partial\Omega$ is of class $C^k$ for some $k \ge 2$, then it admits a second fundamental form $A$, and this is given in our coordinates $\{x^i\}_{i=1}^{n-1}$ by

$$A_{ij} = \frac{u_{ij}}{\sqrt{1+|Du|^2}} \,.$$

In particular,

$$(A_{x_0})_{ij} = u_{ij}(x_0).$$

For convenience, we may further choose the basis $\{e_i\}_{i=1}^{n-1}$ so that each $e_i$ is an eigenvector of $A_{x_0} = D^2u(x_0)$ (called a PRINCIPAL DIRECTION), with eigenvalue $\kappa_i$ (called a PRINCIPAL CURVATURE).

**Proposition 7.5.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with boundary of class $C^k$ for some $k \ge 2$. There exists $\mu > 0$ such that $d \in C^k(\Gamma_\mu)$, where*

$$\Gamma_\mu \doteqdot \{x \in \Omega : d(x) < \mu\} \,.$$

**Proof.** Since $\partial\Omega$ is of class $C^2$ and $\Omega$ is bounded, $\partial\Omega$ satisfies a uniform interior ball condition (see Exercise 7.10). That is, we can find $\mu > 0$ such that $B_\mu(x_0 - \mu\nu(x_0)) \subset \Omega$ for each $x_0 \in \partial\Omega$. In particular, $\kappa_i \le 1/\mu$ for each $i = 1, \dots, n-1$. Moreover, for every $x \in \Gamma_\mu$ there exists a *unique* point $p(x) \in \partial\Omega$ such that $|x - p(x)| = d(x)$. Indeed,

$$p(x) = x + d(x)\nu(x) \,.$$

Now fix $x_0 \in \Gamma_\mu$ and set $p_0 \doteqdot p(x_0)$. As explained above, we may choose a local graphical representation $u \in C^k(T_{x_0}\partial\Omega)$ and local coordinates

$\{x^i\}_{i=1}^{n-1}$ for $\partial\Omega$ in neighbourhood of $p_0$ so that

$$(A_{p_0})_{ij} = u_{ij}(x_0) = \kappa_i\delta_{ij}\,.$$

Define a function $g : T_{p_0}\partial\Omega \times \mathbb{R} \to \mathbb{R}^n$ by moving the point $y + u(y)\nu_0$ on $\partial\Omega$ corresponding to $y \in T_{p_0}\partial\Omega$ a distance $r$ in its inwards normal direction, where $\nu_0 \doteqdot \nu(p_0)$. That is,

$$g(y, r) \doteqdot y + u(y)\nu_0 - r\nu(y + u(y)\nu_0)\,.$$

Note that $g(p_0, d_0) = x_0$, where $d_0 \doteqdot d(x_0)$. Observe that $g$ is of class $C^{k-1}$. Its derivative $Dg$ is given by

$$D_v g = v + D_v u\nu_0 - rA(v)$$

when $v$ is tangent to the first factor, and

$$\partial_r g = -\nu(y + u(y)\nu_0)\,.$$

Since $u(p_0) = Du(p_0) = 0$,

$$Dg_{(p_0,d_0)} = \mathrm{diag}(1 - d_0\kappa_1^0, \ldots, 1 - d_0\kappa_{n-1}^0, -1)\,, \tag{7.16}$$

where $\kappa_i^0 \doteqdot \kappa_i(p_0)$. Since $d_0 < \mu$ and $\kappa_i \leq 1/\mu$, we conclude that $Dg$ is non-degenerate at $(p_0, d_0)$, so the inverse function theorem implies that it admits an inverse $h$ of class $C^{k-1}$ on some neighbourhood $V \subset U$ of $x_0$. This inverse is given explicitly by projecting $x \in \Omega \cap V$ onto $\partial\Omega$, and then onto $T_{x_0}\Omega$. That is,

$$h(x) \doteqdot \big(\pi(p(x)), d(x)\big)\,,$$

where $\pi(x) \doteqdot x - \langle x\,,\, \nu_0\rangle\, \nu_0$ is the projection of $\partial\Omega \cap U$ onto $T_{p_0}\partial\Omega$. This implies that $d$ is of class $C^{k-1}$ near $x_0$. In fact, since

$$d(x) = \langle p(x) - x\,,\, \nu(p(x))\rangle$$

and $D_v p \in T_x\partial\Omega$ for any $v \in \mathbb{R}^n$, we find that

$$D_v d(x) = 0 \ \text{ for } \ v \parallel T_{p(x)}\partial\Omega$$

and

$$D_v d(x) = -1 \ \text{ for } \ v \parallel \nu(p(x))$$

Thus,

$$Dd = -\nu \circ p\,, \tag{7.17}$$

which is of class $C^{k-1}$! So $d$ is actually of class $C^k$ near $x_0$. The claim follows since $x_0 \in \Gamma_\mu$ was arbitrary. $\square$

Observe furthermore that, by (7.17),

$$D_v(Dd)|_x = -A_{p(x)}(Dp(v)) \ \text{ for } \ v \parallel T_{p(x)}\partial\Omega$$

On the other hand, since $|Dd| \equiv 1$, we must have $D_v(Dd) \perp Dd$. So

$$D_v(Dd)|_x = 0 \ \text{ for } \ v \parallel \nu(p(x))\,.$$

By (7.16), we find, with respect to the principal coordinates $\{x^i\}_{i=1}^n$, that

$$\partial_i p(x_0) = \frac{e_i}{1-d_0\kappa_i^0}$$

for each $i=1,\dots,n-1$. Since $x_0\in\Gamma_\mu$ was arbitrary, we conclude that

$$D^2 d = -\sum_{i=1}^{n-1}\frac{\kappa_i}{1-d\kappa_i}e_i\otimes e_i\,,$$

where $\{e_i\}_{i=1}^n$ is a principal frame for $\partial\Omega$ and both $\kappa_i$ and $e_i$ evaluated on $\partial\Omega$ after applying the projection $p$. In particular,

$$\Delta d = -\sum_{i=1}^{n-1}\frac{\kappa_i}{1-d\kappa_i} = -\sum_{i=1}^{n-1}\kappa_i\left(1+\frac{\kappa_i d}{1-\kappa_i d}\right)\le -\sum_{i=1}^{n-1}\kappa_i = -H_{\partial\Omega}\,.$$

Returning now to the construction of boundary barriers, consider a function $\overline{u}:\Gamma_\delta\to\mathbb{R}$ of the form $\overline{u}(x)\doteqdot\phi(x)+\eta(d(x))$ for some to-be-determined $\delta\in(0,\mu)$ and $\eta:\mathbb{R}\to\mathbb{R}$, where $\phi\in C^2(\overline{\Omega})$ is the prescribed boundary data for the Dirichlet problem (7.3). We want $\overline{u}$ to be a supersolution in $\Gamma_\delta$ and take the boundary values $\phi$ on $\partial\Omega$ (so we require $\eta(0)=0$). We also need $\eta$ to exceed any desired height, $K$, (determined by our *a priori* height estimate) at the inner boundary $\partial\Gamma_\delta\setminus\partial\Omega$ (so we require $\eta(\delta)>K-\phi(x)$). Observe that

$$\overline{u}_i = \phi_i+\eta' d_i \;\text{ and }\; \overline{u}_{ij} = \phi_{ij}+\eta' d_{ij}+\eta'' d_i d_j\,.$$

Recalling that $Dd$ is a null eigenvector of $D^2 d$, we thus obtaiin

$$\begin{aligned}
H_{\mathrm{graph}\,\overline{u}} &= \left(\delta_{ij}-\frac{\overline{u}_i\overline{u}_j}{1+|D\overline{u}|^2}\right)\frac{\overline{u}_{ij}}{\sqrt{1+|D\overline{u}|^2}}\\
&= \left(\delta_{ij}-\frac{(\phi_i+\eta' d_i)(\phi_j+\eta' d_j)}{1+|D\phi+\eta' Dd|^2}\right)\frac{\phi_{ij}+\eta'' d_i d_j}{\sqrt{1+|D\phi+\eta' Dd|^2}}\\
&\quad+\left(\delta_{ij}-\frac{\phi_i\phi_j}{1+|D\phi+\eta' Dd|^2}\right)\frac{\eta' d_{ij}}{\sqrt{1+|D\phi+\eta' Dd|^2}}\,.
\end{aligned}$$

We may estimate $\Delta d\le -H_{\partial\Omega}$ and

$$\left(\delta_{ij}-\frac{(\phi_i+\eta' d_i)(\phi_j+\eta' d_j)}{1+|D\phi+\eta' Dd|^2}\right)\phi_{ij}\le |D^2\phi|\,.$$

Moreover, if $\delta\le\mu/2$, then the eigenvalues of $-D^2 d$ satisfy $\frac{\kappa_i}{1-d\kappa_i}\le\frac{2}{\mu}$, so that

$$-d_{ij}\phi_i\phi_j\le\frac{2}{\mu}|D\phi|^2 \;\text{ in }\; \Gamma_\delta\,.$$

Assuming further that $\eta' \geq 0$ and $\eta'' \leq 0$, we may therefore estimate

$$\begin{aligned}\sqrt{1+|D\phi+\eta' Dd|^2}&H_{\operatorname{graph}\overline{u}}\\ &\leq |D^2\phi| - \eta' H_{\partial\Omega} + \frac{\eta''\left(1+|D\phi|^2-(D\phi\cdot Dd)^2\right)-\eta' d_{ij}\phi_i\phi_j}{1+|D\phi+\eta' Dd|^2}\\ &\leq |D^2\phi| - \eta' H_{\partial\Omega} + \frac{\eta''+\frac{2}{\mu}\eta'|D\phi|^2}{1+|D\phi+\eta' Dd|^2}\,.\end{aligned}$$

We can choose $\eta$ so that the final term is non-positive and the second term dominates the first, at least for $\delta$ sufficiently small. Indeed, if we set

$$\eta(r) \doteqdot \frac{1}{\nu}\log(1+kr)$$

for positive $k$ and $\nu$ (to be chosen in a moment), then

$$\eta(0)=0\,,\ \ \eta' = \frac{k}{\nu(1+kr)} > 0 \ \text{ and } \ \eta'' = -\frac{k^2}{\nu(1+kr)^2} < 0\,.$$

If $k>0$ and $\delta \leq \frac{\mu}{2}$ are chosen so that $\frac{k}{1+k\delta} \geq \frac{2}{\mu}\sup_\Omega |D\phi|^2$, then

$$\begin{aligned}\eta''+\frac{2}{\mu}\eta'|D\phi|^2 &= \frac{k}{\nu(1+kd)}\left(\frac{2}{\mu}|D\phi|^2-\frac{k}{1+kd}\right)\\ &\leq 0 \ \text{ in } \ \Gamma_\delta\,.\end{aligned}$$

So it remains to estimate

$$\eta' H_{\partial\Omega} \geq |D^2\phi| + \sqrt{1+|D\overline{u}|^2}\psi(\cdot,\overline{u},D\overline{u})\,.$$

Since $\partial\Omega$ is compact we can find, by the hypothesis (7.15), some $M<\infty$ and $\varepsilon>0$ such that

$$H_{\partial\Omega}(x) \geq \varepsilon + \sqrt{1+\varepsilon^2}\sup_{|p|>M}\psi(x,\phi(x),p)$$

for all $x \in \partial\Omega$; so it suffices to estimate

$$\frac{1}{\eta'}\left(|D^2\phi| + \sqrt{1+|D\overline{u}|^2}\psi(\cdot,\overline{u},D\overline{u})\right) \leq \varepsilon + \sqrt{1+\varepsilon^2}\sup_{|p|>M}\psi(x,\phi(x),p)$$

in $\Gamma_\delta$ for suitable $\delta$, $k$ and $\nu$.

If $\frac{k}{\nu(1+k\delta)} \geq \frac{\sup_\Omega |D^2\phi|}{\varepsilon}$, then we may estimate

$$\frac{|D^2\phi|}{\eta'} \leq \varepsilon \ \text{ in } \ \Gamma_\delta\,.$$

If $\frac{k}{\nu(1+k\delta)} \geq \sup_\Omega |D\phi| + M$, then we may estimate

$$\begin{aligned}
|D\overline{u}|^2 &= |D\phi| + 2\eta' D\phi \cdot Dd + (\eta')^2 \\
&\geq |D\phi|^2 - 2\eta'|D\phi| + (\eta')^2 \\
&= (\eta' - |D\phi|)^2 \\
&\geq M^2 \text{ in } \Gamma_\delta \,.
\end{aligned}$$

On the other hand,

$$\frac{\sqrt{1+|D\overline{u}|^2}}{\eta'} \leq \sqrt{1 + \frac{1+|D\phi|^2 + 2\eta'|D\phi|}{(\eta')^2}}\,,$$

so we can also arrange that

$$\frac{\sqrt{1+|D\overline{u}|^2}}{\eta'} \leq \sqrt{1+\varepsilon^2} \text{ in } \Gamma_\delta$$

so long as

$$\frac{k}{\nu(1+k\delta)} \geq \frac{\sup_\Omega |D\phi| + \sqrt{(1+\varepsilon^2)\sup_\Omega |D\phi|^2 + \varepsilon^2}}{\varepsilon^2}\,.$$

Since $\psi_z \leq 0$, we may also estimate

$$\sup_{|p|>M} \psi(x, \phi(x) + \eta(d(x)), p) \leq \sup_{|p|>M} \psi(x, \phi(x), p)$$

for all $x \in \partial\Omega$.

We conclude that

$$\begin{cases} -H_{\operatorname{graph} \overline{u}} \geq \psi(\cdot, \overline{u}, D\overline{u}) \text{ in } \Gamma_\delta \\ \overline{u} = \phi \text{ on } \partial\Omega \\ \overline{u} \geq K \text{ on } \Omega \cap \partial\Gamma_\delta \end{cases}$$

so long as

$$\frac{k}{1+k\delta} \geq \frac{2}{\mu}|D\phi|^2_{C^0(\Omega)}\,,$$

$$\frac{k}{\nu(1+k\delta)} \geq \max\left\{ \frac{|D^2\phi|_{C^0(\Omega)}}{\varepsilon}, |D\phi|_{C^0(\Omega)} + M, \frac{|D\phi|_{C^0(\Omega)} + \sqrt{(1+\varepsilon^2)|D\phi|^2_{C^0(\Omega)} + \varepsilon^2}}{\varepsilon^2} \right\}\,,$$

and

$$\frac{1}{\nu}\log(1+k\delta) \geq |\phi|_{C^0(\Omega)} + K\,.$$

These conditions can be arranged as follows: if we set $k \doteqdot \frac{4}{\mu}|D\phi|^2_{C^0(\Omega)}$ and $\delta \doteqdot \frac{1}{k}$, then the first condition is satisfied. We may now choose $\nu = \nu(|\phi|_{C^2(\Omega)}, \mu, M, \varepsilon, K)$ so small that the remaining conditions are satisfied.

The construction of a lower barrier is achieved either by an analogous construction, or by replacing $u$ with $-u$ and $\psi(x,z,p)$ with $-\psi(x,-z,-p)$.

These barriers yield the following boundary gradient estimate.

**Proposition 7.6.** *Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ whose boundary is of class $C^2$. Suppose that $\psi \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ and $\phi \in C^2(\overline{\Omega})$ satisfy*

$$\psi_z \leq 0 \;\; \text{and} \;\; H_{\partial\Omega}(x) \geq \varepsilon + \sqrt{1+\varepsilon^2} \sup_{|p|>M} |\psi(x,\phi(x),p)| \;\; \text{for all} \;\; x \in \partial\Omega$$

*for some $\varepsilon > 0$ and $M < \infty$, where $H_{\partial\Omega}$ is the mean curvature of $\partial\Omega$ with respect to its outwards pointing unit normal. If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies*

$$\begin{cases} -H|_{\operatorname{graph} u} = \psi(\cdot, u, Du) & \text{in } \Omega \\ \qquad\quad u = \phi & \text{on } \partial\Omega\,, \end{cases}$$

*then*

$$\sup_{\partial\Omega} |Du| \leq C\left(n, \sup_{\Omega}|u|, |\phi|_{C^2(\Omega)}, \varepsilon, M\right). \tag{7.18}$$

**7.5. A Hölder estimate for the gradient.** Fix a unit direction $e \in S^n$ and consider $v \doteqdot D_e u$. If $u \in C^2(\overline{\Omega})$, then $v \in C^1(\overline{\Omega})$ satisfies the divergence form linear equation (cf. (5.4))

$$-D_i(a^{ij}D_j v) = D_i f^i$$

in the weak sense, where

$$a^{ij} \doteqdot \frac{1}{\sqrt{1+|Du|^2}}\left(\delta_{ij} - \frac{u_i u_j}{1+|Du|^2}\right) \;\; \text{and} \;\; f \doteqdot \psi(\cdot, u, Du)e\,.$$

If $|u|_{C^1(\Omega)} \leq M < \infty$, then

$$a \geq \lambda\delta \;\; \text{and} \;\; |a| \leq \Lambda\,,$$

where $\lambda > 0$ and $\Lambda < \infty$ depend only on $|u|_{C^1(\Omega)}$, so the de Giorgi–Nash Hölder estimate (Theorem 6.5) yields

$$|u|_{C^{1,\alpha}(\Omega')} \leq C$$

for any $\Omega' \Subset \Omega$, where $C = C(n, M, \sup_{\Omega\times[-M,M]\times\overline{B}_M}|\psi|, \Omega, \Omega')$ and $\alpha = \alpha(n, M, \Omega, \Omega')$.

We would like to apply Theorem 6.8 to obtain a Hölder estimate for $Du$ up to the boundary of $\Omega$. This is not immediately possible however, since, roughly speaking, the boundary data only provide an oscillation estimate for $D_e u$ in directions $e$ which are tangent to the boundary. In order to exploit Theorem 6.8, we straighten the boundary in a neighbourhood of a given boundary point $x_0$ using a boundary chart. This results in a modified equation (on a neighbourhood of 0 in the halfspace $\mathbb{R}^{n-1} \times [0, \infty)$) which nonetheless satisfies the hypotheses of Theorem 6.8 (assuming the boundary charts are of class $C^{1,\alpha}$). Theorem 6.8 then yields a Hölder estimate for $D_e u$ at $x_0$ for all directions $e$ tangent to $\partial\Omega$ at $x_0$. The remaining derivative is estimated by a direct argument which exploits *Morrey's inequality*. We omit the details (see [**2**, §13.1]).

The resulting estimate may be stated as follows.

**Proposition 7.7.** *Suppose that $\Omega$ is a bounded open set of class $C^2$, $\phi \in C^2(\overline{\Omega})$, and that $\psi \in C^0(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$. If $u \in C^2(\Omega)$ satisfies $|u|_{C^1(\Omega)} \leq M < \infty$ and solves the Dirichlet problem* (7.3)*, then*

$$|u|_{C^{1,\alpha}} \leq C\,,$$

*where $C = C(n, M, \max_{\overline{\Omega} \times [-M,M] \times \overline{B}_M} |\psi|, |\phi|_{C^2(\Omega)}, \Omega)$ and $\alpha = \alpha(n, M, \Omega)$.*

**7.6. Solving the Dirichlet problem.** We are now able to solve the Dirichlet problem for mean curvature equations using the method of continuity.

**Theorem 7.8.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with boundary of class $C^{2,\alpha}$. Given $\psi \in C^{1,\alpha}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ satisfying $\psi_z \leq 0$ and $\phi \in C^{2,\alpha}(\partial\Omega)$, suppose that*

$$\int \psi(\cdot, 0, D\eta)\eta \leq (1 - \varepsilon) \int |D\eta| \tag{7.19}$$

*for all non-negative/non-positive $\eta \in C^1(\Omega)$ with $\operatorname{spt} \eta \Subset \Omega$ for some $\varepsilon > 0$, and*

$$H|_{\partial\Omega}(x) > \limsup_{|p| \to \infty} |\psi(x, \phi(x), p)| \text{ for all } x \in \partial\Omega\,. \tag{7.20}$$

*The Dirichlet problem*

$$\begin{cases} -H_{\operatorname{graph} u} = \psi(\cdot, u, Du) \text{ in } \Omega \\ \qquad\quad u = \phi \text{ on } \partial\Omega \end{cases} \tag{7.21}$$

*admits a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$.*

**Proof.** Consider, for each $t \in [0, 1]$, the Dirichlet problem

$$\begin{cases} -H_{\operatorname{graph} u} = t\psi(\cdot, u, Du) \text{ in } \Omega \\ \qquad\quad u = t\phi \text{ on } \partial\Omega\,. \end{cases} \tag{7.22}$$

Observe that the problem corresponding to $t = 0$ admits the trivial solution $u \equiv 0$. So it suffices to show that the set $S$ of parameters $t \in [0, 1]$ corresponding to problems which are uniquely soluble in $C^{2,\alpha}(\overline{\Omega})$ is both open and closed in $[0, 1]$.

