CIRCLE PACKING AND DISCRETE ANALYTIC FUNCTION THEORY

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Abstract. Circle packings — configurations of circles with specified patterns of tangency — came to prominence with analysts in 1985 when Thurston conjectured that maps between such configurations would approximate conformal maps. The proof by Rodin and Sullivan launched a topic which has grown steadily as ever more connections with analytic functions and conformal structures have emerged. Indeed, the core ideas have matured to the point that one can fairly claim that circle packing provides a *discrete analytic function theory.*

There are two related but distinct aspects *vis-a-vis* the classical model — analogy and approximation. This survey concentrates on the analogies, with a largely pictorial tour intended for the reader familiar with classical conformal geometry. The companion survey [St99] treats approximation.

Part I of the paper covers circle packing basics and introduces discrete analytic functions as maps between circle packings. Representatives of the standard classes of analytic functions on the sphere, plane, and unit disc are illustrated and the discrete parallels to basic theorems in complex analysis are reviewed. Part II visits several geometric facets of conformal geometry in their discrete encarnations: extremal length, harmonic measure, conformal welding, and conformal structures, among others. The final topic there, conformal tiling, shows both the potential for synergy between the discrete and classical theories and the value of the experimental capabilities available with circle packing. Appendix A comments on computational aspects of circle packing and Appendix B summarizes the circle packing literature.

Introduction

A **circle packing** is a configuration of circles with a specified pattern of tangencies. Early examples may be helpful, so the reader is invited to the small menagerie collected in Figure 1; of course, our packings will involve more features than we are prepared to illustrate at this early stage.

Scattered examples of circle packings can be traced back to antiquity, but our notion was formulated by William Thurston in his famous *Notes* [Thur] for use in constructing certain orbifolds. Recognizing a rigidity reminiscent of that associated with conformal structures, he was led to a conjecture, announced at the 1985 Bieberbach conference [Th85], on the approximation of conformal mappings using circle packings. This conjecture, to which we return in §7, was proven shortly thereafter by Rodin and Sullivan

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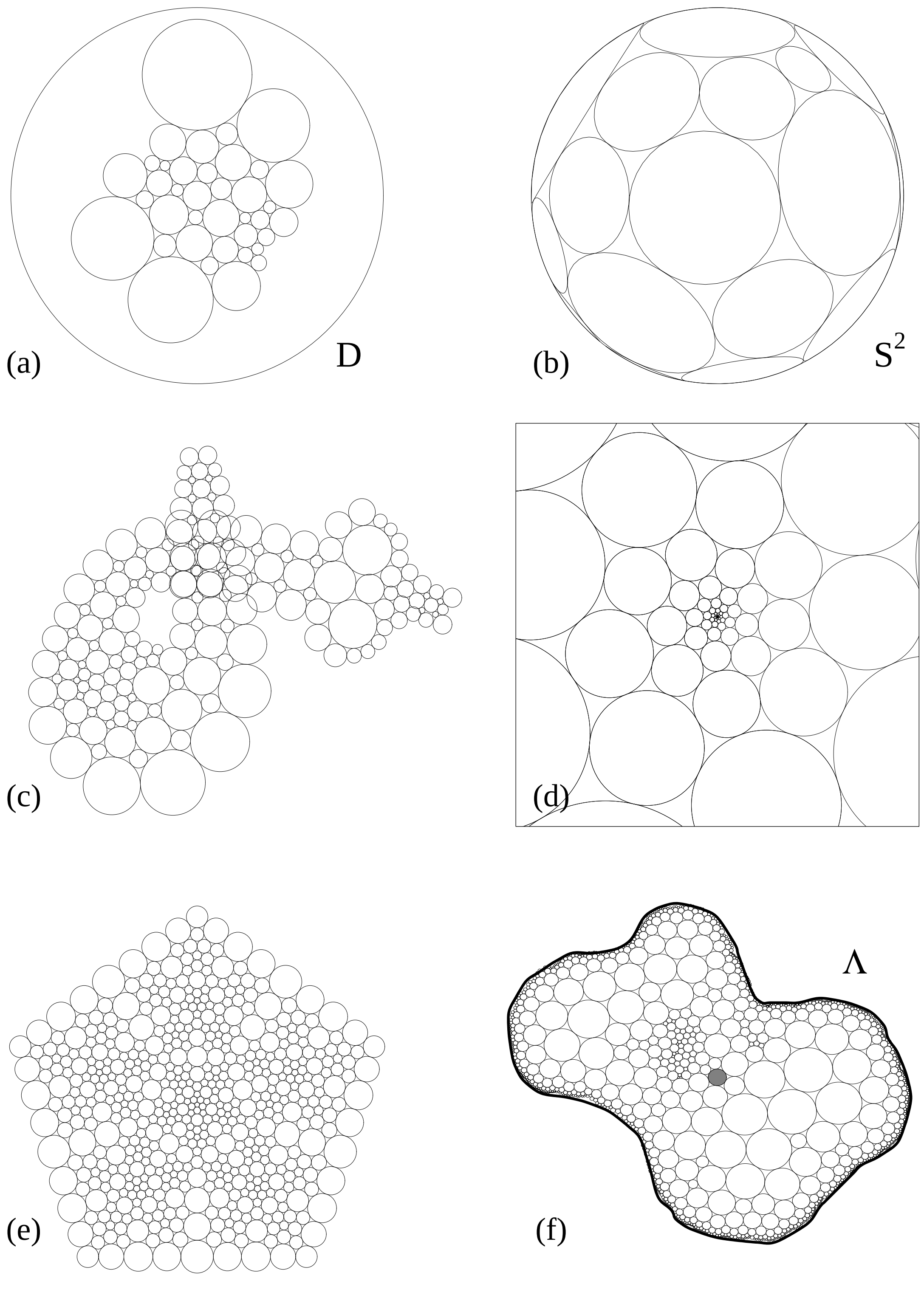


FIGURE 1. A small menagerie of circle packings

[RS87], marking the beginning of the association between circle packings and analytic function theory. (Note that circle packing has essentially **no contact** with the more familiar "sphere packing" studies — how many ping pong balls fit in a boxcar. Here it is the *pattern of tangencies* which is central.)

After seeing the basic notions of circle packing in a moment, I think you will find these configurations a fascinating blend of local geometric rigidity and global geometric flexibility. A mapping between one configuration of circles realizing a certain tangency pattern and another configuration realizing that same pattern will be termed a "discrete" analytic function. These maps are the main objects of study here, and the claim is, quite simply, that *discrete analytic functions behave geometrically like classical analytic functions.* Incidently, I'm using the adjective "classical" simply to distinquish the standard continuous theory from our discrete version.

The emerging discrete theory offers two related but distinct lines of inquiry: namely, **analogy** and **approximation**. The deep and surprisingly comprehensive parallels with the classical theory are the focus in this article, and my central task is to guide the reader in *discretizing* the intuition he/she has developed in the classical setting. In addition, however, the objects of this discrete theory invariably converge under appropriate refinement to their classical counterparts. Such approximations, direct descendents of Thurston's original conjecture, appear here in a supporting role, and the interested reader is encouraged to see the survey article [St99] for more details.

With space at a premium, our survey takes a decidedly graphical approach. The paper has two parts. The first provides the bare minimum of circle packing definitions, notation, and mechanics, with images intended to evoke the central geometric intuition. Motivated by those images we define discrete analytic functions, formulate discrete versions of several fundamental classical results, and illustrate familiar classes of analytic functions in their discrete forms. Parallels with the classical theory remain at the forefront.

With the basic theory in place, the second part of the paper formulates discrete encarnations of various geometric companions of analyticity — type problems, harmonic measure, extremal length, conformal structures, conformal welding, and so forth. In examples and selected applications the reader will see how the discrete and classical theories complement one another, with sometimes one and sometimes the other taking the lead. Our final topic, conformal tiling, shows in particular the powerful synergy which can develop.

A third aspect of circle packing is implicit throughout the paper: namely, circle packings are effectively computable, so the discrete objects can be created and manipulated experimentally. Our illustrations were generated by the software package `CirclePack` (see Appendix A), which implements all the topics discussed here. One of the joys of circle packing is seeing results emerge in live action; we're restricted here to static images, but our approach will emphasize the dynamical nature of the subject and the visual geometric intuition.

This survey is based on talks I gave at the March 1996 *Function Theory and Quasiconformal Mapping* conference in Oberwolfach. My thanks to the organizers, Fred Gehring and Stephan Ruscheweyh, and to the many colleagues in function theory who have encouraged me in pursuing this topic in recent years.

Part I: Discrete Analytic Function Theory

The theory of circle packing began on the sphere with this theorem, proven successively by P. Koebe, E. M. Andreev, and W. Thurston:

Koebe-Andreev-Thurston Theorem. *Given an abstract triangulation $\mathcal{K}$ of a topological sphere, there exists a circle packing P on the Riemann sphere $\mathbb{S}^2$ having the combinatorics of $\mathcal{K}$. P is unique up to Möbius transformations and inversions of $\mathbb{S}^2$.*

The conditions on P mean that each vertex v of $\mathcal{K}$ is represented by a circle $c_v \in P$, with c_v, c_u tangent whenever vertices v and u are connected by an edge in $\mathcal{K}$. The richness of circle packings may be traced to their dual natures: **combinatoric** in the pattern of prescribed tangencies and **geometric** in the realization by actual circles.

1. Circle Packing Basics

We focus attention mainly on circle packings in the familiar metric spaces of constant curvature, namely the plane $\mathbb{C}$, the sphere $\mathbb{S}^2$, and the hyperbolic plane $\mathbb{D}$ (the unit disc with the Poincaré metric), though we have occasion to treat packings on more general Riemann surfaces, accommodated using standard covering arguments. Circle packing requires a surprisingly modest system of bookkeeping. The main objects and terms are as follows:

- ***Complex:*** Combinatorics of a packing are encoded in an abstract simplicial 2-complex $\mathcal{K}$. The conditions are most easily described by requiring that $\mathcal{K}$ be a triangulation of an oriented topological surface.

- ***Label:*** R is a collection $\{R(v)\}$ of positive numbers associated with vertices v of $\mathcal{K}$. If we are working in a metric space $\mathcal{D}$, these represent putative radii, and the label is described as *euclidean*, *spherical*, or *hyperbolic* depending on whether the metric is curvature $0, 1$, or -1, respectively.

• ***Angle Sums:*** A label R determines an angle sum $\theta_R(v)$ at each vertex of $\mathcal{K}$. In particular, for each face $\langle v,u,w\rangle \in \mathcal{K}$, let $\alpha_R(v;u,w)$ denote the angle at c_v in a triangle formed by the centers of a mutually tangent triple $\langle c_v, c_u, c_w\rangle$ of circles of radii $\langle R(v), R(u), R(w)\rangle$ in $\mathbb{S}^2, \mathbb{C}$, or $\mathbb{D}$, depending on curvature; this angle can be computed using the appropriate law of cosines. The angle sum is then $\theta_R(v) = \sum_{\langle v,u,w\rangle} \alpha_R(v;u,w)$, summing over all faces containing v.