Observe that Propositions 7.3, 7.4, 7.6 and 7.7 yield an estimate of the form

$$|u|_{C^{1,\beta}(\Omega)} \le C \tag{7.23}$$

for any solution to (7.22) (independent of $t$) for some $\beta = \beta(n, \psi, \Omega, \phi)$ and $C = C(n, \psi, \Omega, \phi)$. In particular, the gradient is uniformly bounded, so uniqueness of solutions to (7.22) is a consequence of Proposition 7.1.

To see that $S$ is closed, consider a sequence of parameters $t_k$ in $S$ converging to some $t \in [0, 1]$. Let $u_k \in C^{2,\alpha}(\overline{\Omega})$ be the solution to (7.22) corresponding to the parameter $t_k$. If we define $a_k$ and $f_k$ by

$$a_k(x) \doteqdot \frac{1}{\sqrt{1+|Du_k|^2}}\left(\delta - \frac{Du_k \otimes Du_k}{1+|Du_k|^2}\right)\Bigg|_x \quad \text{and} \quad f_k(x) \doteqdot \psi(\cdot, u_k, Du_k)\big|_x ,$$

then $u_k$ satisfies the linear elliptic equation

$$-a_k^{ij} D_i D_j u_k = f_k \ \text{ in } \ \Omega .$$

By (7.23), this equation satisfies the hypotheses of Schauder's estimate. If $\beta \ge \alpha$, then we obtain

$$|u_k|_{C^{2,\alpha}(\Omega)} \le C(n, \psi, \Omega, \phi) . \tag{7.24}$$

If $\beta < \alpha$, then we only obtain an estimate for $|u|_{C^{2,\beta}(\Omega)}$. However, since this implies an estimate for $|u|_{C^{1,\alpha}(\Omega)}$, we obtain (7.24) by applying Schauder's estimate a second time. The Arzelà–Ascoli theorem now provides a subsequence of the solutions $u_k$ which converges in $C^2(\overline{\Omega})$ to a solution $u \in C^{2,\alpha}(\Omega)$ to the problem (7.22) corresponding to the parameter $t$. So $S$ is indeed closed.

To see that $S$ is open, we apply the implicit function theorem and the solvability of the linearized problems. Consider the map $T : C^{2,\alpha}(\overline{\Omega}) \times [0, 1] \to C^{\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\partial\Omega)$ defined by

$$T(u, t) \doteqdot (-H_{\operatorname{graph} u} - t\psi(\cdot, u, Du), u|_{\partial\Omega} - t\phi) .$$

If $t_0 \in S$, then we can find $u_0 \in C^{2,\alpha}(\overline{\Omega})$ such that $T(u_0, t_0) = (0, 0)$. In order to apply the implicit function theorem, we need to show that the Fréchet derivative of $T$ in the first variable at the point $(u_0, t_0)$ is an isomorphism. It suffices to compute the Gateaux derivative $DT : v \mapsto D_v T$, where $D_v T$ is the directional derivative in the direction $v$, so long as this is a continuous linear operator (meaning that $(u, t) \mapsto DT|_{(u,t)}$ is a continuous family of continuous linear maps) in a neighbourhood of $(u_0, t_0)$. So consider, for some $(u,t) \in C^{2,\alpha}(\overline{\Omega}) \times [0,1]$ and $v \in C^{2,\alpha}(\overline{\Omega})$, the directional derivative

$$\begin{aligned} D_vT|_{(u,t)} &\doteqdot \left.\frac{d}{ds}\right|_{s=0} T(u+sv,t) \\ &= \left.\frac{d}{ds}\right|_{s=0} \Big( -H_{\mathrm{graph}(u+sv)} - t\psi(\cdot, u+sv, D(u+sv)), \\ &\qquad\qquad (u+sv)|_{\partial\Omega} - t\phi\Big) \\ &= \big( -L_{(u,t)}v, v|_{\partial\Omega}\big)\,, \end{aligned} \tag{7.25}$$

where the linear map $L_{(u,t)} \doteqdot a^{ij}D_iD_j + b^iD_i + c$ is defined by

$$a^{ij} \doteqdot \frac{1}{\sqrt{1+|Du|^2}}\left(\delta^{ij} - \frac{D_iuD_ju}{1+|Du|^2}\right)\,,$$

$$b^i \doteqdot \sum_{j=1}^n \left[\frac{1}{\sqrt{1+|Du|^2}}\left(\delta^{ij} - \frac{D_iuD_ju}{1+|Du|^2}\right)\right]_j + t\psi_{p_i}(\cdot,u,Du)\,,$$

and

$$c \doteqdot t\psi_z(\cdot,u,Du)\,.$$

The required continuity properties of $L$ follow readily, so the map $v \mapsto (-L_{(u_0,t_0)}v, v|_{\partial\Omega})$ coincides with the Fréchet derivative. By Theorem 4.1, $L_{(u_0,t_0)}$ is an isomorphism, so the implicit function theorem provides some $\delta > 0$ and a function $h : (t_0-\delta, t_0+\delta) \to C^{2,\alpha}(\overline{\Omega})$ such that $T(h(t),t) = (0,0)$ for all $t \in (t_0-\delta, t_0+\delta)$. That is, $(t_0-\delta, t_0+\delta) \subset S$. So $S$ is indeed open. □

Observe that the conclusion of Theorem 12.5 still holds if the condition (7.19) is replaced by the existence of upper and lower barriers for each of the Dirichlet problems (7.22) and/or condition (7.20) is replaced by the existence of upper and lower barriers for the problems (7.22) in a neighbourhood of any point $x \in \partial\Omega$ which take the boundary values at $x$.

In particular, we have proved that it is possible to solve the minimal surface equation and the translator equation (see Exercise 7.4) over any strictly mean convex domain of class $C^{2,\alpha}$. To solve the constant mean curvature or capillary surface problems, we must impose stronger convexity conditions on the boundary (see Exercise 7.3).

Assuming higher (interior or global) regularity of the data, we may obtain correspondingly higher (interior or global, respectively) regularity of the solution by appealing to Schauder's estimates.

**7.7. Epilogue.** In order to apply the method of continuity to solve the Dirichlet problem for mean curvature type equations (Theorem 12.5), we needed two main ingredients. These were the solubility in $C^{k,\alpha}$ of the Dirichlet problem for the linearized operator (12.14) (Theorem 4.1) and an *a priori* estimate in $C^{1,\alpha}$. This followed from the de Giorgi–Nash theory (Theorems 6.5 and 6.8), so long as we are able to obtain an *a priori* estimate in $C^1$. The key tools for proving the latter were the maximum principle, Stampacchia iteration, and the construction of suitable barriers.

### 7.8. Exercises.

**Exercise 7.1.** Let $M$ be the graph of a smooth function equipped with its induced metric $g$ and measure $\mu$. Define the DIVERGENCE of a vector field $V$ on $M$ by

$$\operatorname{div} V \doteqdot \operatorname{tr}(\nabla V) = \partial_i V^i + V^k \Gamma_{ik}{}^i .$$

Prove that the DIVERGENCE THEOREM holds for $\mu$. That is,

$$\int_M \operatorname{div} V = 0$$

for every compactly supported $V$.

**Exercise 7.2.** Suppose that $u : I \to \mathbb{R}$ satisfies the one dimensional translator equation. Show that $u$ is of the form

$$u(x) = y_0 - \log\cos(x - x_0)$$

for some $(x_0, y_0) \in \mathbb{R}^2$. That is, $u$ is part of a Grim Reaper.

**Exercise 7.3.** Given $h \in \mathbb{R}$, show that the Dirichlet problem

$$\begin{cases} H_{\operatorname{graph} u} = h \ \text{ in } \ B_R \\ \qquad\quad u = 0 \ \text{ on } \ \partial B_R \end{cases}$$

for the constant mean curvature equation admits no solution if $|h| > \frac{n}{R}$.

**Exercise 7.4.** Suppose that $n \geq 2$. Show that, for $a$ sufficiently large, the paraboloid $\underline{u}(x) \doteqdot u_0 + \frac{a}{2}\left(|x - x_0|^2 - R^2\right)$ is a subsolution to the translator equation, where $(x_0, u_0) \in \mathbb{R}^{n+1}$ and $R \geq 0$ are arbitrary.

**Exercise 7.5.** Show that the constraint (7.6) holds for the prescribed mean curvature equation

$$-H|_{\operatorname{graph} u} = \psi(\cdot, u) \ \text{ in } \ \Omega$$

if $\psi_z \leq 0$ and $\sup_{x\in\Omega} |\psi(x, 0)| \leq \frac{1}{C}$ for some constant $C = C(n, \Omega, \varepsilon)$.

**Exercise 7.6.** Justify (7.10). *Hint: You will need the* Codazzi identity, *which implies that the tensor $\nabla A$ given by*

$$\nabla_k A_{ij} = \partial_k A_{ij} - \Gamma_{ki}{}^\ell A_{\ell j} - \Gamma_{kj}{}^\ell A_{i\ell}$$

*is totally symmetric.*

**Exercise 7.7.** Verify (7.11) for the translator equation.

**Exercise 7.8.** Let $\{M^n_\varepsilon\}_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ be a smooth one-parameter family of minimal hypersurfaces $M^n_\varepsilon \subset \mathbb{R}^{n+1}$, given as the image of the immersions $X_\varepsilon : M^n \to \mathbb{R}^{n+1}$. Show that the normal component

$$v \doteqdot \langle V\,,\,\nu\rangle$$

of the variation field $V \doteqdot \frac{d}{d\varepsilon}\big|_{\varepsilon=0} X$ satisfies the JACOBI EQUATION

$$-(\Delta + |A|^2)v = 0$$

on $M^n_0$. You may assume that $M_\varepsilon$ are graphs.

**Exercise 7.9.** Let $(M,g)$ be a Riemannian manifold and let $a \in \Gamma(TM\otimes TM)$ be a non-negative definite symmetric tensor, $b \in \Gamma(TM)$ be a smooth vector field and $c \in C^\infty(M)$ be a smooth function on $M$. Suppose that $(M,g)$ admits a strict subsolution $\phi \in C^2(M)$ to the corresponding linear equation. That is,

$$-(a\cdot\nabla^2 + b\cdot\nabla + c)\phi < 0\,,$$

where $\nabla$ is the Levi-Civita covariant differential and $\cdot$ denotes contraction.

(a) Suppose that $v : M \to \mathbb{R}$ satisfies

$$-(a\cdot\nabla^2 + b\cdot\nabla + c)u \le 0\,.$$

Assuming $c \le 0$, show that

$$\max_M u \le \max_{\partial M} u_+\,.$$

(b) Suppose that $v : M \to \mathbb{R}$ is positive and satisfies

$$-(a\cdot\nabla^2 + b\cdot\nabla + c)u \ge 0\,.$$

Assuming $c \ge 0$, show that

$$\min_M u \ge \min_{\partial M} u\,.$$

*Hint: If $\gamma : (-s_0, s_0) \to M$ is the geodesic through $x = \gamma(0)$ with $\gamma'(0) = v$, and $f \in C^2(M)$, then*

$$\frac{d}{ds}\bigg|_{s=0}(f\circ\gamma) = \nabla f\cdot v \;\; \textit{and} \;\; \frac{d^2}{ds^2}\bigg|_{s=0}(f\circ\gamma) = \nabla^2 f(v,v)\,.$$

**Exercise 7.10.** Prove that every bounded open set of class $C^2$ satisfies the interior and exterior ball conditions.

## 8. Fully nonlinear equations — an introduction

Consider now a completely general second order differential equation

$$F(x, u(x), Du(x), D^2u(x)) = 0 \tag{8.1}$$

for a function $u : \Omega \to \mathbb{R}$. Here, $\Omega$ is some subset of $\mathbb{R}^n$ and $F$ may be any function at all which is defined on some subset $\Gamma$ of $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times S^{n\times n}$, where we recall that $S^{n\times n}$ denotes the space of symmetric $n \times n$ matrices. Observe that, in case $\Gamma \subsetneq \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times S^{n\times n}$, the solutions we seek are restricted to those functions which implicitly satisfy the condition $(x, u(x), Du(x), D^2u(x)) \in \Gamma$ for all $x \in \Omega$. Since no linearity properties for $F$ are supposed, such an equation is called FULLY NONLINEAR.

It will be convenient to introduce the $k$-JET of a function $u \in C^k(\Omega)$, which is the map $\mathcal{J}^k u : \Omega \to \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^{n\times n}$ defined by

$$\mathcal{J}^k u(x) \doteqdot (x, u(x), Du(x), \ldots, D^k u(x)) \,.$$

We will make use of the variables $(x, z, p, r)$ to denote points of $\Omega \times \mathbb{R} \times \mathbb{R}^n \times S^{n\times n}$.

An operator $F : \Gamma \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times S^{n\times n} \to \mathbb{R}$ is called ELLIPTIC (or WEAKLY ELLIPTIC) if

$$F(x, z, p, r + A) \geq F(x, z, p, r)$$

for every $(r, p, z, x) \in \Gamma$ and every positive definite $A \in S^{n\times n}$ such that $(x, p, z, r + A) \in \Gamma$, STRICTLY (or LOCALLY UNIFORMLY) ELLIPTIC if the inequality is strict, and UNIFORMLY ELLIPTIC if there exists positive $\lambda > 0$ such that

$$F(x, z, p, r + A) - F(x, z, p, r) \geq \lambda \operatorname{tr}(A)$$

for every $(r, p, z, x) \in \Gamma$ and every positive definite $A \in S^{n\times n}$ such that $(x, p, z, r + A) \in \Gamma$. If $F$ is continuously differentiable with respect to the $r$ variable, then ellipticity, strict ellipticity, and uniform ellipticity are equivalent to

$$\frac{\partial F}{\partial r_{ij}}\xi_i\xi_j \geq 0 \,, \quad \frac{\partial F}{\partial r_{ij}}\xi_i\xi_j > 0 \,, \quad \text{and} \quad \frac{\partial F}{\partial r_{ij}}\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^n \setminus \{0\} \,,$$

respectively, where the partial derivatives of $F$ can be computed from the formula

$$\left.\frac{\partial F}{\partial r_{ij}}\right|_r A_{ij} = \left.\frac{d}{ds}\right|_{s=0} F(r + sA) \,.$$

We will also say that $F$ is (LOCALLY UNIFORMLY/UNIFORMLY) ELLIPTIC AT $u$ for some $u \in C^2(\Omega)$ if $F$ is (locally uniformly/uniformly) elliptic on the two-jet of $u$; that is, on the set $\mathcal{J}^2 u(\Omega)$.

As is generally the case when analyzing nonlinear objects, the *linearization* is a useful tool (both conceptually and analytically). Suppose that $u \in$

$C^2(\Omega)$ is a solution to (8.1) for some $F \in C^1(\Gamma)$, $\Gamma \underset{\text{open}}{\subset} \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times S^{n\times n}$. If $v \in C^2(\Omega)$, then

$$\left.\frac{d}{ds}\right|_{s=0} F(\mathcal{J}^2(u+sv)) = a^{ij}v_{ij} + b^i v_i + cv\,,$$

where

$$a^{ij}(x) \doteqdot \left.\frac{\partial F}{\partial r_{ij}}\right|_{\mathcal{J}^2u(x)}, \quad b^i(x) \doteqdot \left.\frac{\partial F}{\partial p_i}\right|_{\mathcal{J}^2u(x)}, \quad \text{and} \quad c(x) \doteqdot \left.\frac{\partial F}{\partial z}\right|_{\mathcal{J}^2u(x)}.$$

The operator

$$L \doteqdot a^{ij}D_iD_j + b^iD_i + c$$

is called the LINEARIZATION of $F$ at $u$ and the corresponding linear equation the LINEARIZATION of (8.1). Note that (8.1) is (locally uniformly/uniformly) elliptic at $u$ if and only if its linearization at $u$ is (locally uniformly/uniformly) elliptic.

Clearly, the class of fully nonlinear elliptic equations includes all quasilinear elliptic equations, and all linear elliptic equations. Let us list a few less trivial examples.

**Examples 8.1.**

(1) The Monge–Ampère equation:

$$\det(D^2u) = 1\,.$$

The operator $F(r) = \det r$ is elliptic (but not uniformly) on the positive cone

$$S^{n\times n}_+ \doteqdot \{A \in S^{n\times n} : A_{ij}v^iv^j > 0 \,\forall\, v \in \mathbb{R}^n \setminus \{0\}\}\,.$$

(2) Prescribing Gauss curvature:

$$K(x) = f(x, u(x))\,,$$

where

$$K = \frac{\det(D^2u)}{(1+|Du|^2)^{\frac{n+2}{2}}}$$

is the Gauss curvature of the graph of $u$.

(3) (Powers of) Gauss curvature flow translators:

$$K^\alpha = \frac{1}{\sqrt{1+|Du|^2}}\,.$$

(4) $k$-Hessian equations:

$$S_k(D^2u) = 0\,,$$

where $S_k : S^{n\times n} \to \mathbb{R}$ denotes the elementary symmetric polynomial of degree $k$:

$$S_k(r) \doteqdot \binom{n}{k}^{-1} \sum_{1\le i_1\le\cdots\le i_k\le n} \rho_{i_1}\dots\rho_{i_k}$$

with $\rho_i$ denoting the eigenvalues of $r$. $S_k$ is elliptic (but not uniformly) on the cone

$$\Gamma_k \doteqdot \{A \in S^{n\times n} : S_l(A) > 0, 0 < l \le k\}\,.$$

(5) Prescribing $k$-th mean curvature:

$$H_k(x) = f(x, u(x))\,,$$

where

$$H_k = \frac{1}{(1+\|Du\|^2)^{\frac{k}{2}}} S_k\left(\left(I - \frac{Du\otimes Du}{1+\|Du\|^2}\right) D^2u\right)$$

is the $k$-th mean curvature of the graph of $u$.

(6) (Powers of) $k$-th mean curvature flow translators:

$$H_k^\alpha = \frac{1}{\sqrt{1+|Du|^2}}\,.$$

(7) The Bellman equation:

$$\inf_{\alpha\in\mathcal{A}}\{L_\alpha u\} = f\,,$$

where $\{L_\alpha\}_{\alpha\in\mathcal{A}}$ is some family of linear operators

$$L_\alpha u = a_\alpha^{ij} u_{ij} + b_\alpha^k u_k + c_\alpha u\,.$$

The Bellman equation is concave in $(u, Du, D^2u)$ (the infimum of a family of linear maps is concave). Conversely, any equation of the form

$$F(u, Du, D^2u) = f$$

where $F$ is concave in $(u, Du, D^2u)$ can be written as a Bellman equation (this is a consequence of the Hahn–Banach theorem).

(8) The Isaacs equation:

$$\sup_{\alpha\in\mathcal{A}}\inf_{\beta\in\mathcal{B}}\{L_{\alpha\beta} u\} = f\,,$$

where $\{L_{\alpha\beta}\}_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}}$ is some family of linear operators

$$L_{\alpha\beta} u = a_{\alpha\beta}^{ij} u_{ij} + b_{\alpha\beta}^k u_k + c_{\alpha\beta} u\,.$$

In general, the Isaacs equation is neither concave nor convex in any of its arguments. Any fully nonlinear equation of the form

$$F(u, Du, D^2u) = f$$

for which $F$ is Lipschitz in all arguments can be written in the form of the Isaacs equation, since

(1) If $F(r)$ is Lipschitz with constant $\Lambda$, then it is the infimum over $r_0$ of all cones $C(r) = F(r_0) + \Lambda\|r - r_0\|$.
(2) For fixed $r_0$, each cone $C(r)$ is the supremum of all linear functions of the form $L(r) = F(r_0) + \mathrm{tr}(A\cdot(r - r_0))$ for $\|A\| \le \Lambda$.

Suppose that $u$ satisfies (8.1) for some elliptic operator $F$. If $F$ is of class $C^1$, then any derivative $v \doteqdot D_l u$ of $u$ satisfies the *linear elliptic equation*

$$-Lv = f\,, \tag{8.2}$$

where $L$ is the linearization of $F$ at $u$ and

$$f(x) \doteqdot \left.\frac{\partial F}{\partial x^l}\right|_{\mathcal{J}^2u(x)} .$$

If $L$ is uniformly elliptic and $f$ and the coefficients of $L$ are bounded, then *a priori* estimates for the Hölder continuity of solutions to general linear elliptic equations with bounded coefficients (analogous to the de Giorgi–Nash–Moser estimates) would yield Hölder continuity for $Du$. This is the content of the celebrated KRYLOV–SAFONOV THEORY, which we will see in §10. However, note that the coefficients depend also on the *second derivatives* of $u$. Therefore, to apply such a theorem, we would need to bound the coefficients without a bound for the norm of $D^2u$. This is possible in some cases.