• ***Packing:*** A circle packing P for $\mathcal{K}$ is a configuration of circles in the metric space $\mathcal{D}$ satisfying the tangency pattern of $\mathcal{K}$. That is, P is a collection $\{c_v\}$ of circles, one for each vertex v of $\mathcal{K}$, so that c_v is tangent to c_u whenever $\langle v,u\rangle$ is an edge of $\mathcal{K}$ and so that a triple $\langle c_v, c_u, c_w\rangle$ is positively oriented if $\langle v,u,w\rangle$ is an oriented face of $\mathcal{K}$. Were the radii of P listed in a label R, we would write $P \leftrightarrow \mathcal{K}(R)$.

• ***Flower:*** The flower of vertex v refers to the "central" circle c_v and the chain of "petal" circles intended to be tangent to it. Combinatorially, the flower is identified with the "star" of v in $\mathcal{K}$.

• ***Carrier:*** $\mathrm{carr}(P)$ denotes the concrete geometric complex in $\mathcal{D}$ formed by connecting the centers of tangent circles of P with geodesic segments; this is simplicially equivalent to the abstract complex $\mathcal{K}$.

Figure 2 illustrates these basics with a simple finite complex labeled $\mathcal{K}$; (b) is a rather generic hyperbolic packing for $\mathcal{K}$ in $\mathbb{D}$, (c) is a euclidean packing in $\mathbb{C}$ arranged so that its carrier (shaded in the picture) forms a rectangle, and (d) is the maximal packing in $\mathbb{D}$, to be discussed shortly. Note that from one packing to another the circles' *radii* and even the ambient geometry are subject to change — it is the **combinatorics** which these packings share.

The surprise, perhaps, is that for a given complex $\mathcal{K}$ there exist *any* packings whatsoever, much less a potentially huge variety of packings. The local compatability of the labels is key.

Definition. *A label R is a **packing label** for $\mathcal{K}$ if the angle sum $\theta_R(v)$ is a positive integral multiple of 2π for every interior vertex v.*

This is clearly a necessary condition for R to represent the radii of a packing P, since $\theta_R(v) = 2\pi n, n \geq 1$, simply reflects the fact that the petal circles of the flower of v must reach precisely n times around c_v. It is also a sufficient condition if there are no topological obstructions:

Lemma 1. *If $\mathcal{K}$ triangulates a simply connected surface, then a label R for $\mathcal{K}$ represents the radii for a circle packing of $\mathcal{K}$ if and only if R is a packing label.*

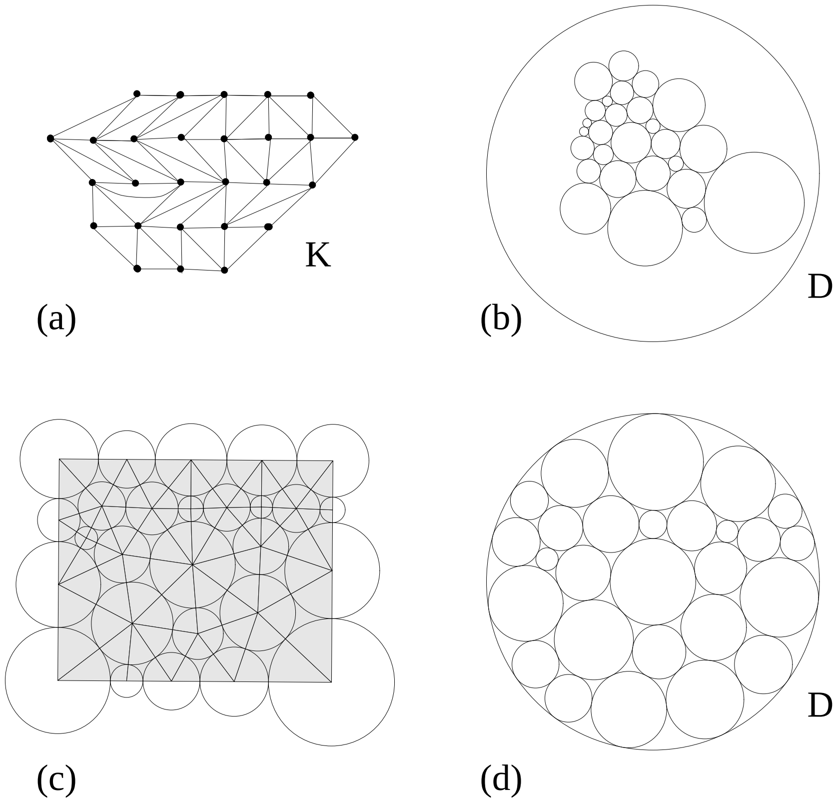


FIGURE 2. Circle packings for a simple complex $\mathcal{K}$

We say little about circle centers because they play a secondary role: once a packing label R is in hand, the circles can be laid out in $\mathcal{D}$ ($\mathbb{S}^2, \mathbb{C}$, or $\mathbb{D}$, as appropriate) using a process akin to analytic continuation. The resulting circle packing $P \leftrightarrow \mathcal{K}(R)$ is *essentially unique* — that is, it is unique up to a conformal automorphism of $\mathcal{D}$.

A bit of terminology: A packing is **univalent** if its circles have mutually disjoint interiors. Univalence fails locally at a **branch** circle (or vertex), that is, at an interior circle whose neighbors wrap two or more times around it (angle sum $4\pi, 6\pi, \cdots$). In Figure 3, the same nine circles which wrap once about the central circle in Figure 3(a) wrap exactly twice around the smaller central circle of Figure 3(b), illustrating branching of order 1. The **branch set** for a packing P, denoted $br(P)$, refers to the indices of its branch circles, repeated according to multiplicities. If all interior angle sums are 2π, then $br(P) = \emptyset$ and the packing is said to be **locally univalent**. Even with local univalence at every circle, global univalence can fail when circles from one part of a packing overlap those from another, as in Figure 1(c).

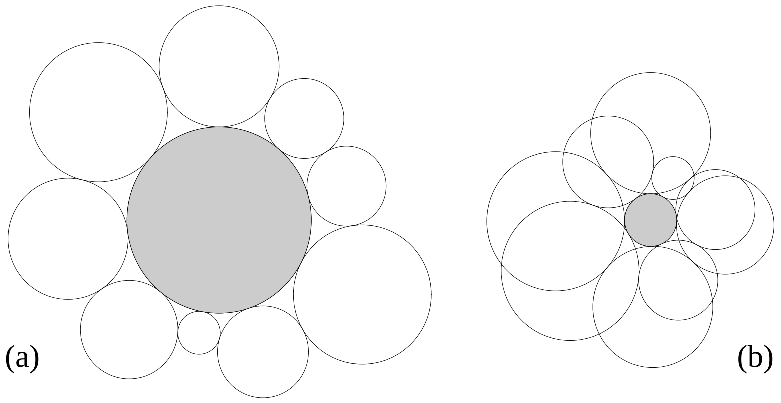


FIGURE 3. Local univalence *versus* branching

2. Maximal Packings

Given a complex $\mathcal{K}$, there are numerous results on the existence and variety of packings for $\mathcal{K}$; we need two main types. Let's begin with the *extreme rigidity* displayed by maximal packings.

Circle Packing Theorem. *Given a complex $\mathcal{K}$, there exists a unique Riemann surface $\mathcal{X}$ and a circle packing $P_{\mathcal{K}}$ for $\mathcal{K}$ in the intrinsic metric of constant curvature on $\mathcal{X}$ so that $P_{\mathcal{K}}$ is univalent and fills $\mathcal{X}$. The packing $P_{\mathcal{K}}$ is unique up to conformal automorphisms of $\mathcal{X}$ and is called the **maximal packing** for $\mathcal{K}$.*

This is just the Koebe-Andreev-Thurston Theorem when $\mathcal{K}$ triangulates a sphere; it was extended by Beardon and Stephenson to arbitrary $\mathcal{K}$ of bounded degree [BSt90] and then by He and Schramm to cases of unbounded degree [HeS93].

Figure 4 illustrates maximal packings for several complexes. Note that each $\mathcal{K}$ "chooses" its own geometry: that is, $\mathcal{K}$, which begins as a topological surface, is endowed with a unique conformal structure and associated metric which support $P_{\mathcal{K}}$. The complex $\mathcal{K}$ is said to be **spherical**, **parabolic**, or **hyperbolic** depending on the curvature of that geometry.

We cannot go into the details of the proof, but it may be helpful to highlight some of the familiar geometric notions which are pivotal. The first case is the key, both in theory and practice.

Case I: *$\mathcal{K}$ triangulates a closed topological disc.*

Here $\mathcal{K}$ is finite, simply connected, and has a boundary. Figure 2(a) is an elementary example, and the extremal nature of its maximal packing, Figure 2(d), is visually evident — the boundary circles are horocycles, circles internally tangent to the unit circle. Indeed, horocycles are naturally interpreted as circles of infinite hyperbolic