Recall that a CONE $\Gamma$ in a real linear space $V$ is a subset with the property

$$x \in \Gamma \implies \lambda x \in \Gamma \text{ for all } \lambda > 0\,.$$

A function $F : \Gamma \to \mathbb{R}$ is $k$-homogeneous, $k \in \mathbb{R}$, if $\Gamma$ is a cone and

$$F(\lambda x) = \lambda^k F(x) \text{ for all } x \in \Gamma \text{ and } \lambda > 0\,.$$

**Example 8.2.** Consider the HESSIAN EQUATION

$$-F(D^2u) = \psi(\cdot, u, Du) \tag{8.3}$$

for some $F \in C^1(\Gamma)$. If $F$ is 1-HOMOGENEOUS, then its derivative $\frac{\partial F}{\partial r_{ij}}$ is 0-HOMOGENEOUS. So in order to obtain uniform estimates for $\frac{\partial F}{\partial r_{ij}}$ it suffices to obtain estimates on the unit sphere $\{r \in \Gamma : \|r\| = 1\}$, since

$$\frac{\partial F}{\partial r_{ij}}(r) = \frac{\partial F}{\partial r_{ij}}\left(\|r\|\,\frac{r}{\|r\|}\right) = \frac{\partial F}{\partial r_{ij}}\left(\frac{r}{\|r\|}\right) .$$

The above example provides a large class of equations which admit $C^{1,\alpha}$ estimates. However, even if the $C^{1,\alpha}$ estimate does apply, this is still not enough to apply the Schauder theory to obtain higher regularity: since the coefficients in (8.2) also depend on *second derivatives* of $u$, we also need to estimate the Hölder continuity of $D^2u$ to obtain a Hölder estimate for the coefficients. This is a more difficult problem than the $C^{1,\alpha}$ theory.

**Example 8.3.** Suppose that $u \in C^4(\Omega)$ satisfies the Hessian equation

$$-F(D^2u) = \psi\,, \tag{8.4}$$

where $\psi \in C^2(\Omega)$ depends only on $x \in \Omega$. After differentiating (8.4) twice, we find that any pure second derivative $v \doteqdot u_{ee}$ of $u$ satisfies

$$-\frac{\partial F}{\partial r_{ij}}v_{ij} - \frac{\partial^2 F}{\partial r_{pq}\partial r_{rs}}u_{epq}u_{ers} = \psi_{ee}\,. \tag{8.5}$$

The second term on the left is problematic: it involves third derivatives of $u$ which cannot be related to first derivatives of $v$; however, if we assume that $F$ is *concave*, then $v$ is a *subsolution* to a linear elliptic equation. It turns out that the Krylov–Safonov theory can then be used to obtain the desired Hölder continuity for $D^2u$. This is the content of the *Evans–Krylov Theorem*, which we will see in §11. Note that these considerations also apply to convex operators by considering the dual operator $F_*(r) \doteqdot -F(-r)$.

Other situations where a $C^{2,\alpha}$ estimate may be obtained are:

(i) (Morrey, Nirenberg) when $n = 2$,

(ii) (Cordes–Nirenberg) for solutions to $F(\cdot, D^2u) = 0$ such that

$$\left\|\frac{\partial F}{\partial r_{ij}}(\cdot, D^2u) - \delta^{ij}\right\| \leq \varepsilon_0\,.$$

(ii) for solutions to INVERSE CONCAVE Hessian equations, for which the dual operator $F_*$ defined by

$$F_*(r^{-1}) \doteqdot F(r)^{-1}$$

is assumed to be concave.

An estimate for the Hölder continuity of second derivatives will be sufficient to apply the Schauder estimate and the method of continuity to obtain existence of smooth ($C^{k+2}$ if $F$ is $C^k$) solutions to (8.1).

In summary, the existence of classical solutions will follow if we can obtain *a priori* estimates in $C^{2,\alpha}$. We have described some situations in which this can be achieved, which raises the question: can it be achieved in general? The following theorem gives a *negative* answer to this question!

**Theorem 8.1** (Nadirashvili *et al*[23] (2007–2012))**.** *In every dimension $n \geq 5$, there is a $C^{1,\alpha}$ function, $\alpha \in (0,1)$, which solves a smooth Hessian equation (in the "viscosity sense") but is not even $C^{1,1}$.*

The reader may have noticed that this theorem still leaves open the possibility that classical solutions always exist in low dimensions. This problem remains open.

**Open Problem 1.** *Are all solutions of* (8.1) *for 'nice' $F$ (smooth and Hessian, say) in dimensions 3 and 4 necessarily smooth ($C^{k+2}$ if $F$ is $C^k$)?*

The construction of the counterexample of Theorem 8.1 is algebraic, and, for algebraic reasons, does not work in dimensions 4 and less. According to Nadirashvili and Vlăduţ, this "suggests strongly that in 4 (and fewer) dimensions there is no homogenous non-classical solutions to uniformly elliptic equations".

### 8.1. Exercises.

**Exercise 8.1.** Consider the function $F : S_+^{n\times n} \to \mathbb{R}$ defined on the positive definite symmetric matrices $S_+^{n\times n}$ by

$$F(r) \doteqdot \log\det r\,.$$

Show that

$$DF|_r = r^{-1}\,.$$

**Exercise 8.2.** Let $F \in C^1(\Gamma)$ be a $k$-homogeneous function. Prove that

$$Df|_z \cdot z = kf(z)\,.$$

This is known as Euler's theorem (for homogeneous functions).

If $f \in C^2(\Gamma)$, deduce that

$$D^2 f|_z(z,z) = k(k-1)f(z)\,.$$

**Exercise 8.3.** Show that the symmetric function $N : S^{n\times n} \to \mathbb{R}$ which gives the norm of a nonzero symmetric matrix,

$$N^2(r) \doteqdot \operatorname{tr}(r^2)$$

is strictly convex in nonradial directions; that is,

$$D^2 N|_r(v,v) > 0$$

for all $r \in S^{n\times n}$ and all $v \in S^{n\times n} \setminus \mathbb{R}r$. (Note that $D^2N|_r(r,r) = 0$ due to Euler's theorem.)

[23] See Nadirashvili and Vlăduţ, *Singular Solutions of Hessian Elliptic Equations in Five Dimensions*.

## 9. The generalized maximum principle of Alexandrov

The key tool in the Krylov–Safanov theory is the generalized maximum principle of Alexandrov. To state Alexandrov's observation, we first need to introduce some natural geometric objects.

First, recall that each non-vertical hyperplane $\Pi$ in $\mathbb{R}^{n+1}$ is the graph of a linear function $\pi : \mathbb{R}^n \to \mathbb{R}$ given by

$$\pi(x) = \pi_0 + p \cdot (x - x_0)$$

for some vector $p \in \mathbb{R}^n$ (the "gradient" of $\Pi$) and some point $(x_0, \pi_0) \in \mathbb{R}^{n+1}$.

**Definition 9.1.** *Let $u : \Omega \to \mathbb{R}$ be a continuous function. The* UPPER CONTACT SET $\Pi_u^+$ *of $u$ is the set*

$$\Pi_u^+ \doteqdot \{y \in \Omega : u(x) \leq u(y) + p \cdot (x - y) \textit{ for all } x \in \Omega \textit{ for some } p \in \mathbb{R}^n\}$$

*and the* NORMAL MAPPING $\chi_u : \Omega \to \mathcal{P}(\mathbb{R}^n)$ *is the mapping defined by*

$$\chi_u(y) \doteqdot \{p \in \mathbb{R}^n : u(x) \leq u(y) + p \cdot (x - y) \textit{ for all } x \in \Omega\}\,.$$

So $\Pi_u^+$ is the set of points $x \in \Omega$ at which $(x, u(x)) \in \operatorname{graph} u$ is a point of "first contact" with hyperplanes which are translated downwards from infinity, and $\chi_u(x)$ is the set of gradients of planes which make first contact at $(x, u(x))$. (Note that the vector $p - e_{n+1}$ is normal to $\Pi$.)

Recall that a SUPPORTING HALF-SPACE for a set $C \subset \mathbb{R}^{n+1}$ at a point $Y \in \partial C$ is any closed half-space $H$ containing $C$ such that $\partial H$ contains $Y$. The upper contact set $\Pi_u^+$ of $u$ is then the set of points $y \in \Omega$ such that the hypograph $\{(x, h) : x \in \Omega, h < u(x)\}$ of $u$ admits at least one supporting half-space at $(y, u(y))$, and $(e_{n+1} - p)/|e_{n+1} - p|$ is the outward pointing unit normal vector to such a half-space.

Clearly,

- $\chi_u(x)$ is non-empty if and only if $x \in \Pi_u^+$.
- $u$ is locally concave on the set $\Pi_u^+$.
- $u$ is concave if and only if $\Pi_u^+ = \Omega$ if and only if the hypograph of $u$ is a convex set.
- If $u \in C^1(\Omega)$, then $\chi_u(x) = \{Du(x)\}$ for all $x \in \Pi_u^+$ and $\Pi_u^+$ is the set of points $x \in \Omega$ for which the tangent plane to $\operatorname{graph} u$ at $(x, u(x))$ lies above $\operatorname{graph} u$.

**Example 9.1.** Consider the function $c_{a,z,R} : B_R(z) \to \mathbb{R}$, defined by

$$c_{a,z,R}(x) \doteqdot a\left(1 - \frac{|x - z|}{R}\right),$$

whose graph is the cone of radius $R$ with base $B_R(z)$ and vertex $(z,a)$. Observe that

$$\Pi^+_{c_{a,z,R}} = B_R(z)$$

and

$$\chi_{c_{a,z,R}}(y) = \begin{cases} -\dfrac{a(y-z)}{R|y-z|} & \text{if} \quad y \neq z \\ \overline{B}_{a/R}(z) & \text{if} \quad y = z\,. \end{cases} \tag{9.1}$$

We can now state the generalized maximum principle of Alexandrov.

**Theorem 9.2.** *Given a bounded open subset $\Omega \subset \mathbb{R}^n$ and any $a : \Omega \to S^{n\times n} \cap GL(n)$, every $u \in C^0(\overline{\Omega}) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$ satisfies*

$$\sup_\Omega u \leq \max_{\partial\Omega} u + \frac{d}{n\omega_n^{\frac{1}{n}}} \left\| \frac{a^{ij}u_{ij}}{(\det a)^{\frac{1}{n}}} \right\|_{L^n(\Pi^+_u)} ,$$

*where $d \doteqdot \mathrm{diam}(\Omega)$ and $\omega_n$ is the area of $\partial B^n_1$.*

By the Sobolev embedding theorem, functions $u \in W^{2,n}(\Omega)$ are continuous in the interior of $\Omega$. Thus, the condition $u \in C^0(\overline{\Omega})$ is no restriction if we simply replace $\max_{\partial\Omega} u$ by $\limsup_{y\to\partial\Omega} u(y)$.

Note also that we have made no restrictions to the coefficients $a$ other than non-degeneracy. However, if the term $a^{ij}u_{ij}/(\det a)^{\frac{1}{n}}$ is not in $L^n(\Pi^+_u)$, then the right hand side is taken to be infinite.

Theorem 9.2 is a consequence of the following beautiful observation.

**Lemma 9.3.** *If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, then*

$$\sup_\Omega u \leq \max_{\partial\Omega} u + d \left( \frac{1}{\omega_n} \int_{\Pi^+_u} |\det D^2 u| \right)^{\frac{1}{n}} ,$$

*where $d \doteqdot \mathrm{diam}(\Omega)$ and $\omega_n$ is the area of $\partial B^n_1$.*

**Proof.** Replacing $u$ by $u - \max_{\partial\Omega} u$, we can assume that $u \leq 0$ on $\partial\Omega$. Observe that $D^2 u \leq 0$ in $\Pi^+_u$. Thus, the Jacobian $J(Du) = \det(D^2 u)$ is non-positive in $\Pi^+_u$. If it were strictly negative, then the classical change of variables formula would allow us rewrite the $n$-dimensional Lebesgue measure of the normal image of $\Omega$ as

$$|\chi_u(\Omega)| = |\chi_u(\Pi^+_u)| = |Du(\Pi^+_u)| = \int_{\Pi^+_u} |\det D^2 u|\,. \tag{9.2}$$

If $\det(D^2 u)$ is not strictly negative, we may still obtain the estimate by applying the above argument to $u_\varepsilon(x) \doteqdot u(x) - \frac{\varepsilon}{2}|x|^2$ and taking $\varepsilon \to 0$ (note that $\chi_{u_\varepsilon} = \chi_u - \varepsilon I$).

It remains to estimate $u$ by $|\chi_u(\Omega)|$. Suppose that $u$ takes a positive maximum at an interior point $x_0 \in \Omega$ (otherwise the claim is already true) and let $c : \Omega \to \mathbb{R}$ be the function whose graph is the cone over $\Omega$ with vertex $(x_0, u(x_0))$. We claim that $\chi_c(\Omega) \subset \chi_u(\Omega)$. Indeed, certainly $Du(x_0) \in \chi_c(x_0)$. On the other hand, each half-space other than $T_{x_0}\,\mathrm{graph}\, u$ which is tangent to the cone $\mathrm{graph}\, c$ intersects $\mathrm{graph}\, u$ at more than one point and, hence, must be parallel to a supporting half-space (simply translate it upwards some finite distance until it detaches completely. The translate at the last point(s) of contact must support $\mathrm{graph}\, u$). This proves the claim. Consider now the function $C$ whose graph is the cone with vertex $(x_0, u(x_0))$ but base $B_d(x_0)$. Then $\chi_C(\Omega) \subset \chi_c(\Omega)$, since each supporting half-space for $\mathrm{graph}\, C$ lies above $\mathrm{graph}\, c$ but intersects it at the vertex. Consequently,

$$|\chi_C(\Omega)| \leq |\chi_u(\Omega)| \, .$$

On the other hand, by (9.1), we can compute $|\chi_C(\Omega)|$ explicitly. Indeed,

$$|\chi_C(\Omega)| = |B_{u(x_0)/d}(x_0)| = \omega_n \left( \frac{u(x_0)}{d} \right)^n \, .$$

Recalling (9.2), this completes the proof. □

If we set $A \doteqdot a$ and $B \doteqdot -D^2 u$, then, provided $u \in C^2(\Omega)$, Theorem 9.2 is an immediate consequence of the arithmetic-geometric mean inequality,

$$\det(AB) \leq \left( \frac{\mathrm{tr}(AB)}{n} \right)^n \, .$$

The general case then follows by approximating $u \in W^{2,n}(\Omega) \cap C^0(\overline{\Omega})$ by a sequence of smooth functions $u_j \in C^2(\Omega) \cap C^0(\overline{\Omega})$.

## 10. The Harnack inequality of Krylov and Safanov — Hölder continuity of solutions to linear elliptic equations of non-divergence form

Consider the general *non-divergence form* linear elliptic equation

$$-Lu \doteqdot -\left(a^{ij}u_{ij} + b^i u_i + cu\right) = f \ \text{ in } \ \Omega\,, \tag{10.1}$$

where $(a, b, c) : \Omega \to S^{n\times n}\times\mathbb{R}^n\times\mathbb{R}$. Our ultimate goal is the Krylov–Safonov Harnack inequality and Hölder estimate, which provides an analogue of the de Giorgi–Nash–Moser theory for linear elliptic equations of *divergence form.*

Other than uniform ellipticity, we will require only very weak conditions of the coefficients. This is, of course, crucial when we want to apply the theory to the derivatives of solutions to fully nonlinear equations.

In the case of divergence form equations, we were able to consider weak solutions which admit a weak first derivative. For equations of non-divergence form, we need *two* weak derivatives. We say that a function $u$ is a (STRONG) SOLUTION to (10.1) if it has two weak derivatives and satisfies (10.1) pointwise almost everywhere. Strong sub- and super-solutions are defined analogously.

Just as in the de Giorgi–Nash–Moser theory, the Krylov–Safonov theory is based on a Harnack inequality derived from two complimentary estimates: a *mean value inequality* for subsolutions and a *weak Harnack inequality* for positive supersolutions.

**10.1. The mean value inequality.** We begin with the mean value inequality.

**Theorem 10.1.** *There exists $C = C(n, \gamma, \nu) < \infty$ with the following property. Suppose that $(a, b, c) : B_{2R}(y) \to S^{n\times n} \times \mathbb{R}^n \times \mathbb{R}$ satisfy, for some $\lambda > 0$,*

$$\lambda\delta^{ij} \leq a^{ij} \leq \gamma\lambda\delta^{ij} \quad \textit{and} \quad \left(\frac{|b|}{\lambda}\right)^2 + \frac{|c|}{\lambda} \leq \frac{\nu}{R^2}\,. \tag{10.2}$$

*Given $f \in L^n(B_{2R}(y))$ and $p \leq n$, any subsolution $u \in W^{2,n}(B_{2R}(y)) \cap C^0(\overline{B}_{2R}(y))$ to*

$$-\left(a^{ij}u_{ij} + b^i u_i + cu\right) = f \quad \textit{in} \quad B_{2R}(y) \tag{10.3}$$

*satisfies*

$$\sup_{B_R(y)} u \leq C\left[\left(\fint_{B_{2R}(y)} u_+^p\right)^{\frac{1}{p}} + \frac{R}{\lambda}\|f\|_{L^n(B_{2R}(y))}\right].$$

**Proof.** We will prove the theorem for the case $B_{2R}(y) = B_1(0)$ and $u \in C^2(B_1) \cap C^0(\overline{B}_1)$. The general case then follows by approximating $u \in W^{2,n}(B_1) \cap C^0(\overline{B}_1)$ by a sequence of functions $u_k \in C^2(B_1) \cap C^0(\overline{B}_1)$ and considering the rescaled function $u_R(x) \doteqdot \frac{1}{R} u\left(\frac{x-y}{R}\right)$.

For $\beta \geq 1$ to be chosen momentarily, consider the smooth cut-off function $\eta : B_1 \to \mathbb{R}_+$ given by

$$\eta(x) \doteqdot (1 - |x|^2)^\beta \,.$$

The function $v \doteqdot \eta u$ is in $C^2(B_1) \cap C^0(\overline{B}_1)$ and satisfies $v \equiv 0$ on $\partial B_1$.