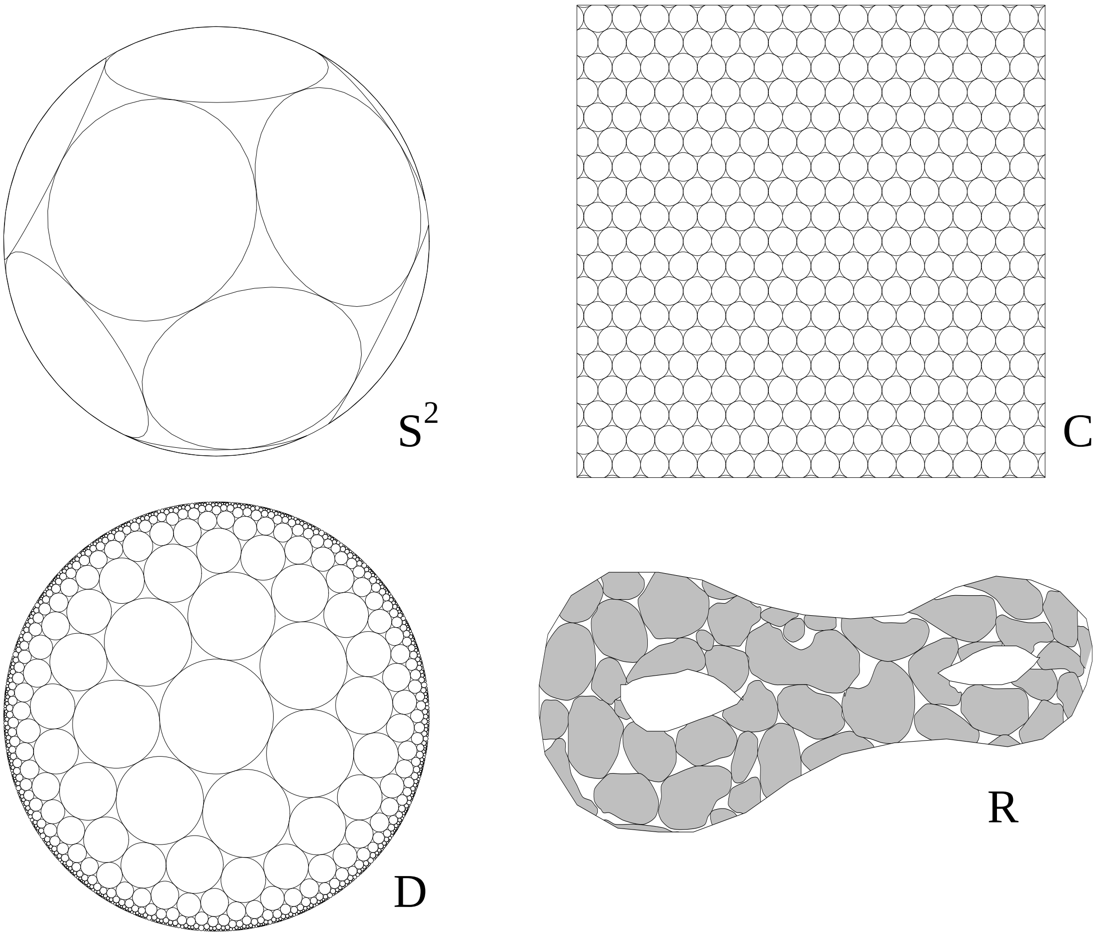


FIGURE 4. Maximal packings

radius, so our task reduces to finding a hyperbolic packing label for $\mathcal{K}$ whose boundary labels are infinite.

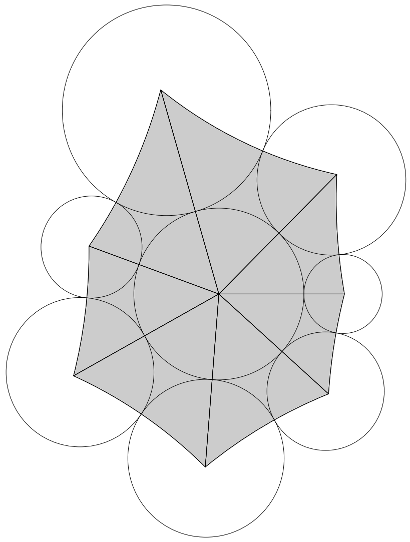
Formally, the proof relies on monotonicity results and a "Perron" method. One defines Φ as the collection of **subpacking** labels for $\mathcal{K}$, that is, (hyperbolic) labels R with the property that the associated angle sums for interior vertices are greater than or equal to 2π. One shows that Φ is non-empty, closed under maxima, and uniformly bounded for interior v. The upper envelop, the label $\widehat{R}$ defined by

$$\widehat{R}(v) = \sup_{R \in \Phi}\{R(v)\},$$

can be shown to be a packing label with infinite boundary radii. With these radii in hand, successively laying out the circles gives $P_{\mathcal{K}}$.

It is more informative (and practical) to approach the packing problem algorithmically. The geometric ingredients are surprisingly elementary. Consider a univalent flower in $\mathbb{D}$ as shown below, with central (hyperbolic) radius r and m petal radii $r_1, r_2, \cdots, r_n$; the (hyperbolic) faces are shown for reference. Then:

- *Monotonicity:* If r is increased, then the angle sum at the center circle goes **down** while the angle sum at each of the petal circles goes **up**.
- *Rodin/Sullivan Ring Lemma:* There's a constant $C = C(m)$ so that $r_j/r \geq C$; that is, in a (univalent) flower, no petal circle can be too much smaller than the center circle.
- *Bound:* $r \leq -\log(\sin(\dfrac{\pi}{m}))$. In the hyperbolic plane, a circle cannot be too large if m petal circles can wrap at least once around it.



To get some taste for the reasoning, suppose we were to grant the existence of *some* initial hyperbolic packing P_0 for $\mathcal{K}$. I ask the reader to imagine the effect of increasing one of its boundary circles. It's an interesting exercise depending solely on *monotonicity*; deduce an *upward* pressure reverberating through *all* of the interior radii as they adjust in order to keep their angles sums at 2π. The adjustments ultimately lead to a new packing for $\mathcal{K}$ which accommodates the increased boundary circle. Iterating this *increment/repack* cycle allows one to push the boundary radii to ∞; monotonicity and our bound force the interior radii to converge to finite limiting values. The result is the maximal packing label $R_{\mathcal{K}}$. The argument is quite striking when implemented live on a computer screen. Figure 5(a) is an initial packing P_0 for the complex of Figure 2. Incrementally increasing the boundary radii generates a succession of intermediate packings, several of which are superimposed in Figure 5(b). One can see the monotonicity at work as the maximal packing, isolated in Figure 5(c), emerges.

The bound on radii for interior circles, which fails in euclidean geometry, along with the availability of circles (horocycles) having infinite radius are what make the proof click in the hyperbolic setting. The uniqueness of $P_{\mathcal{K}}$ (up to automorphisms of $\mathbb{D}$) can be established using monotonicity along with the association in hyperbolic geometry between "angles" and "area". (It also follows from fixed point arguments due to He and Schramm, which we'll comment on later.) The fact that the circles of the maximal packing have mutually disjoint interiors is a consequence of local univalence and a standard topological argument principle.

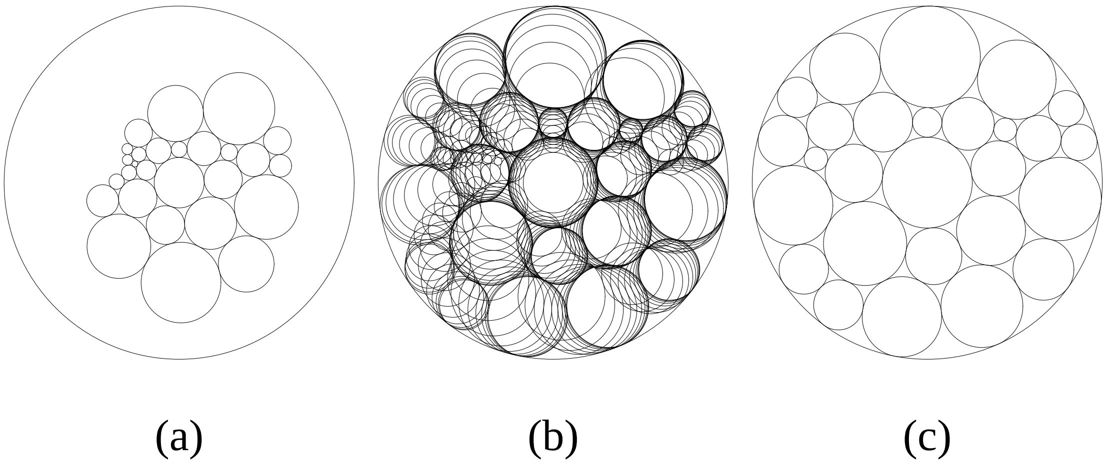


Figure 5. Growing to the max

Let's register an important observation about $R_{\mathcal{K}}$ which can be spun off from the proof.

Lemma 2. *Suppose $\mathcal{K}$ triangulates a closed topological disc. If R is any hyperbolic packing label for $\mathcal{K}$, then $R \le R_{\mathcal{K}}$, meaning that $R(v) \le R_{\mathcal{K}}(v)$ for every vertex $v \in \mathcal{K}$. Moreover, equality holds at a single interior vertex if and only if $R = R_{\mathcal{K}}$.*

This justifies the adjective "maximal" for $P_{\mathcal{K}}$ and will shortly become the key ingredient for handling infinite complexes.

Case II: *$\mathcal{K}$ triangulates a sphere.*

The spherical case is an easy consequence of Case I. Choosing a vertex v_∞ of $\mathcal{K}$, form a reduced complex $\mathcal{K}'$ by removing v_∞ and all edges and faces containing v_∞. Then $\mathcal{K}'$ triangulates a closed disc, so Case I yields a maximal packing $P' = P_{\mathcal{K}'}$ in $\mathbb{D}$. Stereographic projection to $\mathbb{S}^2$ carries P' to a circle packing in the southern hemisphere. All its boundary circles, as projections of horocycles in $\mathbb{D}$, will be tangent to the northern hemisphere, so by including that hemisphere as the circle c_∞ for v_∞, we have a spherical packing in $\mathbb{S}^2$ for $\mathcal{K}$ itself. A Möbius transformation can be applied for any desired normalization.

This stereographic projection argument is Thurston's and was originally used to prove Case I *from* Case II. However, as you will see in the next case, the hyperbolic arguments give added value.

Case III: *$\mathcal{K}$ triangulates an open topological disc.*

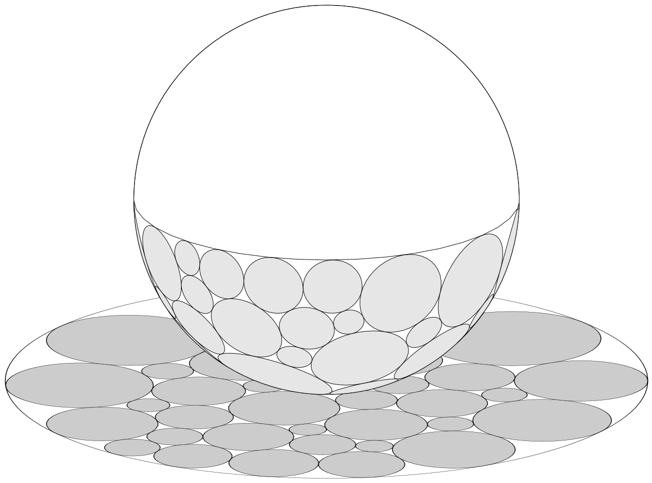


FIGURE 6. Projection to the sphere

Here $\mathcal{K}$ is infinite, simply connected, and without boundary. Designate some interior vertex v_0 and exhaust $\mathcal{K}$ by a nested sequence of finite simply connected complexes $v_0 \in \mathcal{K}_1 \subset \mathcal{K}_2 \subset \cdots \longrightarrow \mathcal{K}$. For each j, write P_j for the maximal packing of $\mathcal{K}_j$, R_j for the maximal packing label, and $r_0^{(j)}$ for the label at v_0. Note that within the maximal packing P_{j+1} lies a packing for $\mathcal{K}_j$. By Lemma 2, therefore, $r_0^{(j+1)} < r_0^{(j)}$; that is, the labels at v_0 are strictly decreasing,

$$r_0^{(1)} > r_0^{(2)} > r_0^{(3)} > \cdots .$$

In other words, as more circles of $\mathcal{K}$ are incorporated, the circle at the origin gets smaller. That brings us to a crucial

$$\textbf{Dichotomy:} \quad \begin{cases} \text{(P):} \quad r_0^{(j)} \longrightarrow 0 \\ \\ \text{(H):} \quad r_0^{(j)} \longrightarrow r_0^{(\infty)} \text{ for some } r_0^{(\infty)} > 0 \end{cases} .$$

We will find that alternatives (P) and (H) correspond to *parabolic* and *hyperbolic*, respectively. It might be best to illustrate with a pair of particularly clean examples. Let $\mathcal{K}^6$ denote the familiar hexagonal (constant 6-degree) complex and $\mathcal{K}^7$, the heptagonal (constant 7-degree) complex. In each instance choose $\mathcal{K}_j$ to consist of vertices within j-generations of v_0 and normalize P_j so that v_0 is at the origin and some designated neighbor v_1 is centered on the positive imaginary axis.

In Figure 7(a) I have superimposed the first few packings P_j associated with $\mathcal{K}^6$. Note that the circle for v_0 at the origin is shrinking rather rapidly — this is alternative (P). In contrast, in Figure 7(b) I have superimposed the first few packings associated with $\mathcal{K}^7$; the circle at the origin again gets smaller, but very quickly stablizes at a positive value — this is alternative (H).