Note that

$$\begin{aligned} \eta_i &= -2\beta\eta^{1-\frac{1}{\beta}} x_i \\ \text{and} \quad \eta_{ij} &= -2\beta\eta^{1-\frac{1}{\beta}}\delta_{ij} + \left(1 - \frac{1}{\beta}\right)\frac{\eta_i\eta_j}{\eta} \\ &= -2\beta(1-|x|^2)^{\beta-2}\left[(1-|x|^2)\delta_{ij} - 2(\beta-1)x_ix_j\right] \,. \end{aligned}$$

By the the maximum principle of Alexandrov (Theorem 9.2), we can estimate $v$ by estimating $-a^{ij}v_{ij}$ on the upper contact set $\Pi_v^+$ of $v$. So consider

$$\begin{aligned} -a^{ij}v_{ij} &= -\eta a^{ij}u_{ij} - 2a^{ij}\eta_i u_j - a^{ij}\eta_{ij}u \\ &\leq -\eta a^{ij}u_{ij} + 4\beta\eta^{1-\frac{1}{\beta}}a^{ij}x_iu_j + 2\beta u\eta^{1-\frac{1}{\beta}}a^{ij}\delta_{ij} \\ &\leq \eta(f + b^iu_i + cu) + 4\beta\eta^{1-\frac{1}{\beta}}a^{ij}x_iu_j + 2\beta u\,\eta^{1-\frac{1}{\beta}}a^{ij}\delta_{ij} \\ &\leq \eta(f + |b||Du| + cu) + 4\beta\gamma\lambda\eta^{1-\frac{1}{\beta}}|x||Du| + 2\beta n\gamma\lambda u\,\eta^{1-\frac{1}{\beta}} \,. \end{aligned} \tag{10.4}$$

Since $v$ is of class $C^1$, we can estimate

$$v(y) \leq v(x) + Dv|_x \cdot (y - x)$$

for any $x \in \Pi_v^+$ and any $y \in \overline{B}_1$. If $Dv|_x \neq 0$, we can choose $y \in \partial B_1$ so that

$$y - x = -|y - x|\frac{Dv(x)}{|Dv(x)|} \,.$$

Since $|y - x| \geq d(x, \partial B_1) = 1 - |x|$, this yields

$$|Dv(x)| \leq \frac{v(x)}{1 - |x|} \,.$$

Note also that $v > 0$, and hence $u > 0$, on $\Pi_v^+$ (this is because $v \equiv 0$ on $\partial B_1$ and $\Pi_v^+$ is the set of points $y$ for which the hypograph of $v$ admits a supporting hyperplane at $(y, u(y))$). We can now estimate

$$\begin{aligned}\eta|Du| &= |Dv - uD\eta| \\ &\leq |Dv| + u|D\eta| \\ &\leq \frac{v}{1-|x|} + u|D\eta| \\ &\leq 2(1+\beta)\eta^{1-\frac{1}{\beta}}u\,. \end{aligned} \tag{10.5}$$

Putting (10.4) and (10.5) together yields

$$\begin{aligned}-a^{ij}v_{ij} &\leq \eta f + \left(\eta^{\frac{2}{\beta}}\frac{|c|}{\lambda} + 2(1+\beta)\eta^{\frac{1}{\beta}}\frac{|b|}{\lambda} + 8\beta(1+\beta)\gamma + 2\beta n\gamma\eta^{\frac{1}{\beta}}\right)\eta^{-\frac{2}{\beta}}\lambda v \\ &\leq f + C\eta^{-\frac{2}{\beta}}\lambda v \quad \text{on} \quad \Pi_v^+\,,\end{aligned}$$

where $C = C(n, \beta, \gamma, \nu)$. Applying the Alexandrov maximum principle (Theorem 9.2), we obtain

$$\begin{aligned}\sup_{B_1} v &\leq C\left(\left\|\eta^{-\frac{2}{\beta}}v\right\|_{L^n(B_1)} + \frac{1}{\lambda}\|f\|_{L^n(B_1)}\right) \\ &\leq C\left((\sup_{B_1} v)^{1-\frac{2}{\beta}}\|u^{\frac{2}{\beta}}\|_{L^n(B_1)} + \frac{1}{\lambda}\|f\|_{L^n(B_1)}\right)\,,\end{aligned}$$

where $C = C(n, \beta, \gamma, \nu)$. If $\beta \geq 2$, writing $p = \frac{2n}{\beta}$ and applying Young's inequality yields

$$\sup_{B_1} v \leq C\left(\|u\|^p_{L^p(B_1)} + \frac{1}{\lambda}\|f\|_{L^n(B_1)}\right)\,.$$

The claim now follows by estimating $\eta$ from below on $B_{1/2}$. $\square$

**10.2. The weak Harnack inequality.** We next prove a weak Harnack inequality.

**Theorem 10.2.** *There are constants $\sigma = \sigma(n, \gamma, \nu) \in (0, n]$ and $C = C(n, \gamma, \nu) < \infty$ with the following property. Suppose that $(a, b, c) : B_{2R}(y) \to S^{n\times n} \times \mathbb{R}^n \times \mathbb{R}$ satisfy (10.2) for some $\lambda > 0$. Given $f \in L^n(B_{2R}(y))$, any non-negative supersolution $u \in W^{2,n}(B_{2R}(y)) \cap C^0(\overline{B}_{2R}(y))$ to (10.3) satisfies*

$$\left(\fint_{B_R(y)} u^\sigma\right)^{\frac{1}{\sigma}} \leq C\left(\inf_{B_R(y)} u + \frac{R}{\lambda}\|f\|_{L^n(B_{2R}(y))}\right)\,.$$

**Proof.** We may assume that $B_{2R}(y) = B_1 \doteqdot B_1(0)$ and $u \in C^2(B_1) \cap C^0(\overline{B}_1)$. Replacing $L$ and $f$ by $\lambda^{-1}L$ and $\lambda^{-1}f$, we may also assume that

$\lambda = 1$. Set, for any $\varepsilon > 0$,

$$\overline{u} \doteqdot u + \varepsilon + \|f\|_{L^n(B_1)} \,, \quad w \doteqdot -\log \overline{u}\,, \quad v \doteqdot \eta w \quad \text{and} \quad g \doteqdot -\frac{f}{\overline{u}}\,,$$

where, as in the proof of the mean value inequality, $\eta \doteqdot (1-|x|^2)^\beta$ for $\beta \geq 1$ to be chosen momentarily. We wish to apply the Alexandrov maximum principle to $v$. Noting that

$$w_i = -\frac{u_i}{\overline{u}} \;\text{ and }\; w_{ij} = -\frac{u_{ij}}{\overline{u}} + w_i w_j\,,$$

we find that

$$-a^{ij}v_{ij} = \eta\left(b^i w_i - c\frac{u}{\overline{u}} + g - a^{ij}w_iw_j\right) - 2a^{ij}\eta_i w_j - w a^{ij}\eta_{ij}\,.$$

Estimating, via Young's inequality,

$$-2a^{ij}\eta_i w_j \leq \frac{2}{\eta}a^{ij}\eta_i\eta_j + \frac{\eta}{2}a^{ij}w_iw_j$$

and

$$b^i w_i \leq |b|^2 + \frac{1}{4}|Dw|^2\,,$$

and using the assumption $a^{ij} \geq \delta^{ij}$ yields

$$-a^{ij}v_{ij} \leq \eta\left(|b|^2 + |c| + g\right) + \frac{2}{\eta}a^{ij}\eta_i\eta_j - w a^{ij}\eta_{ij}\,.$$

Note that

$$-a^{ij}\eta_{ij} = 2\beta(1-|x|^2)^{\beta-2}\left[(1-|x|^2)a^{ij}\delta_{ij} - 2(\beta-1)a^{ij}x_ix_j\right]\,.$$

If we fix $\alpha \in (0,1)$ and choose $\beta$ so that

$$\beta \geq \frac{n\gamma}{2\alpha^2}\,,$$

then, whenever $|x| \geq \alpha$, we obtain

$$a^{ij}\delta_{ij} \leq n\gamma \leq 2\alpha^2\beta \leq |x|^2 a^{ij}\delta_{ij} + 2(\beta-1)a^{ij}x_ix_j$$

and we conclude

$$-a^{ij}\eta_{ij} \leq 0 \quad \text{in} \quad B_1 \setminus B_\alpha\,.$$

On the other hand, we can crudely estimate

$$\frac{-a^{ij}\eta_{ij}}{\eta} \leq \frac{2n\beta\gamma}{1-\alpha^2} \quad \text{in} \quad B_\alpha\,.$$

Thus, on the set $B_1^+ \doteqdot \{x \in B_1 : v > 0\}$, we obtain

$$\begin{aligned} -a^{ij} v_{ij} &\leq \eta \left(|b|^2 + |c| + g\right) + v\, \chi_{B_\alpha} \sup_{B_\alpha} \left( \frac{-a^{ij}\eta_{ij}}{\eta} \right) \\ &\quad + 4\beta^2 (1 - |x|^2)^{\beta - 2} a^{ij} x_i x_j \\ &\leq |b|^2 + |c| + g + 4\beta^2 \gamma + \frac{2n\beta\gamma}{1-\alpha^2}\, v\, \chi_{B_\alpha}\,, \end{aligned}$$

where $\chi_{B_\alpha}$ is the indicator function of the ball $B_\alpha$. Alexandrov's maximum principle (Theorem 9.2) now yields

$$\sup_{B_1} v \leq C(1 + \|v_+\|_{L^n(B_\alpha)})\,, \tag{10.6}$$

where $C = C(n, \alpha, \gamma, \nu)$.

This implies a bound for $v_+$ on $B_\alpha$ so long as $\alpha$ is small enough. In order to exploit a Calderon–Zygmund type cube decomposition, we will need to phrase this as an estimate on cubes. Given $z \in \mathbb{R}^n$ and $R > 0$, denote by

$$K_R(z) \doteqdot \{x \in \mathbb{R}^n : x_i \in (z_i - R, z_i + R)\}$$

the open cube parallel to the coordinate axes with centre $z$ and side length $2R$. Observe that, if $\alpha \leq \frac{1}{\sqrt{n}}$, then $B_\alpha \subset K_\alpha \doteqdot K_\alpha(0) \subset B_1$. Thus, (10.6) yields

$$\begin{aligned} \sup_{B_1} v &\leq C(1 + \|v_+\|_{L^n(K_\alpha)}) \\ &\leq C\left(1 + |K_\alpha^+|^{\frac{1}{n}} \sup_{B_1} v_+\right)\,, \end{aligned}$$

where $K_\alpha^+ \doteqdot \{x \in K_\alpha : v(x) > 0\}$. Thus, if (note that $|K_\alpha| = (2\alpha)^n$)

$$\frac{|K_\alpha^+|}{|K_\alpha|} \leq \vartheta \doteqdot \frac{1}{(2C)^n (2\alpha)^n}\,,$$

then we obtain

$$\sup_{B_1} v \leq 2C$$

with the same constant $C = C(n, \alpha, \gamma, \nu)$ from (10.6).

If we now fix $\alpha \doteqdot \frac{1}{3n}$ (and $\vartheta$ accordingly), then $K_{3\alpha} \Subset B_1$. So $\eta$ is bounded from below on $K_{3\alpha}$ and we obtain

$$\sup_{K_{3\alpha}} w \leq C(n, \gamma, \nu)\,.$$

In fact, by an appropriate change of variables (namely, $x \mapsto \alpha(x - z)/r$), we obtain

$$\sup_{K_{3r}(z)} w \leq C(n, \gamma, \nu) \tag{10.7}$$

for any $z \in B_1$ and $r > 0$ such that

$$B_{3r}(z) \subset B_1 \quad \text{and} \quad |K_r^+(z)| \le \vartheta |K_r(z)|\,.$$

Moreover, by replacing $w$ by $w - k$ in the arguments leading to (10.7), we obtain

$$\sup_{K_{3r}(z)} (w-k) \le C(n,\gamma,\nu)$$

for any $z \in B_1$ and $r > 0$ such that

$$B_{3r}(z) \subset B_1 \quad \text{and} \quad |K_r^+(z)| \le \vartheta |K_r(z)|\,,$$

where now $K_R^+(z) \doteqdot \{x \in K(z) : w - k > 0\}$.

Consider the set

$$U_k \doteqdot \{x \in K_0 : w(x) \le k\}\,.$$

If we set $\delta \doteqdot 1 - \vartheta$ and $K_0 \doteqdot K_\alpha(0)$ (where $\alpha = \frac{1}{3n}$), then

$$|K_r^+| \le \vartheta |K_r| \quad \Longrightarrow \quad \delta |K_r| \le |K_r| - |K_r^+| = |K_r \cap U_k|\,.$$

**Lemma 10.3.** *Given a cube $K_0 \subset \mathbb{R}^n$, any $w \in L^n(K_0)$, and any $k \in \mathbb{R}$, set*

$$U_k \doteqdot \{x \in K_0 : w(x) \le k\}\,.$$

*Suppose there exist positive constants $\delta < 1$ and $C < \infty$ such that, for any $K_r(z) \subset K_0$,*

$$|K_r(z) \cap U_k| \ge \delta |K_r(z)| \quad \Longrightarrow \quad \sup_{K_0 \cap K_{3r}(z)} (w-k) \le C\,. \tag{10.8}$$

*If $|U_k| > 0$, then*

$$\sup_{K_0} (w-k) \le C\left(1 + \frac{\log(|U_k|/|K_0|)}{\log \delta}\right)\,.$$

**Proof of Lemma 10.3.** It will suffice to show that

$$|U_k| \ge \delta^m |K_0| \quad \Longrightarrow \quad \sup_{K_0} (w-k) \le mC \tag{10.9}$$

for any $m \in \mathbb{N}$. Indeed, if $|U_k| > 0$, then we can always choose $m$ large enough that $|U_k| \ge \delta^m |K_0|$.

Certainly (10.9) holds when $m = 1$. So suppose that (10.9) holds for some $m \in \mathbb{N}$ and that $|U_k| \ge \delta^{m+1}|K_0|$. Set

$$\widetilde{U}_k \doteqdot \bigcup \{K_{3r}(z) \cap K_0 : |K_r(z) \cap U_k| \ge \delta K_r(z)\}\,.$$

By (10.8), $w - k \le C$ on $\widetilde{U}_k$ and hence $\widetilde{U}_k \subset U_{k+C}$. We claim that

$$|\widetilde{U}_k| \ge \delta^m |K_0|\,. \tag{10.10}$$

It will then follow from the induction hypothesis that

$$\sup_{K_0}(w-k) \leq (m+1)C\,,$$

which proves (10.9) (and hence the Lemma).

To prove (10.10), we perform a cube decomposition. First, bisect the edges of $K_0$ to obtain a family of $2^n$ congruent subcubes; denote this family of cubes by $\mathcal{K}_1$. Set $u_k \doteqdot \chi_{U_k}$ and let $\mathcal{K}_1^+$ be the subset of these cubes $K$ satisfying

$$\int_K u_k = |U_k \cap K| > \delta|K|\,. \tag{10.11}$$

Denote by $\mathcal{K}_1^- = \mathcal{K}_1 \setminus \mathcal{K}_1^+$ the collection of remaining cubes; that is, those cubes $K$ satisfying

$$\int_K u_k = |U_k \cap K| \leq \delta|K|\,. \tag{10.12}$$

Next, bisect the edges of $\mathcal{K}_1^-$ to obtain a second generation of cubes, $\mathcal{K}_2$, which are separated into the subset $\mathcal{K}_2^+$ satisfying (10.11) and the set $\mathcal{K}_2^-$ satisfying (10.12). Continuing in this way, we obtain a countable family $\mathcal{K}^+ \doteqdot \cup_{i=0}^\infty \mathcal{K}_i^+$ of subcubes satisfying (10.11). Each $K^+ \in \mathcal{K}^+$ lies in some $\mathcal{K}_i^+$ and hence (unless $i=1$) has some 'parent' cube $K \in \mathcal{K}_{i-1}^-$. We denote the set of all parent cubes of cubes in $\mathcal{K}^+$ by $\mathcal{K}^-$. Since each point of $K_0 \setminus \mathcal{K}^+$ lies in a nested sequence of cubes satisfying (10.12) with diameters tending to zero, Lebesgue's differentiation theorem implies that

$$u_k \leq \delta \quad \text{a.e. in} \quad K_0 \setminus \mathcal{K}^+\,.$$

Since $\delta < 1$ and $u_k$ is an indicator function, this implies that

$$u_k = 0 \quad \text{a.e. in} \quad K_0 \setminus \mathcal{K}^+\,;$$

that is, up to a set of measure zero, $U_k \subset \mathcal{K}^+$. It follows that, up to a set of measure zero, we can cover $U_k$ by a family $\{K_i^-\}_{i\in\mathcal{I}}$ of disjoint cubes in $K_i^- \in \mathcal{K}^-$, and hence obtain

$$|U_k| = \left|U_k \cap \left(\bigcup_{i\in\mathcal{I}} K_i^-\right)\right| \leq \delta \left|\bigcup_{i\in\mathcal{I}} K_i^-\right|\,.$$

Since each parent $K$ of a cube $K_r(z)$ lies inside the larger cube $K_{3r}(z)$, we see that $\cup_{i\in\mathcal{I}} K_i^- \subset \widetilde{U}_k$. We thereby conclude that

$$|U_k| \leq \delta|\widetilde{U}_k|\,.$$

This proves (10.10), and, by choosing $m$ appropriately, completes the proof of Lemma 10.3. $\square$

Denote by

$$\mu_t \doteqdot |\{x \in K_0 : \overline{u}(x) > t\}|$$

the distribution function of $\overline{u}$ in $K_0$. This is related to $|U_k|$ via the change of variables

$$\mu_t = |U_k| \,, \quad t = \mathrm{e}^{-k} \,.$$

Applying Lemma 10.3 yields

$$\begin{aligned} -\log \inf_{K_0}\left(\frac{\overline{u}}{\mathrm{e}^{-k}}\right) &= \sup_{K_0}\left(-\log\left(\frac{\overline{u}}{\mathrm{e}^{-k}}\right)\right) \\ &= \sup_{K_0}\,(w-k) \\ &\le C\left(1+\frac{\log(\mu_{\mathrm{e}^{-k}}/|K_0|)}{\log\delta}\right) . \end{aligned}$$

Exponentiating both sides and replacing $\mathrm{e}^{-k} = t$, we obtain

$$\begin{aligned} \inf_{K_0}\left(\frac{\overline{u}}{t}\right) &\ge \mathrm{e}^{-C}\left[\exp\left(\log\left(\frac{\mu_t}{|K_0|}\right)\right)\right]^{-\frac{C}{\log\delta}} \\ &= \mathrm{e}^{-C}\left(\frac{\mu_t}{|K_0|}\right)^{\frac{C}{-\log\delta}} . \end{aligned}$$

We conclude that there are positive constants

$$C' \doteqdot \delta|K_0| = C'(n,\gamma,\nu) \quad \text{and} \quad \kappa \doteqdot \frac{-\log\delta}{C} = \kappa(n,\gamma,\nu)$$

such that

$$\mu_t \le C'\left(\inf_{K_0}\frac{\overline{u}}{t}\right)^{\kappa} . \tag{10.13}$$

Note that this holds for all $t > 0$.

On the other hand, Fubini's theorem yields

$$\begin{aligned} \int_K |\overline{u}|^{\sigma} &= \int_{K_0}\int_0^{|\overline{u}(x)|^{\sigma}} dt\,dx \\ &= \sigma\int_{K_0}\int_0^{|\overline{u}(x)|} t^{\sigma-1}dt\,dx \\ &= \sigma\int_{K_0}\int_0^{\infty} t^{\sigma-1}\chi_{(0,|\overline{u}(x)|)}dt\,dx \\ &= \sigma\int_0^{\infty} t^{\sigma-1}\int_{K_0}\chi_{(0,|\overline{u}(x)|)}dx\,dt \\ &= \sigma\int_0^{\infty} t^{\sigma-1}\mu_t\,dt \end{aligned} \tag{10.14}$$

for any $\sigma > 0$, where $\chi_{(0,|u(x)|)}$ is the characteristic function of the interval $(0, |u(x)|)$. Combining (10.13) and (10.14), we obtain, for any $0 < \sigma < \kappa$,

$$\begin{aligned}\int_{K_0} \overline{u}^\sigma &\leq \sigma \int_0^{\inf_{K_0} \overline{u}} t^{\sigma-1}\mu_t\, dt + \sigma C' \big(\inf_{K_0} \overline{u}\big)^\kappa \int_{\inf_{K_0} \overline{u}}^\infty t^{\sigma-\kappa-1} dt \\ &\leq |K_0| \big(\inf_{K_0} \overline{u}\big)^\sigma + \frac{\sigma C'}{\kappa - \sigma}\big(\inf_{K_0} \overline{u}\big)^\sigma .\end{aligned}$$

Since $B_\alpha \subset K_0$, fixing $\sigma = \kappa/2$, say, and taking $\varepsilon \to 0$, we conclude that

$$\left(\int_{B_\alpha} u^\sigma\right)^{\frac{1}{\sigma}} \leq C \left(\inf_{B_\alpha} u + \|f\|_{L^n(B_1)}\right) ,$$

where $C = C(n, \gamma, \nu)$. The estimate can now be obtained with $B_\alpha$ replaced by $B_{1/2}$ via a scaling and covering argument. □

**10.3. The Harnack inequality.** Combining Theorems 10.1 and 10.2 yields a Harnack inequality for non-negative solutions to $-Lu = f$.

**Corollary 10.4.** *There is a constant $C = C(n, \gamma, \nu) < \infty$ with the following property. Suppose that $(a, b, c) : B_{R_0}(y) \to S^{n\times n} \times \mathbb{R}^n \times \mathbb{R}$ satisfy* (10.2) *for some $\lambda > 0$. Given $f \in L^n(B_{2R}(y))$, any non-negative solution $u \in W^{2,n}(B_{2R}(y))$ to* (10.3) *satisfies*

$$\sup_{B_R(y)} u \leq C \left(\inf_{B_R(y)} u + \frac{R}{\lambda}\|f\|_{L^n(B_{2R}(y))}\right) .$$

**10.4. The Hölder estimate.** Just as in the case of divergence form equations, the Harnack inequality yields an interior Hölder estimate.

**Theorem 10.5.** *There exist $\alpha = \alpha(n, \gamma, \nu) \in (0, 1)$ and $C = C(n, \gamma, \nu) < \infty$ with the following property. Suppose that $(a, b, c) : B_R(y) \to S^{n\times n} \times \mathbb{R}^n \times \mathbb{R}$ satisfy* (10.2) *for some $\lambda > 0$. Given $f \in L^n(B_R(y))$, any solution $u \in W^{2,n}(B_R(y))$ to* (10.3) *in $B_R(y)$ satisfies*

$$[u]_{C^\alpha(B_{R/2}(y))} \leq C \left(|u|_{C^0(B_R(y))} + R\, \|f - cu\|_{L^n(B_R(y))}\right) .$$

**10.5. Estimates up to the boundary.** We will also need suitable estimates up to the boundary.

The boundary version of the mean value inequality is almost identical to the interior case.