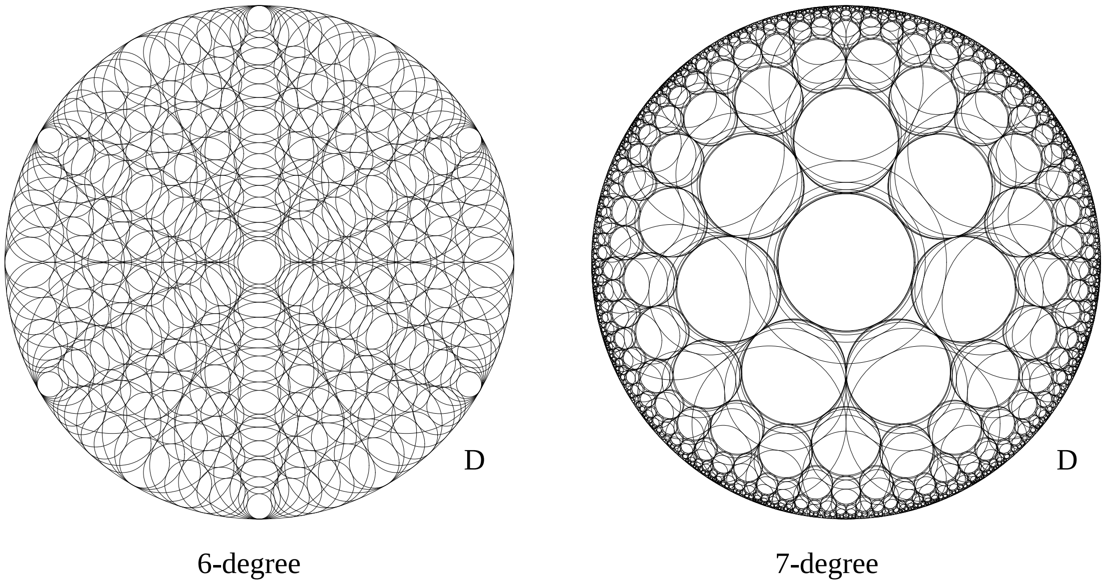


FIGURE 7. The hyperbolic/parabolic dichotomy

The maximal packing $P_{\mathcal{K}}$ in each case results from a "geometric" limiting process. In the haptagonal case, Figure 7(b), one can almost see the limit packing emerging. Formally, one uses the Ring Lemma to prove existence of positive limit radii for *every* vertex v and diagonalization to deduce limits for circle centers. It's relatively easy to show that the limit packing P is univalent and fills $\mathbb{D}$. In other words, $\mathcal{K}^7$ is *hyperbolic* and the packings P_j converge to $P_{\mathcal{K}^7}$.

On the other hand, it's certainly difficult to see a penny packing emerging in Figure 7(a). Indeed, in alternative (P), the Ring Lemma implies that *all* circle radii decrease to zero. Consequently, we shift perspective, treating each P_j as a euclidean packing and scaling it so the circle for v_0 is the unit circle. Now the Ring Lemma yields positive and finite limits for all the (euclidean) radii and again diagonalization provides us with a univalent limit packing P. Thus $\mathcal{K}^6$ is *parabolic* and $P = P_{\mathcal{K}^6}$, the familiar "penny" packing. The proof that carr $P_{\mathcal{K}^6} = \mathbb{C}$ in the parabolic case is more difficult than it might seem. It was confirmed for $\mathcal{K}$ having bounded degree in [BSt90] using quasiconformal arguments. The proof in the general case is even more subtle and was provided by He and Schramm in [HeS93]. Their key observation? *Distinct circles can intersect in at most 2 points!* I can't give details, but their arguments deserve mention not only for their elegance but for the powerful tools they bring to the discrete setting — versions of the winding number arguments so central in classical complex analysis.

If $\mathcal{K}$ is infinite, simply connected, but has boundary, then it falls under the *hyperbolic* alternative (H). With this observation, we find that we've taken care of all simply connected complexes, Case I being *hyperbolic*, Case II *spherical*, and Case III either *hyperbolic* or *parabolic* depending on combinatorics. Before going on, it's important to note the essential uniqueness of all the extremal packings obtained so far: *when $\mathcal{K}$ is simply connected, its maximal packing $P_{\mathcal{K}}$ is unique up to Möbius transformations of the sphere, plane, or disc, as appropriate.*

Case IV: *$\mathcal{K}$ triangulates a surface S.*

We assume that S is an oriented topological surface and, in view of our earlier cases, that S is not simply connected. It is well known that a triangulation $\mathcal{K}$ of S can be lifted to a complex $\widetilde{\mathcal{K}}$ triangulating the universal covering surface $\widetilde{S}$ of S. There is an associated simplicial projection $p : \widetilde{\mathcal{K}} \longrightarrow \mathcal{K}$ and a group G of simplicial automorphisms $g : \widetilde{\mathcal{K}} \longrightarrow \widetilde{\mathcal{K}}$ satisfying $p \circ g \equiv p$.

$\widetilde{\mathcal{K}}$ is simply connected, and since the sphere covers only itself, $\widetilde{\mathcal{K}}$ must be and infinite triangulation of a topological disc, Case III. Let $\mathcal{D}$ denote the plane or disc, depending on whether $\widetilde{\mathcal{K}}$ is parabolic or hyperbolic, and let $\widetilde{P} = P_{\widetilde{\mathcal{K}}}$ denote the maximal packing in $\mathcal{D}$ for $\widetilde{\mathcal{K}}$. The situation is illustrated in Figure 8 for a hyperbolic case. The circle packing shown in $\mathbb{D}$ is just the part of the infinite packing $\widetilde{P}$ associated with a fundamental domain for the covering p.