**Theorem 10.6.** *There exists $C = C(n, \gamma, \nu) < \infty$ with the following property. Given an open set $\Omega \subset \mathbb{R}^n$, suppose that $(a, b, c) : \Omega \to S^{n\times n} \times \mathbb{R}^n \times \mathbb{R}$*

*satisfy* (10.2) *for some* $\lambda > 0$. *Given* $f \in L^n(\Omega)$ *and* $p \leq n$, *any subsolution* $u \in W^{2,n}(\Omega) \cap C^0(\overline{\Omega})$ *to* (10.3) *with* $u \leq 0$ *on* $\partial\Omega \cap B_{2R}(y)$ *satisfies*

$$\sup_{\Omega \cap B_R(y)} u \leq C \left[ \left( \frac{1}{R^n} \int_{\Omega \cap B_{2R}(y)} u_+^p \right)^{\frac{1}{p}} + \frac{R}{\lambda} \|f\|_{L^n(\Omega \cap B_{2R}(y))} \right].$$

**Proof.** Assuming (without loss of generality) that $R = 1/2$, $y = 1/2$ and $u \in C^2(\Omega)$, the theorem may be obtained by applying the argument of Theorem 10.1 to the extension

$$v(x) \doteqdot \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega \end{cases}$$

of $u$ to $B_1$. (Indeed, despite the fact that $v$ may not be of class $C^2$ in $B_1$, its upper contact set lies in $B_1 \cap \Omega$.) $\square$

The boundary version of the weak Harnack inequality is as follows.

**Theorem 10.7.** *There are constants* $\sigma = \sigma(n, \gamma, \nu) \in (0, n]$ *and* $C = C(n, \gamma, \nu) < \infty$ *with the following property. Given an open set* $\Omega \subset \mathbb{R}^n$, *suppose that* $(a, b, c) : \Omega \to S^{n\times n} \times \mathbb{R}^n \times \mathbb{R}$ *satisfy* (10.2) *for some* $\lambda > 0$. *Given* $f \in L^n(\Omega)$, *any non-negative supersolution* $u \in W^{2,n}(\Omega) \cap C^0(\overline{\Omega})$ *to* (10.3) *with* $\inf_{B_R(y)\cap\partial\Omega} u \geq m$ *satisfies*

$$\left( \frac{1}{R^n} \int_{B_R(y)} u_m^\sigma \right)^{\frac{1}{\sigma}} \leq C \left( \inf_{\Omega \cap B_R(y)} u_m + \frac{R}{\lambda} \|f\|_{L^n(\Omega \cap B_{2R}(y))} \right).$$

*where*

$$u_m \doteqdot \begin{cases} \min\{u(x), m\} & \text{if } x \in \Omega \\ m & \text{if } x \notin \Omega\,. \end{cases}$$

**Proof.** The claim is proved by proceeding more or less as in Theorem 10.2 with $u$ replaced by $u_m$. $\square$

As a consequence, we obtain the following global Hölder estimate.

**Theorem 10.8.** *There exist constants* $\alpha = \alpha(n, \gamma, \nu, \alpha_0, K, L, \vartheta_0) \in (0, 1)$ *and* $C = C(n, \gamma, \nu, \alpha_0, K, L, \vartheta_0, R) < \infty$ *with the following property. Let* $\Omega \subset \mathbb{R}^n$ *be a connected open set with* $\operatorname{diam}(\Omega) \leq R$ *satisfying the uniform exterior cone condition* $\min_{x\in\partial\Omega} \vartheta(C_x) \geq \vartheta_0 > 0$ *and let* $(a, b, c) : \Omega \to S^{n\times n} \times \mathbb{R}^n \times \mathbb{R}$ *satisfy* (10.2) *for some* $\lambda > 0$. *Given* $f \in L^n(\Omega)$ *and* $\phi \in C^{\alpha_0}(\overline{\Omega})$ *with* $|\phi|_{C^{\alpha_0}(\Omega)} \leq L$, *any solution* $u \in W^{2,n}(\Omega)$ *to*

$$\begin{cases} -\left(a^{ij}u_{ij} + b^i u_i + cu\right) = f & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases}$$

*with* $|u|_{C^0(\Omega)} \leq K$ *satisfies*

$$|u|_{C^\alpha(\Omega)} \leq C\,.$$

**Proof.** See [**2**, Corollary 9.29]. $\square$

## 11. Hölder continuity of second derivatives for concave Hessian equations

We want to apply the Krylov–Safonov theory to obtain classical solutions of fully nonlinear elliptic PDE (via the method of continuity). By the Schauder theory, it suffices to obtain an *a priori* estimate in $C^{2,\alpha}$. Recall, however, that the counterexamples of Nadirashvilli *et al* ensure that such an estimate cannot hold in general. Thus (at least in dimensions 5 and above) some additional assumption is necessary. The first breakthrough, due to Evans and Krylov (which appeared soon after the paper of Krylov and Safonov), came for *concave* operators.

Our goal then is to establish a Hölder estimate for second derivatives of solutions $u$ to the equation

$$F(\cdot, u, Du, D^2u) = 0 \quad \text{in} \quad \Omega\,,$$

where $F$ is defined on some open subset $\Gamma$ of $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times S^{n\times n}$ containing $(\cdot, u, Du, D^2u)$ at each point of $\Omega$. We begin by differentiating the PDE. Assuming that $u \in C^4(\Omega)$, fix $l \in \{1, \dots, n\}$ and set $w \doteqdot u_{ll}$. Assuming further that $F \in C^2(\Gamma)$, we find that

$$-(a^{ij}w_{ij} + b^i w_i + cw) = d + e\,,$$

where

$$a^{ij} \doteqdot \frac{\partial F}{\partial r_{ij}}\,,\quad b^i \doteqdot \frac{\partial F}{\partial p_i}\,,\quad c \doteqdot \frac{\partial F}{\partial z}\,,$$

$$d \doteqdot \frac{\partial^2 F}{\partial p_i \partial p_j}u_{il}u_{jl} + 2\frac{\partial^2 F}{\partial p_i \partial z}u_{il}u_l + 2\frac{\partial^2 F}{\partial p_i \partial x^l}u_{il} + \frac{\partial^2 F}{\partial z \partial z}u_l u_l + 2\frac{\partial^2 F}{\partial z \partial x^l}u_l + \frac{\partial^2 F}{\partial x^l \partial x^l}$$

and

$$e \doteqdot \frac{\partial^2 F}{\partial r_{pq}\partial r_{rs}}u_{lpq}u_{lrs} + 2\left(\frac{\partial^2 F}{\partial r_{ij}\partial x^l} + \frac{\partial^2 F}{\partial r_{ij}\partial z}u_l + \frac{\partial^2 F}{\partial r_{ij}\partial p_k}u_{lk}\right)u_{lij}\,,$$

and, of course, each of the derivatives of $F$ are evaluated at $(\cdot, u, Du, D^2u)$. To make things more manageable, we will only consider *concave Hessian equations*[24],

$$-F(D^2u) = \psi(\cdot, u, Du)\,,$$

with $F$ a concave function of $D^2u$. That is, abusing notation,

$$F(x, z, p, r) = F(r) + \psi(x, z, p)\,,$$

[24] It is, however, possible to relax this assumption (see [**2**, §17.4]); that is, the linear term of the component $e$ involving the third derivatives of $u$ can be controlled. All of the difficulties in the theory seem to lie in controlling the quadratic term (which we discard by assuming concavity of $F$).

where, writing $\Gamma = \Gamma_{xzp} \times \Gamma_r \subset (\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n) \times S^{n\times n}$, $F : \Gamma_r \to \mathbb{R}$ is concave and $\psi : \Gamma_{xzp} \to \mathbb{R}$. This ensures that the mixed derivatives involving one derivative with respect to $r$ vanish, so that the term $e$ involving third derivatives of $u$ reduces to the quadratic term, which is now under control due to the concavity hypothesis. That is,

$$e = \frac{\partial^2 F}{\partial r_{pq} \partial r_{rs}} u_{lpq} u_{lrs} \leq 0 \, .$$

Note also that the inhomogeneous terms, encompassed by $d$, are bounded by a constant depending only on $\|u\|_{C^2(\Omega)}$ and $\sup_{\mathcal{J}^1 u(\Omega)} |D^2 f|$ and $c$ is bounded by $\sup_{\mathcal{J}^1 u(\Omega)} \left|\frac{\partial f}{\partial z}\right|$.

We want to apply the weak Harnack inequality. So consider a family of concentric balls $B_r = B_r(x_0)$ such that $B_{2R} \subset \Omega$. Setting $M_r \doteqdot \sup_{B_r} w$, we find that the function $v \doteqdot M_{2R} - w$ is a positive supersolution to the linear equation

$$-(a^{ij} v_{ij} + b^i v_i + cv) = d \quad \text{in} \quad B_{2R} \, ,$$

where

$$a^{ij}(x) \doteqdot \left.\frac{\partial F}{\partial r_{ij}}\right|_{D^2 u(x)} \, , \quad b^i(x) \doteqdot \left.\frac{\partial \psi}{\partial p_i}\right|_{\mathcal{J}^1 u(x)} \, , \quad c(x) \doteqdot \left.\frac{\partial \psi}{\partial z}\right|_{\mathcal{J}^1 u(x)}$$

and

$$\begin{aligned} d(x) \doteqdot \Bigg( & \frac{\partial^2 \psi}{\partial p_i \partial p_j} u_{il} u_{jl} + 2 \frac{\partial^2 \psi}{\partial p_i \partial z} u_{il} u_l + 2 \frac{\partial^2 \psi}{\partial p_i \partial x^l} u_{il} + \frac{\partial^2 \psi}{\partial z \partial z} u_l u_l \\ & + 2 \frac{\partial^2 \psi}{\partial z \partial x^l} u_l + \frac{\partial^2 \psi}{\partial x^l \partial x^l} \Bigg)\Bigg|_{\mathcal{J}^1 u(x)} \, . \end{aligned}$$

If we can find $\lambda > 0$ and $\gamma < \infty$ (depending only on $DF|_{D^2 u(\Omega)}$) such that

$$\lambda \delta^{ij} \leq a^{ij} \leq \gamma \lambda \delta^{ij}$$

and $\nu > 0$ (depending only on $\operatorname{diam}(\Omega)$, $\lambda$, $\|u\|_{C^2(B_{2R})}$, $\sup_{\mathcal{J}^1 u(\Omega)} |\frac{\partial \psi}{\partial z}|$ and $\sup_{\mathcal{J}^1 u(\Omega)} |D^2 \psi|$) such that

$$\left(\frac{|b|}{\lambda}\right)^2 + \frac{|c|}{\lambda} \leq \frac{\nu}{R^2} \, ,$$

then the weak Harnack inequality (Theorem 10.2) yields

$$\left(R^{-n} \int_{B_R} (M_{2R} - w)^\sigma \right)^{\frac{1}{\sigma}} \leq C \left(M_{2R} - M_R + R^2\right) \, , \tag{11.1}$$

where $C$ and $\sigma$ depend only on $n$, $\gamma$ and $\nu$.

To obtain a complementary inequality, we use the concavity of $F$ to obtain a functional relation between the second derivatives of $u$. Namely, if we set

$$g(x) \doteqdot -\psi(x, u(x), Du(x)),$$

then concavity of $F$ implies that, for any $x$, $y \in \Omega$,

$$\begin{aligned} DF|_{D^2u(y)} \cdot (D^2u(y) - D^2u(x)) &\leq F(D^2u(y)) - F(D^2u(x)) \\ &= g(y) - g(x). \end{aligned} \tag{11.2}$$

We would like to "diagonalize" this inequality to obtain a relation between pure second derivatives. Unfortunately, it is not in general possible to mutually diagonalize a family of symmetric matrices (although in this case we can mutually diagonalize two of the three matrices involved in the inequality; namely, $DF|_{D^2u(y)}$ and $D^2u(y)$). However, the following lemma shows that we can indeed obtain a diagonal expression if we are willing to pay the cost of adding a finite number of additional "diagonal components".

**Lemma 11.1.** *Given $0 < \lambda \leq \Lambda < \infty$, denote by $S^{n\times n}_{\lambda,\Lambda}$ the space of symmetric matrices with eigenvalues all lying in the interval $[\lambda, \Lambda]$. There exist $N \in N$ and $0 < \lambda_* \leq \Lambda_* < \infty$ (depending only on $n$, $\lambda$ and $\Lambda$) and unit vectors $\{\gamma_i\}_{i=1}^N$ such that any $A \in S^{n\times n}_{\lambda,\Lambda}$ can be written in the form*

$$\sum_{i=1}^{N} \beta^i \gamma_i \otimes \gamma_i$$

*such that $\beta^i \in [\lambda_*, \Lambda_*]$ for each $i = 1, \ldots, N$. The set of directions $\{\gamma_i\}_{i=1}^N$ can be arranged to include the basis directions $\{e_i\}_{i=1}^n$ as well as the directions $\{(e_i \pm e_j)/\sqrt{2}\}_{i<j=1}^n$.*

**Proof.** See [**2**, Lemma 17.13]. □

Combining Lemma 11.1 with the inequality (11.2), we obtain constants $N \in \mathbb{N}$, $\lambda_* > 0$ and $\gamma_* > 0$ (depending only on $n$, $\lambda$ and $\gamma$) and, for each $l = 1, \ldots, N$, vector fields $\gamma_l : \Omega \to S^n$ and functions $\varphi^l :\in \Omega \to [\lambda_*, \gamma_*\lambda_*]$ such that

$$\sum_{l=1}^{N} \varphi^l_y (w_l(y) - w_l(x)) \leq g(y) - g(x), \tag{11.3}$$

where $w_l \doteqdot D_{\gamma_l} D_{\gamma_l} u$.

Set, for each $r < 2R$ and $l = 1, \ldots, N$,

$$M_{r,l} \doteqdot \sup_{B_r} w_l \quad \text{and} \quad m_{r,l} \doteqdot \inf_{B_r} w_l .$$

By (11.1),

$$\begin{aligned}\left(R^{-n}\int_{B_R}\left[\sum_{l=1}^{N}(M_{2R,l}-w_l)\right]^{\sigma}\right)^{\frac{1}{\sigma}} &\leq N^{\frac{1}{\sigma}}\sum_{l=1}^{N}\left(R^{-n}\int_{B_R}(M_{2R,l}-w_l)^{\sigma}\right)^{\frac{1}{\sigma}}\\ &\leq C\left(\sum_{l=1}^{N}(M_{2R,l}-M_{R,l})+R^2\right)\\ &\leq C\left(\omega(2R)-\omega(R)+R^2\right)\,, \qquad (11.4)\end{aligned}$$

where, for $r \leq 2R$,

$$\omega(r) \doteqdot \sum_{l=1}^{N}\operatorname{osc}_{B_r} w_l = \sum_{l=1}^{N}(M_{r,l}-m_{r,l})$$

and $C$ depends only on the data $n$, $\lambda$, $\gamma$, $\operatorname{diam}(\Omega)$, $\|u\|_{C^2(B_{2R})}$, $\sup_{\mathcal{J}^1u(\Omega)}|\frac{\partial\psi}{\partial z}|$ and $\sup_{\mathcal{J}^1u(\Omega)}|D^2\psi|$.

On the other hand, by (11.3), for any $x \in B_{2R}$ and $y \in B_R$, and any fixed $l \in \{1,\dots,N\}$,

$$\varphi_y^l(w_l(y)-w_l(x)) \leq g(x)-g(y)+\sum_{k\neq l}\varphi_y^k(w_k(x)-w_k(y))$$

so that

$$w_l(y)-m_{2R,l} \leq \frac{1}{\lambda_*}\left(R\,\|Dg\|_{C^0(B_{2R})}+\gamma_*\lambda_*\sum_{k\neq l}(M_{2R,k}-w_k)\right)\,.$$

Estimating $\|Dg\|_{C^0(B_{2R})}$ by a constant depending only on $\|u\|_{C^2(B_{2R})}$ and $\sup_{\mathcal{J}^1u(B_{2R})}|D\psi|$, and recalling (11.4), we now obtain

$$\left(R^{-n}\int_{B_R}(w_l-m_{2R,l})^{\sigma}\right)^{\frac{1}{\sigma}} \leq C\left(\omega(2R)-\omega(R)\right)+R+R^2\right)\,, \qquad (11.5)$$

where $C$ depends only on $n$, $\lambda$, $\gamma$, $\operatorname{diam}(\Omega)$, $\|u\|_{C^2(B_{2R})}$, $\sup_{\mathcal{J}^1u(\Omega)}|\frac{\partial\psi}{\partial z}|$ and $\sup_{\mathcal{J}^1u(\Omega)}|D^2\psi|$.

Finally, combining (11.5) with (11.1), we obtain

$$\begin{aligned}
\omega(2R) &= \sum_{l=1}^{n}(M_{2R,l} - m_{2R,l}) \\
&= \left(\fint \left[\sum_{l=1}^{n}(M_{2R,l} - w_l + w_l - m_{2R,l})\right]^{\sigma}\right)^{\frac{1}{\sigma}} \\
&\le \sum_{l=1}^{n}\left[\left(\fint (M_{2R,l} - w_l)^{\sigma}\right)^{\frac{1}{\sigma}} + \left(\fint (w_l - m_{2R,l})^{\sigma}\right)^{\frac{1}{\sigma}}\right] \\
&\le C(\omega(2R) - \omega(R) + R + R^2)\,.
\end{aligned}$$

Writing $\delta \doteqdot 1 - C^{-1}$, we conclude that

$$\omega(R) \le \delta\omega(2R) + R + R^2\,.$$

Oscillation estimates for the functions $w_l$ now follow from Lemma 6.4. By polarization, the final claim of Lemma 11.1 yields an oscillation estimate for $D^2u$; namely,

$$\operatorname{osc}_{B_R} D^2u \le C\left(\frac{R}{R_0}\right)^{\alpha}\left(\operatorname{osc}_{B_{R_0}} D^2u + R_0 + R_0^2\right)\,,$$

where $C$ and $\alpha$ depend only on the data $n$, $\lambda$, $\gamma$, $\operatorname{diam}(\Omega)$, $\|u\|_{C^2(B_{2R})}$, $\sup_{\mathcal{J}^1u(\Omega)}|\frac{\partial\psi}{\partial z}|$ and $\sup_{\mathcal{J}^1u(\Omega)}|D^2\psi|$. Dividing by $R^\alpha$ and taking the supremum over $R < R_0$ then yields the desired Hölder estimate.

**Theorem 11.2.** *There are constants* $\alpha = \alpha(n, \lambda, \Lambda, K, M, D, d) \in (0, 1)$ *and* $C = C(n, \lambda, \Lambda, K, M, D, d) < \infty$ *with the following property. Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set such that* $\operatorname{diam}(\Omega) \le D$*, let* $\Gamma_{zp} \subset \mathbb{R} \times \mathbb{R}^n$ *and* $\Gamma_r \subset S^{n\times n}$ *be open sets, and let* $F \in C^2(\Gamma_r)$ *and* $\psi \in C^2(\Omega \times \Gamma_{zp})$ *be smooth functions. If* $F$ *is concave, then, given any open subset* $\Omega' \Subset \Omega$ *satisfying* $\operatorname{dist}(\Omega', \partial\Omega) > d$*, any solution* $u \in C^4(\Omega)$ *to*

$$-F(D^2u) = \psi(\cdot, u, Du) \quad \textit{in} \quad \Omega$$

*satisfying*

$$\begin{aligned}
|u|_{C^2(\Omega)} &\le K \\
\left|\psi_z|_{\mathcal{J}^1u}\right|,\ \left|D^2\psi|_{\mathcal{J}^1u}\right| &\le M \\
\textit{and}\quad \lambda\delta^{ij} \le \left.\frac{\partial F}{\partial r_{ij}}\right|_{D^2u} &\le \Lambda\delta^{ij}
\end{aligned}$$

*satisfies*

$$[D^2u]_{C^\alpha(\Omega')} \le C\,.$$

In order to apply the method of continuity to obtain existence of solutions, we need to reduce the regularity hypothesis in Theorem 11.2.

**Proposition 11.3.** *Suppose that $u \in C^{2,\beta}(\Omega)$, $\beta \in (0,1)$, satisfies*

$$F(\cdot, u, Du, D^2u) = 0 \ \text{ in } \ \Omega\,,$$

*where $F : \Gamma \subset \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^{n\times n} \to \mathbb{R}$ is strictly elliptic. If $F \in C^{k,\alpha}(\Gamma)$ for some $k \geq 1$ and $0 < \alpha < 1$, then $u \in C^{k+2,\alpha}(\Omega)$.*

**Proof.** Given $e \in S^n$ and $h \in \mathbb{R}$, denote by $\delta_e^h u$ the difference quotioent

$$\delta_e^h u(x) \doteqdot \frac{1}{h}\big(u(x+he) - u(x)\big)\,.$$

Given any $\Omega' \Subset \Omega$ there exists some $h_0 > 0$ such that $\delta_e^h u$ is defined in $\Omega'$ for all $h < h_0$, and satisfies in $\Omega'$ the linear elliptic equation

$$-(a^{ij}D_iD_j + b^jD_i + c)(\delta_e^h u) = f\,,$$

where

$$a^{ij}(x) \doteqdot \int_0^1 F_{r_{ij}}\big(s\mathcal{J}^2u(x+he) + (1-s)\mathcal{J}^2u(x)\big)\,ds\,,$$

$$b^i(x) \doteqdot \int_0^1 F_{p_i}\big(s\mathcal{J}^2u(x+he) + (1-s)\mathcal{J}^2u(x)\big)\,ds\,,$$

$$c(x) \doteqdot \int_0^1 F_z\big(s\mathcal{J}^2u(x+he) + (1-s)\mathcal{J}^2u(x)\big)\,ds\,,$$

and

$$f(x) \doteqdot \int_0^1 e^iF_{x^i}\big(s\mathcal{J}^2u(x+he) + (1-s)\mathcal{J}^2u(x)\big)\,ds\,.$$

The claim now follows from Schauder's theorem (Theorem 3.1) and a bootstrapping argument. $\square$

We also need an estimate up to the boundary.

**Theorem 11.4.** *There are constants $\alpha = \alpha(n, \lambda, \Lambda, K, L, M, \Omega) \in (0,1)$ and $C = C(n, \lambda, \Lambda, K, L, M, \Omega) < \infty$ with the following property. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class $C^3$, let $\Gamma_{zp} \subset \mathbb{R} \times \mathbb{R}^n$ and $\Gamma_r \subset S^{n\times n}$ be open sets, and let $F \in C^2(\Gamma_r)$ and $\psi \in C^2\left(\overline{\Omega} \times \Gamma_{zp}\right)$ be smooth functions. If $F$ is concave, then, given any $\varphi \in C^3(\overline{\Omega})$ with $\|\varphi\|_{C^3(\overline{\Omega})} \leq L$, any solution $u \in C^3(\overline{\Omega}) \cap C^4(\Omega)$ to*

$$\begin{cases} -F(D^2u) = \psi(\cdot, u, Du) \ \text{ in } \ \Omega \\ \qquad\quad u = \varphi \ \text{ on } \ \partial\Omega \end{cases}$$

*satisfying*

$$|u|_{C^2(\Omega)} \le K$$
$$\left|\psi_z|_{\mathcal{J}^1 u}\right|, \ \left|D^2\psi|_{\mathcal{J}^1 u}\right| \le M$$
$$\textit{and} \quad \lambda\delta^{ij} \le \left.\frac{\partial F}{\partial r_{ij}}\right|_{D^2 u} \le \Lambda\delta^{ij}$$

*satisfies*

$$[D^2 u]_{C^\alpha(\Omega)} \le C\,.$$

**Proof.** See [**2**, Theorem 17.26']. □

The regularity hypothesis can be relaxed by a difference quotient argument using the boundary Schauder theory, similarly as in Proposition 11.3.

Note that, in particular, Theorem 11.2 applies immediately to quasilinear equations. Further examples include equations of Bellman-type, equations of Monge–Ampère type, and, more generally, equations of $k$-Hessian type, which include the equation of prescribed Gauss curvature and, more generally, the equations of prescribed $k$-th mean curvature. Furthermore, by making the transformation

$$F_*(x, z, p, r) \doteqdot -F(z, z, p, -r)\,,$$

the theorem also applies to *convex* operators. To mention just one interesting family, these include equations of the form

$$-F(D^2 u) = \psi(\cdot, u, Du)\,,$$

where $F : S_+^{n\times n} \to \mathbb{R}$ is given by

$$F(r) \doteqdot \|r\|_p \doteqdot \operatorname{tr}(r^p)^{\frac{1}{p}}$$

for any $p \ge 1$.

## 12. Equations of Monge–Ampère/Gauss curvature type

We will apply the Krylov–Safanov theory to obtain solutions to equations of Monge–Ampère type, under suitable conditions. So consider the Dirichlet problem

$$\begin{cases} -\det(D^2u) = f(\cdot, u, Du) \text{ in } \Omega \\ u = \phi \text{ on } \partial\Omega \end{cases} \tag{12.1}$$

for suitable domains $\Omega \subset \mathbb{R}^n$ and data $f$ and $\phi$.

This class of equations includes the Monge–Ampère equation

$$\det(D^2u) = 1\,.$$

It also includes equations of prescribed Gauss curvature, as well as translator equations for flows by powers of the Gauss curvature. Indeed, let $M^n \subset \mathbb{R}^{n+1}$ be a smooth hypersurface. Suppose that $M^n \cap B_1 = \operatorname{graph}(u) \cap B_1$ for some smooth function $u : \mathbb{R}^n \to \mathbb{R}$. Recall that, at any point $X = (x, u(x)) \in M^n \cap B_1$, we have the following formulae for the downward pointing unit normal $\nu$, the metric $g$ and the second fundamental form $h$ of $M^n$:

$$\nu = \frac{(Du, -1)}{\sqrt{1+|Du|^2}} \quad g_{ij} = \delta_{ij} + u_i u_j \quad \text{and} \quad h_{ij} = \frac{u_{ij}}{\sqrt{1+|Du|^2}}\,.$$

If we denote by $g_u$ and $h_u$ the corresponding component matrices, then the Gauss curvature of $M^n$ at the point $(x, u(x))$ is

$$K = \det(g_u^{-1} \cdot h_u) = \frac{\det h_u}{\det g_u} = \frac{\det(D^2u)}{(1+|Du|^2)^{\frac{n+2}{2}}}\,.$$

Thus, the problem of constructing (locally) hypersurfaces with prescribed Gauss curvature $f : B_1 \to \mathbb{R}$ gives rise to the equation

$$\frac{\det(D^2u)}{(1+|Du|^2)^{\frac{n+2}{2}}} = f(\cdot, u)\,.$$

Surfaces satisfying $K = \langle \nu\,,\, e_{n+1}\rangle$ translate with constant velocity $e_{n+1}$ under the Gauss curvature flow. This gives rise to the (graphical) Gauss curvature flow translator equation

$$\frac{\det(D^2u)}{(1+|Du|^2)^{\frac{n+2}{2}}} = \frac{1}{\sqrt{1+|Du|^2}}\,.$$

A similar equation is obtained for graphical translators of flows by powers of the Gauss curvature.

We first note that ellipticity of (12.1) is not guaranteed in general, but does hold in the class of locally uniformly convex functions since, for positive definite symmetric matrices,

$$\frac{\partial}{\partial r_{ij}} \det r = \det r \, r^{ij} > 0 \,,$$

where $r^{ij}$ are the components of $r^{-1}$. We shall seek solutions within the class of locally uniformly convex functions, and hence consider only negative right hand sides $f$. We say that a solution (or subsolution) $u \in C^2(\Omega)$ to

$$-\det(D^2 u) = f(\cdot, u, Du)$$

is *admissible* if $D^2u(x) > 0$ for each $x \in \Omega$.

We will establish *a priori* estimates for $|u|$, $|Du|$ and $|D^2u|$ assuming the existence of suitable barriers. Ellipticity estimates then follow from the structure of the equation, so a Hölder estimate for $|D^2u|$ follows from the Krylov–Safanov theory of §11. We will then be able to obtain solutions by the method of continuity.

We note that, while the presence of barriers is a subtle issue, some such condition is necessary. Indeed, consider the prescribed Gauss curvature equation

$$K_{\operatorname{graph} u} = f(\cdot, u) \;\text{ in }\; \Omega \subset \mathbb{R}^n \,,$$

where $f$ is a positive function on $\mathbb{R}^{n+1}$. We claim that no solution exists if the inradius of $\Omega$ exceeds $(\inf_{\Omega\times\mathbb{R}} f)^{-\frac{1}{n}}$. Indeed, if the inradius of $\Omega$ exceeds $(\inf_{\Omega\times\mathbb{R}} f)^{-\frac{1}{n}}$, then we can fit a sphere of radius $R > (\inf_{\Omega\times\mathbb{R}} f)^{-\frac{1}{n}}$ inside the cylinder $\Omega \times \mathbb{R}$. Moving such a sphere downwards from infinity, we eventually make contact with $\operatorname{graph} u$, at an interior point $p$. But then, since the Gauss curvature of a sphere of radius $R$ is $R^{-n}$,

$$K_{\operatorname{graph} u}(p) \le R^{-n} < \inf_{\Omega\times\mathbb{R}} f \le K_{\operatorname{graph} u}(p) \,,$$

which is absurd.

**12.1.** $C^0$**-estimate.** Assuming the existence of a lower barrier, a $C^0$ estimate will follow from the comparison principle.

**Proposition 12.1** (Comparison Principle)**.** *Let $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be locally uniformly convex in $\Omega$ with at least one of them uniformly convex. Suppose that $f \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ satisfies $\frac{\partial f}{\partial z} \le 0$. If*

$$\begin{cases} -\det(D^2u) - f(\cdot, u, Du) \le -\det(D^2v) - f(\cdot, v, Dv) & \text{in } \Omega \\ u \le v & \text{on } \partial\Omega \,, \end{cases}$$

*then $u \le v$ in $\Omega$.*

**Proof.** Set $u_\vartheta \doteqdot \vartheta u + (1-\vartheta)v$. Observe that

$$\begin{aligned}0 \leq\ & \det(D^2u) - \det(D^2v) + f(\cdot, u, Du) - f(\cdot, v, Dv)\\ &= \int_0^1 \frac{d}{d\vartheta}\left(\det(D^2u_\vartheta) + f(\cdot, u_\vartheta, Du_\vartheta)\right) d\vartheta\\ &= a^{ij}w_{ij} + b^k w_k + cw\,,\end{aligned}$$

where

$$w = u - v$$

and, denoting by $u^{ij}$ the components of the inverse matrix of $D^2u$,

$$\begin{aligned}a^{ij}(x) &\doteqdot \int_0^1 \det(D^2u_\vartheta(x))u_\vartheta^{ij}(x)d\vartheta\,,\\ b^k(x) &\doteqdot \int_0^1 \frac{\partial f}{\partial p_k}(x, u_\vartheta(x), Du_\vartheta(x))d\vartheta \quad \text{and}\\ c(x) &\doteqdot \int_0^1 \frac{\partial f}{\partial z}(x, u_\vartheta(x), Du_\vartheta(x))d\vartheta\,.\end{aligned}$$

Since at least one of $u$ or $v$ is locally uniformly convex, $a^{ij}$ is uniformly elliptic. The claim now follows from the maximum principle since $c$ is non-positive due to the monotonicity of $f$. □

Since locally uniformly convex functions attain their maxima at the boundary, a $C^0$ estimate follows, assuming the presence of a lower barrier.

**Proposition 12.2** ($C^0$ estimate)**.** *Suppose that $f \in C^1(\Omega\times\mathbb{R}\times\mathbb{R}^n)$ satisfies $\frac{\partial f}{\partial z} \leq 0$. If there exists an admissible subsolution $\underline{u} \in C^2(\Omega)\cap C^0(\overline{\Omega})$ to* (12.1) *with $\underline{u} \leq \phi$ on $\partial\Omega$, then any admissible solution $u \in C^2(\Omega)\cap C^0(\overline{\Omega})$ to* (12.1) *satisfies*

$$\max_\Omega |u| \leq C(\max_\Omega |\underline{u}|, \max_{\partial\Omega} \phi)\,.$$

**Proof.** Since $D^2u > 0$, $u$ cannot attain an interior maximum, and hence $u \leq \sup_{\partial\Omega} u$. On the other hand, by the comparison principle, $u \geq \underline{u}$. □

**12.2. $C^1$-estimate.** If $v \doteqdot \frac{1}{2}|Du|^2$ attains a local interior maximum at the point $x$, then

$$0 = D_w v = 2\left\langle D_w Du\,,\, Du\right\rangle$$

at $x$ for all $w \in \mathbb{R}^n$. So $Du$ is a null eigenvector of $D^2u$ at $x$. But this contradicts local strict convexity of $u$. We conclude that $|Du|$ attains its maximum on the boundary.

We may estimate the gradient of $u$ at the boundary under the assumption that (12.1) admits a subsolution $\underline{u}$ which attains the boundary data $\phi$.

12.2.1. *Tangential derivatives.* Since $(u-\underline{u})\circ\gamma \equiv 0$ for any curve $\gamma : (-s_0, s_0) \to \partial\Omega$, we obtain $D_v(u-\underline{u})(x) = 0$ for any vector $v$ tangent to $\partial\Omega$ at $x \in \partial\Omega$.

12.2.2. *Normal derivatives.* Denote by $\nu$ the outward pointing unit normal field to $\partial\Omega$. Since $\underline{u} \le u$ in $\Omega$ and $\underline{u} \equiv u$ on $\partial\Omega$, we immediately obtain

$$D_\nu(u-\underline{u}) \le 0 \Longrightarrow D_\nu u \le D_\nu \underline{u}\,.$$

To bound $D_\nu u$ from below, fix $x \in \partial\Omega$ and consider the line $\ell(s) \doteqdot x - s\nu(x)$ through $x$ in the direction $-\nu(x)$. Since $\Omega$ is bounded, the line must reach a point $y = x - s_0\nu(x) \in \partial\Omega$ at some time $s_0 > 0$. We take $s_0$ to be the first such time. Since $u$ is (by assumption) locally convex in $\Omega$, we obtain, from Taylor's theorem,

$$u(y) \ge u(x) + \langle Du(x)\,,\, y - x\rangle\ .$$

That is,

$$\begin{aligned}-D_\nu u(x) \le \frac{u(y)-u(x)}{|y-x|} = \frac{\underline{u}(y)-\underline{u}(x)}{|y-x|} &= \frac{d_{\partial\Omega}(x,y)}{|y-x|}\cdot\frac{\underline{u}(y)-\underline{u}(x)}{d_{\partial\Omega}(x,y)}\\ &\le C(\|\underline{u}\|_{C^1(\partial\Omega)}, \partial\Omega)\,,\end{aligned}$$

where $d_{\partial\Omega}$ denotes the intrinsic distance on $\partial\Omega$.

Putting this together, we obtain the following.

**Proposition 12.3** ($C^1$-estimate)**.** *Suppose that $f \in C^1(\Omega\times\mathbb{R}\times\mathbb{R}^n)$ satisfies $\frac{\partial f}{\partial z} \le 0$, $\partial\Omega$ and $\phi$ are of class $C^1$, and that there exists an admissible subsolution $\underline{u} \in C^2(\Omega)\cap C^1(\overline{\Omega})$ to* (12.1) *with $\underline{u}_{\partial\Omega} \equiv \phi|_{\partial\Omega}$. Any smooth admissible solution $u \in C^2(\Omega)\cap C^0(\overline{\Omega})$ to* (12.1) *satisfies*

$$\sup_\Omega |Du| \le C(|\underline{u}|_{C^1(\partial\Omega)}, \partial\Omega)\,.$$

**12.3. $C^2$-estimate.** If we write $F(D^2u) \doteqdot \log\det(D^2u)$ and $\hat{f} \doteqdot -\log(-f)$, then (12.1) becomes

$$\begin{cases} -F(D^2u) = \hat{f}(\cdot, u, Du) \;\text{ in }\; \Omega\subset\mathbb{R}^n \\ \qquad\quad u = \phi \;\text{ on }\; \partial\Omega\,. \end{cases} \tag{12.2}$$

Note that $F : S^{n\times n}_+ \to \mathbb{R}$ is elliptic, since

$$\frac{\partial F}{\partial r_{kl}} = r^{kl}\,.$$

Moreover, $F$ is a concave function since both log and det are concave. Indeed,

$$\frac{\partial^2 F}{\partial r_{pq}\partial r_{rs}} = -r^{pr}r^{qs}\,.$$

Thus, differentiating (12.2) in some direction $w \in S^n$ (without loss of generality, and abusing notation, we may take $w = e_w$) yields

$$
\begin{aligned}
-u^{ij}u_{wij} &= D_w\Big(\hat{f}(\cdot, u, Du)\Big) \\
&= \frac{\partial \hat{f}}{\partial p_k}u_{wk} + \frac{\partial \hat{f}}{\partial z}u_w + \frac{\partial \hat{f}}{\partial x^w}
\end{aligned}
\tag{12.3}
$$

and

$$
\begin{aligned}
-u^{ij}u_{wwij} = &\; - u^{pr}u^{qs}u_{wpq}u_{wrs} + D_wD_w\Big(\hat{f}(\cdot, u, Du)\Big) \\
= &\; - u^{pr}u^{qs}u_{wpq}u_{wrs} + \frac{\partial^2 \hat{f}}{\partial p_k \partial p_l}u_{kw}u_{lw} + 2\frac{\partial^2 \hat{f}}{\partial p_k \partial z}u_{kw}u_w \\
&+ \frac{\partial^2 \hat{f}}{\partial p_k \partial x^w}u_{kw} + \frac{\partial \hat{f}}{\partial p_k}u_{kww} + \frac{\partial^2 \hat{f}}{\partial z^2}(u_w)^2 + 2\frac{\partial^2 \hat{f}}{\partial z \partial x^w}u_w \\
&+ \frac{\partial \hat{f}}{\partial z}u_{ww} + \frac{\partial^2 \hat{f}}{\partial x^w \partial x^w}\,.
\end{aligned}
$$

For the Monge–Ampère equation (where $f \equiv -1$), we have $\hat{f} \equiv 0$, so this becomes

$$
-u^{ij}u_{wwij} \leq 0
$$

and the maximum principle implies that $u_{ww}$ can have no interior maximum. Since $w \in S^n$ was arbitrary, we conclude that $\max_{(x,w)\in\overline{\Omega}\times S^n} D^2u|_x(w,w)$ occurs at a pair $(x,w)$ with $x$ in the boundary of $\Omega$. This reduces the $C^2$ estimate to the boundary case, since we automatically have the lower bound $D^2u(w,w) \geq 0$ by local convexity of $u$.

For non-trivial $f$, we need to work a bit harder. Note that, if a vector $w$ maximizes a quadratic form $r(w,w)$ with respect to all unit vectors, then it is an eigenvector with eigenvalue $r(w,w)$. Thus, if $w$ maximizes $D^2u(w,w)$, then $u^{ij} \geq \frac{1}{u_{ww}}\delta^{ij}$. After rotating so that $w = e_w$, this implies that

$$
u^{pr}u^{qs}u_{wpq}u_{wrs} \geq u^{ij}\frac{u_{iww}u_{jww}}{u_{ww}}\,.
$$

Assuming further that $u_{ww} \geq 1$, we can estimate

$$
-\frac{u^{ij}}{u_{ww}}u_{wwij} \leq \; - u^{kl}\frac{u_{kww}u_{lww}}{u_{ww}^2} + C\,(u_{ww}+1) + \frac{\partial \hat{f}}{\partial p_k}\frac{u_{kww}}{u_{ww}}\,, \tag{12.4}
$$

where $C$ depends on bounds for $f$ and $Du$.

Define a function $v : \Omega \times S^{n-1} \to \mathbb{R}$ by

$$
v(x,w) \doteqdot \log(D_wD_wu) + \frac{\beta}{2}|Du|^2\,.
$$

Suppose that $v$ takes its maximum at a pair $(x, w)$ with $x$ in the interior of $\Omega$. After a coordinate rotation, we can arrange that $w = e_1$. Differentiating $v$ with respect to $x$ yields

$$0 = v_i = \frac{u_{11i}}{u_{11}} + \beta u_{ik} u_k \tag{12.5}$$

at $(x, e_1)$ and

$$v_{ij} = \frac{u_{11ij}}{u_{11}} - \frac{u_{i11} u_{j11}}{u_{11}^2} + \beta \left( u_{ijk} u_k + u_{ik} u_{jk} \right)$$

at $(x, e_1)$. Thus, applying (12.4) and the maximality of $(x, e_1)$, and assuming, without loss of generality, that $u_{11} \geq 1$, we find

$$0 \leq -u^{ij} v_{ij} \leq C \left( u_{11} + 1 \right) + \frac{\partial \hat{f}}{\partial p_k} \frac{u_{k11}}{u_{11}} - \beta u^{ij} \Big( u_{kij} u_k + u_{ik} u_{jk} \Big)$$

at $(x, e_1)$, where $C = C(\|u\|_{C^1(\Omega)}, f)$. Observe that

$$u^{ij} u_{ik} u_{jk} = \Delta u \,.$$

Thus, Replacing $u^{ij} u_{kij}$ using (12.3) and $u_{k11}$ using (12.5), we obtain

$$\begin{aligned} 0 \leq -u^{ij} v_{ij} &\leq C \left( u_{11} + 1 \right) - \beta \frac{\partial \hat{f}}{\partial p_k} u_{kl} u_l \\ &\qquad + \beta \left( \frac{\partial \hat{f}}{\partial p_l} u_{lk} + \frac{\partial \hat{f}}{\partial z} u_k + \frac{\partial \hat{f}}{\partial x^k} \right) u_k - \beta \Delta u \\ &= C \left( u_{11} + 1 \right) + \beta \left( \frac{\partial \hat{f}}{\partial z} u_k + \frac{\partial \hat{f}}{\partial x^k} \right) u_k - \beta \Delta u \\ &\leq (C - \beta) u_{11} + C(1 + \beta) \,, \end{aligned}$$

for a constant $C$ depending only on $\widehat{f}$ and $\|Du\|_{C^0}$. If we choose $\beta \doteqdot 2C$, we obtain

$$u_{11} \leq 1 + 2C \,.$$

We have shown that $0 \leq D_w D_w u \leq 1 + 2C = C(\|Du\|_{C^0}, f)$ for all $(x, v) \in \Omega \times S^n$ if the function $v$ takes its maximum in the interior of $\Omega \times S^n$. If, on the other hand, $v$ takes its maximum at a point $(x, w)$ with $x$ on the boundary of $\Omega$, then we obtain

$$\max_{\|w\|=1} D_w D_w u \leq \max_{\partial\Omega} \max_{\|w\|=1} D_w D_w u + C(\|Du\|_{C^0}) \,.$$

By the polarization identity, this reduces the $C^2$ estimate to the boundary case.

12.3.1. *Tangential derivatives.* Suppose that $\Omega$ is of class $C^2$. Then $\Sigma \doteqdot \partial\Omega$ is a compact, embedded $C^2$-hypersurface of $\mathbb{R}^n$. Fix a point $x \in \Sigma$ and a vector $v$ tangent to $\Sigma$ at $x$ and let $\gamma : (-s_0, s_0) \to \Sigma$ be the geodesic in $\Sigma$ with $(\gamma(0), \gamma'(0)) = (x, v)$. That is,

$$\begin{cases} \gamma'' + h(\gamma', \gamma')\nu \circ \gamma = 0 & s \in (s_0, s_0)\,, \\ (\gamma, \gamma') = (x, v) & s = 0\,, \end{cases}$$

where $h : T\Sigma \otimes T\Sigma \to \mathbb{R}$ is the second fundamental form of $\Sigma$,

$$h(v, w) \doteqdot -\langle D_v w\,,\, \nu\rangle\ .$$

Observe that

$$0 = \frac{d^2}{ds^2}\Big|_{s=0} (u - \underline{u}) \circ \gamma = D_v D_v u - D_v D_v \underline{u} - h(v, v) D_\nu (u - \underline{u})$$

and hence

$$|D_v D_v u| \le |D_v D_v \underline{u}| + |h(v, v)|\,(|D_\nu u| + |D_\nu \underline{u}|) \le C(|\underline{u}|_{C^2(\partial\Omega)}, \partial\Omega)\|v\|^2$$

at $x$. Since $D^2 u$ is symmetric, the polarization identity gives similar bounds for mixed tangential derivatives.

12.3.2. *Mixed derivatives.* Next, we want to bound the mixed derivatives $D^2 u|_{x_0}(\tau, \nu)$ at each point $x_0 \in \partial\Omega$, where $\tau$ is a non-zero vector tangent to $\partial\Omega$ at $x_0$ and $\nu$ is its outward pointing normal at $x_0$. To do this, we will extend $\tau$ to a vector field $T$ defined on some neighbourhood of $\Sigma$ in $\overline{\Omega}$ and use a barrier argument to bound $Tu$ by (an appropriate function of) the distance to $\partial\Omega$.

So assume that $\Sigma \doteqdot \partial\Omega$ is compact and of class $C^2$. Then, by the tubular neighbourhood theorem, there exists $\delta_0 > 0$ such that, for each $\delta \in (-\delta_0, \delta_0)$, the normal map $N_\delta : \Sigma \to \mathbb{R}^n$, defined by

$$N_\delta(x) \doteqdot x - \delta\nu(x)\,,$$

is a diffeomorphism. Set $\Sigma_\delta \doteqdot N_\delta(\Sigma)$. By choosing $\delta_0$ possibly smaller, we can arrange that $\Sigma_\delta \subset \Omega$ for $\delta \in (0, \delta_0)$. Fix a point $x_0 \in \Sigma$ and let $T$ be a $C^2$ vector field on $\Sigma$ which is equal to $\tau$ at $x_0$. Then we can extend $T$ to a $C^2$ vector field on the tubular neighbourhood $\Omega_{\delta_0} \doteqdot \cup_{0<\delta<\delta_0} \Sigma_\delta$ via

$$T(N_\delta(x)) \doteqdot \frac{d}{dt}\Big|_{t=0} N_\delta(\phi(x, t))\,,$$

where $\phi$ is the flow of $T$,

$$T(\phi(x, t)) = \frac{d}{dt}\phi(x, t)\,.$$

Note that $T$ depends only on $\Sigma = \partial\Omega$.

Define the linear operator

$$Lv \doteqdot u^{ij}v_{ij} + \frac{\partial \hat{f}}{\partial p_k}v_k\,.$$

Note that, differentiating (12.2),

$$Lu_i = -\frac{\partial \hat{f}}{\partial z}u_i - \frac{\partial \hat{f}}{\partial x^i}\,, \tag{12.6}$$

which we can bound using the gradient estimate.

Set

$$v \doteqdot D_T(u-\underline{u}) = T^k(u-\underline{u})_k\,.$$

Note that, by construction,

$$v \equiv 0 \quad \text{on} \quad \partial\Omega\,.$$

We claim that

$$|Lv| \leq C(1+u^{ij}\delta_{ij})\,, \tag{12.7}$$

where the constant $C$ depends on $\underline{u}$, $f$ and $\partial\Omega$. Indeed,

$$\begin{aligned}
Lv &= u^{ij}(T^k(u-\underline{u})_k)_{ij} + \frac{\partial \hat{f}}{\partial p_i}(T^k(u-\underline{u})_k)_i\\
&= u^{ij}\left(T^k(u-\underline{u})_{kij} + T^k_{ij}(u-\underline{u})_k + T^k_i(u-\underline{u})_{kj} + T^k_j(u-\underline{u})_{ki}\right)\\
&\quad + \frac{\partial \hat{f}}{\partial p_i}\left(T^k(u-\underline{u})_{ki} + T^k_i(u-\underline{u})_k\right)\\
&= T^kLu_k + u^{ij}\left(-T^k\underline{u}_{kij} + T^k_{ij}(u-\underline{u})_k + T^k_j(u-\underline{u})_{ki} + T^k_i(u-\underline{u})_{kj}\right)\\
&\quad + \frac{\partial \hat{f}}{\partial p_i}\left(-T^k\underline{u}_{ki} + T^k_i(u-\underline{u})_k\right)\,.
\end{aligned}$$

The first term and the terms of the second line can be bounded, via the gradient estimate (recall (12.6)), by a constant depending on $\partial\Omega$, $\underline{u}$ and $f$. We cannot bound the term $u^{ij}$, but we can bound the matrix on which it is contracted by a multiple (depending on $\partial\Omega$ and $\underline{u}$) of the identity matrix. This yields the claim.

Now consider the function

$$\vartheta \doteqdot (u-\underline{u}) + \alpha d - \mu d^2$$

for some yet to be chosen constants $0 < \alpha \ll 1$ and $\mu \gg 1$, where $d$ is the distance function to $\partial\Omega$. After possibly choosing $\delta_0$ smaller, we can assume that $d$ is smooth and bounded in $C^2$ on $\Omega_{\delta_0}$ (recall Proposition 7.5). So we may compute

$$\begin{aligned} L\vartheta = {} & u^{ij}\left((u-\underline{u})_{ij} + \alpha d_{ij} - 2\mu(dd_{ij} + d_i d_j)\right) \\ & + \frac{\partial \hat{f}}{\partial p_k}\left((u-\underline{u})_k + \alpha d_k - \mu d d_k\right) . \end{aligned}$$

Since $\underline{u}$ is locally uniformly convex on $\overline{\Omega}$ and $\Omega$ is bounded, there is a constant $\varepsilon > 0$ such that $\underline{u}_{ij} \geq 4\varepsilon\delta_{ij}$. We can use this term to control $\alpha d_{ij}$ since, choosing $\alpha$ small enough, we can assume that $\alpha d_{ij} \leq \varepsilon\delta_{ij}$. Estimating also $d \leq \delta_0$, we obtain

$$L\vartheta \leq C(1+\mu\delta_0) + u^{ij}\left((C\mu\delta_0 - 3\varepsilon)\delta_{ij} - 2\mu d_i d_j\right) ,$$

where the constant $C$ depends on $n$, $\partial\Omega$, $\underline{u}$ and $f$. We claim that $\mu$ can be chosen large enough that

$$2C - u^{ij}\left(\varepsilon\delta_{ij} + 2\mu d_i d_j\right) \leq 0 .$$

This would imply

$$L\vartheta \leq C(\mu\delta_0 - 1) + (C\mu\delta_0 - 2\varepsilon)u^{ij}\delta_{ij} .$$

We can assume without loss of generality that $2\varepsilon < C$. Then, choosing $\delta_0$ small enough that $C\mu\delta_0 < \varepsilon$, we would obtain

$$L\vartheta \leq \; -\varepsilon\left(1 + u^{ij}\delta_{ij}\right) . \tag{12.8}$$

To prove the claim, write $A \doteqdot [D^2u]^{-1}$ and $B \doteqdot \varepsilon I + 2\mu Dd \otimes Dd$. Note that, by the arithmetic-geometric mean inequality,

$$\det A \det B \leq \left(\frac{\operatorname{tr}(AB)}{n}\right)^n .$$

On the other hand, the $C^1$ estimate yields

$$\det A = \frac{1}{\det(D^2u)} = -\frac{1}{f} \geq \frac{1}{C}$$

and, since $|Dd| = 1$,

$$\det B = \varepsilon^n\left(1 + 2\frac{\mu}{\varepsilon}\right) .$$

We conclude

$$\operatorname{tr}(AB) \geq n(\det A \det B)^{\frac{1}{n}} \geq nC^{\frac{1}{n}}\varepsilon\left(1 + 2\frac{\mu}{\varepsilon}\right)^{\frac{1}{n}} .$$

The claim follows.

We can now compare $T(u-\underline{u})$ to $\vartheta$: First, observe that (possibly choosing $\delta_0$ even smaller, so that $\alpha \geq 2\mu\delta_0$) we can arrange that

$$\vartheta \geq \frac{\alpha\delta_0}{2} \quad \text{on} \quad \partial\Omega_{\delta_0} \setminus \partial\Omega .$$

Since $\vartheta \geq 0$ on $\Omega_{\delta_0}$ and $T(u-\underline{u}) = 0$ on $\partial\Omega$, we can find a constant $R > 0$ (depending on $\partial\Omega$ and $\underline{u}$) such that

$$\pm T(u-\underline{u}) \leq R\vartheta \quad \text{on} \quad \partial\Omega_{\delta_0}\,.$$

On the other hand, by (12.7) and (12.8), choosing $R$ possibly larger, we can arrange that

$$-L(\pm v - R\theta) \leq 0\,.$$

It follows that

$$\pm T(u-\underline{u}) \leq R(u-\underline{u}+\alpha d-\mu d^2) \quad \text{in} \quad \Omega_{\delta_0}\,.$$

Dividing by $d$ and taking the limit $x \to x_0$ along a curve tangent to $\nu(x_0)$ at $x_0$ yields the desired estimate for the mixed derivatives.

12.3.3. *Normal derivatives.* Since $f$ is a negative function, the global gradient estimate allows us to bound $f(x,u,Du)$ from below and above by negative constants. We thereby obtain

$$C^{-1} \leq \det D^2u \leq C \quad \text{in} \quad \overline{\Omega}\,.$$

At a boundary point $x_0 \in \partial\Omega$, we can decompose $D^2u$ into tangential and normal components. Up to a change of basis, we can arrange that $\nu(x_0) = e_1$ and $D^2u|_{T_x\partial\Omega\times T_x\partial\Omega}$ is diagonal, so that

$$\det D^2u = \prod_{i=1}^{n} u_{ii} - \sum_{j=2}^{n} u_{1j}^2 \prod_{\substack{i\neq j\\ i\neq 1}} u_{ii}\,. \tag{12.9}$$

From this, we see that a *lower* bound for pure tangential derivatives combined with the upper bounds for the mixed and pure tangential derivatives will provide an upper bound for the pure normal derivatives. On the other hand, if $D^2u|_x(\tau,\tau)$ becomes too small for some $\tau \in S_x\partial\Omega$, then we shall be able to construct suitable barriers in order to bound $D^2(\nu,\nu)$.

Consider the function $v : S\Sigma \to \mathbb{R}$ defined by

$$v(x,\tau) \doteqdot D^2u|_{x_0}(\tau,\tau)\,,$$

where $S\Sigma = \sqcup_{x\in\Sigma} S_x\Sigma$ denotes the unit tangent bundle of $\Sigma$. Since $S\Sigma$ is compact, the infimum of $v$ is attained by some pair $(x_0,\tau_0) \in S\Sigma$. Let $T$ be a tangent vector field on $\Sigma$ such that $T_{x_0} = \tau_0$. Extend $T$ into the tubular neighbourhood as above but keeping also its norm constant. By choosing $\delta_0$ smaller, this can be done in $\Omega_{\delta_0} \doteqdot \Omega \cap B_{\delta_0}(x_0)$. Then

$$D^2u(T,T) \geq D^2u|_{x_0}(\tau_0,\tau_0) \quad \text{on} \quad \partial\Omega_{\delta_0} \cap \partial\Omega$$

with equality at $x_0$.

Recall that

$$D^2(u-\underline{u})(\tau,\tau) = h(\tau,\tau)D_\nu(u-\underline{u})$$

for any tangent vector $\tau \in T\Sigma$. We may suppose that

$$h(\tau_0,\tau_0)D_\nu(u-\underline{u})|_{x_0} = D^2(u-\underline{u})|_{x_0}(\tau_0,\tau_0) \le -\frac{1}{2}D^2\underline{u}|_{x_0}(\tau_0,\tau_0)\,,$$

since, otherwise, we have a global lower bound for pure tangential derivatives, and the desired upper bound for the normal derivatives would follow from (12.9).

With this assumption, $h(\tau_0,\tau_0) \neq 0$. Choosing $\delta_0$ smaller, we can arrange that $h(T,T) > 0$ on $\Omega_{\delta_0}$, so that the function $\Psi : \Omega_{\delta_0} \to \mathbb{R}$ given by

$$\Psi(x) \doteqdot \frac{1}{h(T,T)}\left(D^2\underline{u}|_x(T,T) - D^2u|_{x_0}(\tau_0,\tau_0)\right)$$

is well-defined. Note that

$$-\Psi \le D_\nu(u-\underline{u}) \quad \text{on} \quad \partial\Omega_{\delta_0}\cap\partial\Omega$$

with equality at $x_0$. We can define a vector field $N$ on $\Omega_{\delta_0}$ by parallel translating $\nu$; that is, $N(x-\delta\nu(x)) \doteqdot \nu(x)$ for any $x \in \partial\Omega\cap\Omega_{\delta_0}$. Then for $\delta_0$ sufficiently small, there is a constant $B$ such that

$$\Psi + N(u-\underline{u}) + B\|x-x_0\|^2 \ge 0 \quad \text{on} \quad \partial\Omega_{\delta_0}$$

We claim (as an exercise) that $A$, $\alpha$, $\mu$ and $\delta$ can be chosen so that

$$\begin{cases} -L\left(\Psi + N(u-\underline{u}) + B\|x-x_0\|^2 + A\vartheta\right) \ge 0 & \text{in} \quad \Omega_{\delta_0} \\ \Psi + N(u-\underline{u}) + B\|x-x_0\|^2 + A\vartheta \ge 0 & \text{on} \quad \partial\Omega_{\delta_0} \end{cases} \tag{12.10}$$

By the maximum principle, we then obtain

$$\Psi + N(u-\underline{u}) + B\|x-x_0\|^2 + A\vartheta \ge 0 \quad \text{in} \quad \Omega_{\delta_0}$$

with equality at $x_0$. This yields

$$D_\nu D_\nu(u-\underline{u}) \le -D_\nu\Psi - A\,D_\nu\vartheta$$

at $x_0$ and rearranging leads to the desired estimate.

**Proposition 12.4** ($C^2$-estimate)**.** *Suppose that $f \in C^1(\Omega\times\mathbb{R}\times\mathbb{R}^n)$ satisfies $\frac{\partial f}{\partial z} \le 0$, that $\partial\Omega$ and $\phi$ are of class $C^2$, and that there exists an admissible subsolution $\underline{u} \in C^2(\overline{\Omega})$ to (12.1) with $\underline{u} = \phi$ on $\partial\Omega$. Any smooth admissible solution $u \in C^1(\Omega)\cap C^0(\overline{\Omega})$ to (12.1) with $u_{\partial\Omega} \equiv \phi$ satisfies*

$$|u|_{C^2(\Omega)} \le C(n,\underline{u},f,\partial\Omega)\,.$$

**12.4. Uniform ellipticity and a Hölder estimate for the Hessian.** Recall that $F(r) \doteqdot \log\det r$, so that $\frac{\partial F}{\partial r_{ij}} = r^{ij}$. So, for any solution $u \in C^2(\Omega)$ to (12.2) and any $\xi \in S^n$,

$$\frac{\partial F}{\partial r_{ij}}(D^2u)\xi_i\xi_j = u^{ij}\xi_i\xi_j \geq |D^2u|_{C^0}^{-1} \doteqdot \lambda\,.$$

On the other hand, since

$$\det(D^2u) = -f(\cdot, u, Du)\,,$$

we may estimate, for any $\xi \in S^n$,

$$D^2u(\xi,\xi) \geq \frac{\det D^2u}{(\max_{\xi\in S^n} D^2u(\xi,\xi))^{n-1}} \geq \lambda^{n-1} \min_{\mathcal{J}^1u(\overline{\Omega})} |f| \doteqdot \Lambda^{-1}$$

and hence

$$\frac{\partial F}{\partial r_{ij}}(D^2u)\xi_i\xi_j \leq \Lambda\,.$$

A Hölder estimate for $D^2u$ now follows from Theorem 11.2.

**12.5. Solving the Dirichlet problem.** We may now solve the Dirichlet problem for equations of Monge–Ampère type, assuming the existence of a subsolution taking the boundary values.

**Theorem 12.5.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open convex set with boundary of class $C^3$. Suppose that $f \in C^2(\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n)$ satisfies $f < 0$ and $f_z \leq 0$. If $\phi \in C^{2,\alpha}(\overline{\Omega})$ is locally uniformly convex and satisfies*

$$-\det(D^2\phi) \leq f(\cdot, \phi, D\phi) \;\text{ in }\; \Omega\,,$$

*then the Dirichlet problem*

$$\begin{cases} -\det(D^2u) = f(\cdot, u, Du) & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases} \tag{12.11}$$

*admits a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$.*

**Proof.** Consider, for each $t \in [0,1]$, the Dirichlet problem

$$\begin{cases} -F(D^2u) - \psi(\cdot, u, Du) = -t\left(F(D^2\phi) + \psi(\cdot,\phi,D\phi)\right) & \text{in } \Omega \\ u = \phi \;\text{ on }\; \partial\Omega\,, \end{cases} \tag{12.12}$$

where

$$F(r) \doteqdot \log\det r \;\text{ and }\; \psi(x,z,p) \doteqdot -\log(-f(x,z,p))\,.$$

Observe that the problem corresponding to $t = 1$ admits the solution $u = \phi$, while the problem corresponding to $t = 0$ is equivalent to (12.11). So it suffices to show that the set $S$ of parameters $t \in [0,1]$ corresponding to problems which are soluble in $C^{2,\alpha}(\overline{\Omega})$ is both open and closed in $[0,1]$ (uniqueness of solutions to (12.12) is a consequence of Proposition 12.1).

Since $\phi$ is a subsolution to (12.12), Propositions 12.2, 12.3 and 12.4 and Theorem 11.4 yield an estimate of the form

$$|u|_{C^{2,\beta}(\Omega)} \leq C \tag{12.13}$$

for any solution to (12.12) (independent of $t$) for some $\beta = \beta(n, f, \Omega, \phi)$ and $C = C(n, f, \Omega, \phi)$.

To see that $S$ is closed, consider a sequence of parameters $t_k$ in $S$ converging to some $t \in [0,1]$. Let $u_k \in C^{2,\alpha}(\overline{\Omega})$ be the solution to (7.22) corresponding to the parameter $t_k$. Due to the uniform estimate (12.13), the Arzelà–Ascoli theorem provides a subsequence of the solutions $u_k$ which converges in $C^2(\overline{\Omega})$ to a solution $u \in C^{2,\beta}(\Omega)$ to the problem (7.22) corresponding to the parameter $t$. By Proposition 11.3, $u \in C^{3,\alpha}(\Omega) \subset C^{2,\alpha}(\Omega)$. So $S$ is indeed closed.

To see that $S$ is open, we apply the implicit function theorem and the solvability of the linearized problems. Consider the map $T : C^{2,\alpha}(\overline{\Omega}) \times [0,1] \to C^{\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\partial\Omega)$ defined by

$$T(u,t) \doteqdot \big(-F(D^2u) - \psi(\cdot, u, D^2u) + t(F(D^2\phi) + \psi(\cdot, \phi, D\phi)), u|_{\partial\Omega} - \phi\big)\ .$$

If $t_0 \in S$, then we can find $u_0 \in C^{2,\alpha}(\overline{\Omega})$ such that $T(u_0, t_0) = (0,0)$. In order to apply the implicit function theorem, we need to show that the Fréchet derivative of $T$ in the first variable at the point $(u_0, t_0)$ is an isomorphism. It suffices to compute the Gateaux derivative $v \mapsto D_vT$, where $D_vT$ is the directional derivative in the direction $v$, so long as this is a continuous linear operator. So consider, for some $v \in C^{2,\alpha}(\overline{\Omega})$, the directional derivative

$$\begin{aligned} D_vT|_{(u_0,t_0)} &\doteqdot \left.\frac{d}{ds}\right|_{s=0} T(u_0 + sv, t_0) \\ &= \big(-Lv, v|_{\partial\Omega}\big)\,, \end{aligned} \tag{12.14}$$

where the linear operator $L \doteqdot a^{ij}D_iD_j + b^iD_i + c$ is defined by

$$a^{ij} \doteqdot u_0^{ij} > 0\,,\ \ b^i \doteqdot \psi_{p_i}(\cdot, u_0, Du_0)\,,\ \ \text{and}\ \ c \doteqdot \psi_z(\cdot, u_0, Du_0) \leq 0\,.$$

Since $u_0 \in C^{2,\alpha}(\overline{\Omega})$, $L$ is a continuous linear operator, and hence the map $v \mapsto (-Lv, v|_{\partial\Omega})$ coincides with the Fréchet derivative. Since $u_0^{ij} > 0$, Theorem 4.1 implies that the map is an isomorphism, and hence the implicit function theorem guarantees that there exist $\delta > 0$ and a function $h : (t_0-\delta, t_0+\delta) \to C^{2,\alpha}(\overline{\Omega})$ such that $T(h(t),t) = (0,0)$ for all $t \in (t_0-\delta, t_0+\delta)$. That is, $(t_0-\delta, t_0+\delta) \subset S$. So $S$ is indeed open. The theorem is proved. □

**12.6. Exercises.**

**Exercise 12.1.** Derive (12.10).

**Exercise 12.2.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $u \in C^2(\overline{\Omega})$ be a solution to the Monge–Ampère type equation

$$-\det(D^2u) = f(\cdot, u, Du) \text{ in } \Omega$$

with $f \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ negative. Suppose that $\Omega$ is locally uniformly convex; that is, $\partial\Omega$ is of class $C^2$ and its second fundamental form $A$ is positive definite at all points. Under the assumption

$$|u|_{C^1(\overline{\Omega})} + \max_{(x,\tau)\in S\partial\Omega} |D^2u_x(\tau,\tau)| + \max_{(x,\tau)\in S\partial\Omega} |D^2u_x(\tau,\nu)| \leq K\,,$$

where $\nu$ is the outward unit normal field to $\partial\Omega$ and $S\partial\Omega \doteqdot \sqcup_{x\in\partial\Omega} S_x\partial\Omega$ is the unit tangent bundle to $\partial\Omega$, show that

$$D^2u_x(\nu,\nu) \leq C(n, K, f, \partial\Omega)$$

for all $x \in \partial\Omega$ and all unit tangent vectors $\tau \in S_x\partial\Omega$.

## 13. Level set flows: an invitation to degenerate nonlinear equations

**13.1. (Inverse) mean curvature flows.** Roughly speaking, a smooth family of smooth hypersurfaces $\{M_t\}_{t\in I}$ evolves by the *$\alpha$-mean curvature flow*, $\alpha \in \mathbb{R}\setminus\{0\}$, if at each "time" $t$ each point of $M_t$ moves in its normal direction $\nu$ with speed $-\operatorname{sign}(\alpha)H^\alpha$, where $H$ is the mean curvature with respect to $\nu$. More precisely, this means that for each $t_0 \in I$ and $p_0 \in M_{t_0}$, we can find open neighbourhoods $J \subset I$ about $t_0$, $U \subset \mathbb{R}^n$ about 0, and $V \subset \mathbb{R}^{n+1}$ about $p_0$, and a smooth (*local parametrization*) map $X : U\times J \to \mathbb{R}^{n+1}$ such that $X(U,t)\cap V = M_t \cap V$ (this is just what it means for $\{M_t\}_{t\in I}$ to be a smooth family of smooth hypersurfaces) and

$$\frac{\partial X}{\partial t} = -\operatorname{sign}(\alpha)H^\alpha\nu\,. \tag{13.1}$$

When $\alpha = 1$, equation (13.1) is called the *mean curvature flow*. When $\alpha = -1$, it is called the *inverse mean curvature flow*.

**Example 13.1.** The family of spheres $\{S^n_{r_\alpha(t)}\}_{t\in I_\alpha}$, where

$$I_\alpha \doteqdot \begin{cases} (-\infty,0) & \text{if } \alpha \notin [-1,0) \\ (-\infty,\infty) & \text{if } \alpha = -1 \\ (0,\infty) & \text{if } \alpha \in (-1,0) \end{cases} \quad \text{and } r_\alpha(t) \doteqdot \begin{cases} \mathrm{e}^{-nt} & \text{if } \alpha = -1 \\ (-(\alpha+1)nt)^{\frac{1}{\alpha+1}} & \text{if } \alpha \neq -1 \end{cases}$$

evolves by $\alpha$-mean curvature flow. Observe that, for $\alpha \notin [-1,0)$, the flow "breaks down" as $t \to 0$: when $\alpha > 0$, the "spheres disappear in a point", while when $\alpha < 0$, they "disappear at infinity". In each case, the speed "blows up" as $t \to 0$. This behaviour, finite time "blow-up" turns out to be a general feature of the flows.

**Example 13.2.** Consider a dumbell shaped surface (approximately two large spheres of radius $\sim R$ joined by a thin neck-like bridge of radius $\sim r$). The mean curvature of the spherical regions is $\sim \frac{2}{R}$ while that of the neck region is $\sim \frac{1}{r}$. Thus, for $r \ll R$, the mean curvature in the neck is much larger than in the spherical regions. Thus, under mean curvature flow ($\alpha = 1$), we expect the neck to shrink quickly, eventually pinching off with infinite curvature, while the spheres remain relatively stationary. One can prove this picture rigorously using a barrier argument.

**Example 13.3.** Consider a torus of revolution with circular profile. If the radius of revolution is sufficiently large compared to the radius of the profile circle, the torus will have positive mean curvature. Under inverse mean curvature flow ($\alpha = -1$), it will tend to expand outwards, closing the "hole", until the mean curvature tends to zero, at which point the inverse mean curvature tends to infinity. Again, one can demonstrate this picture rigorously using barriers.

**13.2. The level sets of a smooth function.** We have typically been (locally) representing smoothly embedded hypersurfaces $M$ in $\mathbb{R}^{n+1}$ as graphs over their tangent planes. It is also possible to represent $M$ locally by the zero set of a smooth function. Indeed, each $p \in M$ admits a neighbourhood $U \underset{\text{open}}{\subset} \mathbb{R}^{n+1}$ on which the distance function

$$d(x) \doteqdot \min q \in M|x - q|$$

is smooth. If we choose $V \underset{\text{open}}{\subset} \mathbb{R}^{n+1}$ such that $p \in V$ and $\overline{V} \subset U$, and choose a smooth function $\eta$ which is equal to one on $V$ and zero on $\mathbb{R}^{n+1} \setminus U$, then the function $u = \eta d$ is smoothly defined on $\mathbb{R}^{n+1}$ and we have

$$M \cap V = \{x \in \mathbb{R}^{n+1} : u(x) = 0\} \cap V \,.$$

Conversely, by the implicit function theorem (surjective version), the zero set $M \doteqdot \{x \in \mathbb{R}^{n+1} : u(x) = 0\}$ of a smooth function $u : \mathbb{R}^{n+1} \to \mathbb{R}$ is a smoothly embedded hypersurface in the neighbourhood of any regular point $p \in M \cap \{x \in \mathbb{R}^{n+1} : |Du(x)| \neq 0\}$.

**13.3. Level set flows.** Observe that the family of spheres defined above may be described by the $t \in I_\alpha$ level sets of the function $u_\alpha : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by

$$u_\alpha(X) \doteqdot \begin{cases} -\dfrac{\log|x|}{n} & \text{if } \alpha = -1 \\ -\dfrac{|x|^{\alpha+1}}{(\alpha+1)n} & \text{if } \alpha \neq -1\,. \end{cases}$$

Note that $u_\alpha$ is continuously differentiable at $x = 0$ when $\alpha \notin [-1, 0)$, even though its zero set is "singular" (since $Du_\alpha(0) = 0$). In fact, when $\alpha \geq 1$, $u_\alpha$ is smooth at the origin.

Consider a general $\alpha$-mean curvature flow $\{M_t\}_{t\in I}$. If $H(p_0) > 0$, $p_0 \in M_{t_0}$, then we may choose a local level set representation $u : \mathbb{R}^{n+1} \to \mathbb{R}$ for $\{M_t\}_{t\in I}$ about $p_0$. Observe that $Du(p_0) \perp T_{p_0}M_{t_0}$. Indeed, if $X : U \times J \to \mathbb{R}^{n+1}$ is a local parametrization for $\{M_t\}_{t\in I}$ about $p_0 \in M_{t_0}$, then

$$0 = \frac{\partial}{\partial x^i}\bigg|_{x=0} u(X(x, t_0)) = Du(p_0) \cdot X_i(0, t_0)\,,$$

where $X_i \doteqdot \frac{\partial X}{\partial x^i}$. So we may define a local (outward pointing) unit normal field $\nu$ by

$$\nu = -\operatorname{sign}\alpha \frac{Du}{|Du|}$$

for points $p \in M_t$ near $p_0$ with $t$ near $t_0$. Observe that, as a map from $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$,

$$\begin{aligned} D_V\nu = & -\operatorname{sign}\alpha\frac{1}{|Du|}\left(D_VDu - \frac{(D_VDu\cdot Du)Du}{|Du|^2}\right) \\ = & -\operatorname{sign}\alpha\frac{(D_VDu)^\top}{|Du|}\,, \end{aligned}$$

where $\cdot^\top$ denotes projection onto the tangent space to $M_t$. In other words, with respect to an orthonormal basis with $\nu(p) = e_{n+1}$, $D\nu|_p$ takes the form

$$D\nu = \begin{pmatrix} A_p & W_p \\ 0 & 0 \end{pmatrix}$$

for some vector $W_p \in T_pM_t$, where $A_p : T_pM_t \to T_pM_t$ is the shape operator for $M_t$ at $p$. In particular, the mean curvature is given by

$$H = -\operatorname{sign}\alpha\operatorname{div}\left(\frac{Du}{|Du|}\right).$$

On the other hand, differentiating the equation $u(X(x,t)) = t$, we obtain

$$1 = \frac{\partial}{\partial t}(u\circ X) = Du\cdot\frac{\partial X}{\partial t} = |Du|H^\alpha$$

for $p \in M_t$ near $p_0$ with $t$ near $t_0$. We conclude that $u$ satisfies

$$-|Du|^{\frac{1}{\alpha}}\operatorname{div}\left(\frac{Du}{|Du|}\right) = \operatorname{sign}\alpha \tag{13.2}$$

in a neighbourhood of any $p \in M_t$ at which $H(p) > 0$.

Equation (13.2) is called the *level set $\alpha$-mean curvature flow* (or simply the *level set mean curvature flow* when $\alpha = 1$ and the *level set inverse mean curvature flow* when $\alpha = -1$). Observe that (13.2) is a nonlinear elliptic equation (since the tangential projection is non-negative definite) but is not strictly elliptic (since $Du$ is a null eigenvector of the tangential projection).

Now, if a function $u : \Omega \to \mathbb{R}$ satisfies (13.2) at all *regular points*

$$p \in \operatorname{reg} u \doteqdot \{q \in \Omega : |Du(q)| \neq 0\}\,,$$

then, by the above arguments, the level sets $M_t \doteqdot \{p \in \mathbb{R}^{n+1} : u(p) = t\}$ satisfy (13.1) at all *regular values*

$$t \in \{s \in \mathbb{R} : M_s \subset \operatorname{reg} u\}\,.$$

Since the level sets $M_t$ are defined even for critical values of $u$, the level set formulation (13.2) provides a generalization (or “weak formulation”) of (13.1).

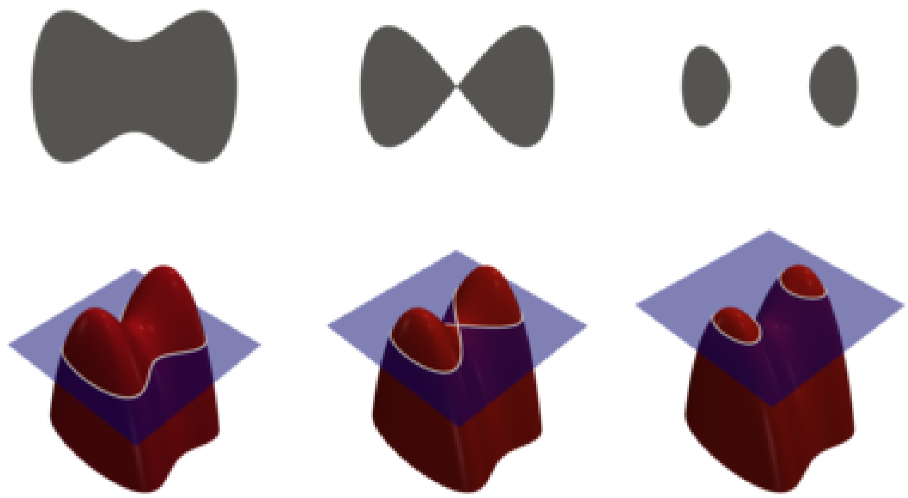

**Figure 1.** The level sets of a smooth function passing continuously through a "singularity". (Source: Wikipedia. User: Nicoguaro.)

The punchline is that, if $M_0$ bounds a domain $\Omega \subset \mathbb{R}^{n+1}$ and we are able to find a solution $u : \Omega \to \mathbb{R}$ to (13.2) with $u|_{\partial\Omega} \equiv 0$, then the level sets of $u$ define an extension of the classical (smooth) flow through any singularities. Unfortunately, since (13.2) is degenerate, the theory that we have thus far established does not apply (at least not directly).

**13.4. Solving the Dirichlet problem for the level set flow.** We wish to solve the Dirichlet problem

$$\begin{cases} -|Du|^{\frac{1}{\alpha}} \operatorname{div}\left(\dfrac{Du}{|Du|}\right) = \operatorname{sign}\alpha & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{13.3}$$

over an open domain $\Omega$ with prescribed boundary $\partial\Omega = M_0$. By the above discussion, we only seek solutions whose level sets are strictly mean convex (i.e. have positive mean curvature), so we must assume that $\partial\Omega$ is of class $C^2$ and mean convex. Moreover, since $M_0$ will move inward when $\alpha > 0$ and outward when $\alpha < 0$, we require $\Omega$ to be a bounded set when $\alpha > 0$ and the compliment of a bounded set when $\alpha < 0$.

In fact, even under these conditions, we cannot expect to be able to obtain classical solutions, since any $u \in C^1(\Omega)$ with $u|_{\partial\Omega} \equiv 0$ will attain an interior local maximum or minimum (at least when $\Omega$ is bounded), at which $Du = 0$, and hence the equation breaks down.

Another issue is the phenomenon of "fattening", whereby the level sets of $u$ develop an interior in $\mathbb{R}^{n+1}$ (e.g. when $u$ is constant on an open set of $\mathbb{R}^{n+1}$) and hence do not have any useful interpretation as "hypersurfaces". To deal with this phenomenon, it is convenient to work with the boundaries $\partial\{p \in \Omega : u(p) > t\}$ of the sublevel sets of $u$, instead of the level sets themselves.

Our strategy for solving the problem (13.3) (in a suitable generalized sense) is to instead seek solutions to the "desingularized" problems

$$\begin{cases} -\left(\varepsilon^2 + |Du|^2\right)^{\frac{1}{2\alpha}} \operatorname{div}\left(\dfrac{Du}{\sqrt{\varepsilon^2 + |Du|^2}}\right) = \operatorname{sign}\alpha & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{13.4}$$

and take $\varepsilon \to 0$. Observe that, dividing both sides by $\varepsilon^{\frac{1}{\alpha}}$ and making the substitution $u \mapsto \varepsilon u$, the problem (13.4) becomes

$$\begin{cases} -\left(1 + |Du|^2\right)^{\frac{1}{2\alpha}} \operatorname{div}\left(\dfrac{Du}{\sqrt{1 + |Du|^2}}\right) = \operatorname{sign}\alpha\varepsilon^{-\frac{1}{\alpha}} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{13.5}$$

which is just the equation for an $\alpha$-mean curvature flow *translator* in $\mathbb{R}^{n+2}$ (with velocity $-\operatorname{sign}\alpha\varepsilon^{-1}e_{n+1}$)! Indeed, if $u^\varepsilon$ satisfies (13.5), then the function $U^\varepsilon$ defined by $U^\varepsilon(X,t) \doteqdot u^\varepsilon(X) - \varepsilon^{-1}t$ satisfies the *graphical* $\alpha$-mean curvature flow.

When $\Omega$ is bounded, the results of §7 yield a solution $u^\varepsilon \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ to (13.5) for each $\varepsilon > 0$. It turns out that the same methods still yield a solution in case $\alpha < 0$, even though $\Omega$ is unbounded in this case. (The idea is to impose the Dirichlet boundary condition $u|_{\partial K_L} = L$ on the boundary of a large compact set $K_L$, solve this "compactified" problem, and then take $L \to \infty$ with $K_L \to \mathbb{R}^{n+1}$. Of course, one requires uniform-in-$L$ estimates for solutions to the compactified problems.) Moreover, we actually end up with a gradient estimate for $\varepsilon u^\varepsilon$ which is independent of $\varepsilon$! (The higher order estimates depend on $\varepsilon$, however.) By applying the Arzelà–Ascoli theorem, we can now obtain a limit $\varepsilon_j u_{\varepsilon_j} \to u \in C^{0,1}(\overline{\Omega})$ in the $C^{0,1}_{\mathrm{loc}}(\overline{\Omega})$ topology along some sequence $\varepsilon_j \to 0$. This convergence ensures that $u|_{\partial\Omega} \equiv 0$, but is not strong enough to conclude that $u$ satisfies (13.2) except in the tautological sense that it is a limit as $\varepsilon_j \to 0$ of solutions to (13.4). It turns out that this notion of weak solution actually inherits a number of further properties (such as coincidence with the smooth flow whenever the latter exists), at least for certain values of $\alpha$. Let us only make a couple of further observations.

Recall that the family of hypersrufaces $M_t^\varepsilon \doteqdot \operatorname{graph} U^\varepsilon(\cdot,t)$ evolves by $\alpha$-mean curvature flow in $\mathbb{R}^{n+2}$. Observe that, for $t>0$, the Hausdorff limit $\lim_{\varepsilon\to 0} M_t^\varepsilon$ is the vertical cylinder over the $t$-level set of $u$! Since the mean curvature of a the cylinder $M\times\mathbb{R}$ in $\mathbb{R}^{n+2}$ over the hypersurface $M$ of $\mathbb{R}^{n+1}$ is just the lift of the mean curvature of $M$ (i.e. $H_{M\times\mathbb{R}}(x,t)=H_M(x)$), we would be able to conclude that the level sets of the limit $u$ satisfy the $\alpha$-mean curvature flow if we could establish this for the hypersurfaces $\lim_{\varepsilon\to 0} M_t^\varepsilon$.

Alternatively, we can exploit the fact that (13.2) admits a variational structure: consider the functional

$$J(v) \doteqdot \int (|Dv| - \operatorname{sign}\alpha v|Du|^{\frac{1}{\alpha}})\,,$$

where $u\in C^{0,1}_{\mathrm{loc}}(\overline{\Omega})$ is the limit of the solutions $\varepsilon u^\varepsilon$ to the approximating problems (13.4) obtained above. Observe that, for $v\in C^\infty(\Omega)$ and any compactly supported $w\in C^\infty(\Omega)$,

$$\begin{aligned}\frac{d}{ds}\bigg|_{s=0} J(v+sw) &= \int\left(\frac{Dv\cdot Dw}{|Dv|} - \operatorname{sign}\alpha w|Du|^{\frac{1}{\alpha}}\right)\\ &= -\int w\left(\operatorname{div}\left(\frac{Dv}{|Dv|}\right) + \operatorname{sign}\alpha|Du|^{\frac{1}{\alpha}}\right)\,.\end{aligned}$$

In particular, if it can be established that $u$ is a critical point of $J$, then we conclude that it must satisfy the $\alpha$-mean curvature level set flow in a weak sense.

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