The essential uniqueness of $\widetilde{P}$ in $\mathcal{D}$ becomes the key ingredient as we deploy a standard arrow-chasing argument. Briefly, each simplicial automorphism g of $\widetilde{\mathcal{K}}$ must induce a Möbius transformation M_g of $\mathcal{D}$ which maps the packing $\widetilde{P}$ to itself. $\Gamma = \{M_g : g \in G\}$ is a discrete group of Möbius transformations of $\mathcal{D}$ isomorphic to G. Let $\mathcal{X}$ denote the Riemann surface $\mathcal{D}/\Gamma$ obtained in the classical manner from $\mathcal{D}$ by identifying all points equivalent modulo Γ, and write $\pi : \mathcal{D} \longrightarrow \mathcal{X}$ for the analytic covering projection. As topological surfaces, $\mathcal{X}$ and S are homeomorphic, however $\mathcal{X}$ inherits a conformal structure and conformal metric from $\mathcal{D}$ under π. This "intrinsic" metric is either hyperbolic or euclidean, depending on $\mathcal{D}$, and each circle in $\mathcal{D}$ projects, *a fortiori*, to a "circle" in this intrinsic metric on $\mathcal{X}$. Clearly, the projected circles $\pi(\widetilde{P})$ in $\mathcal{X}$ provide an *in situ* packing for $\mathcal{K}$. This is precisely the maximal packing $P_{\mathcal{K}}$ we have been looking for.

This concludes our overview of maximal packings. Even in this last case, note that $\mathcal{X}$ is uniquely determined among all Riemann surfaces homeomorphic to S based purely on the combinatorics of $\mathcal{K}$, so the take-home message is that $\mathcal{K}$ again "chooses" the appropriate geometry for its maximal packing.

3. Packing Variety

In stark contrast to the *rigidity* of maximal packings, one finds tremendous *variety* among general packings. In Figure 9, for example, all packings share one complex $\mathcal{K}$, the lower right image being its maximal packing. The variety comes not from

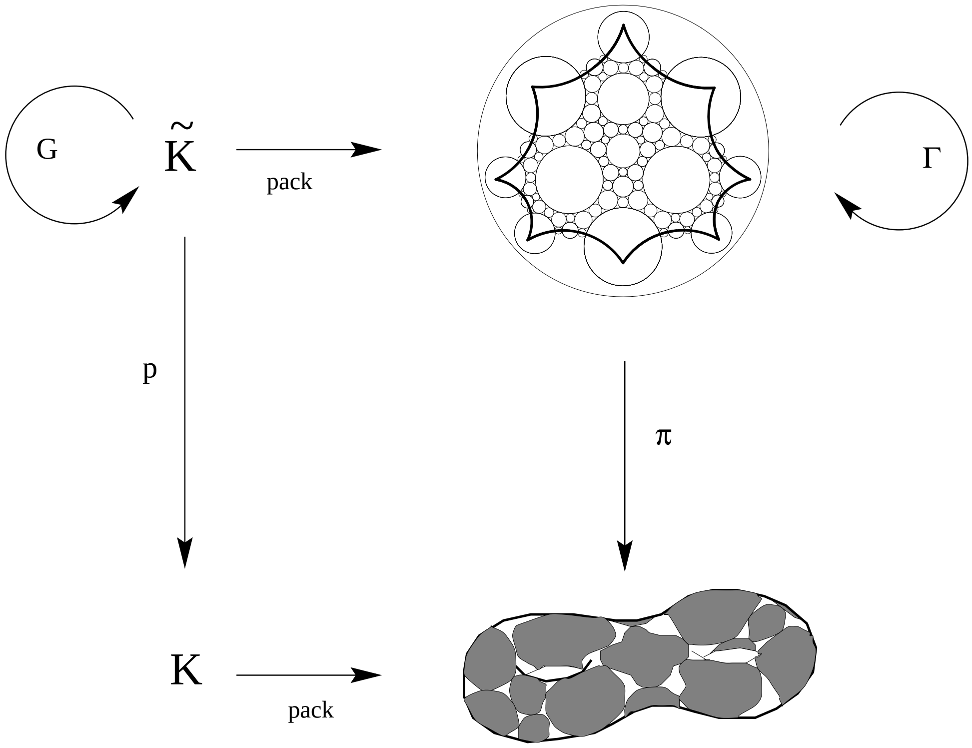


FIGURE 8. Discrete covering map

combinatorics, but from solving packing problems with various prescribed boundary value and branching conditions.

Assume for a moment that $\mathcal{K}$ triangulates a closed topological disc and that $\beta = \{b_1, \cdots, b_k\}$ is a list (perhaps empty, perhaps with repititions) of interior vertices of $\mathcal{K}$ satisfying the following property:

> *If γ is a simple closed edge path in $\mathcal{K}$ with m edges, then γ encloses no more than $(m-1)/2$ points of β.*

Dubejko [Dub95] and Bowers [Bow93] have shown that this purely combinatoric condition is necessary and sufficient for β to be the branch set of a packing for $\mathcal{K}$. Writing $\partial\mathcal{K}$ for the boundary vertices of $\mathcal{K}$, we have

Theorem 3. *Let $\mathcal{K}$, β, and a boundary label $g : \partial\mathcal{K} \longrightarrow (0, \infty)$ be given. Then there exists a unique euclidean packing label R for $\mathcal{K}$ with branch set β and satisfying $R(w) = g(w), w \in \partial\mathcal{K}$. The same result holds in hyperbolic geometry, where g is also allowed to assume the value $+\infty$.*

This means that in principle one can find all euclidean and hyperbolic packings for $\mathcal{K}$. For instance, the maximal packing for $\mathcal{K}$ is simply the hyperbolic packing having empty

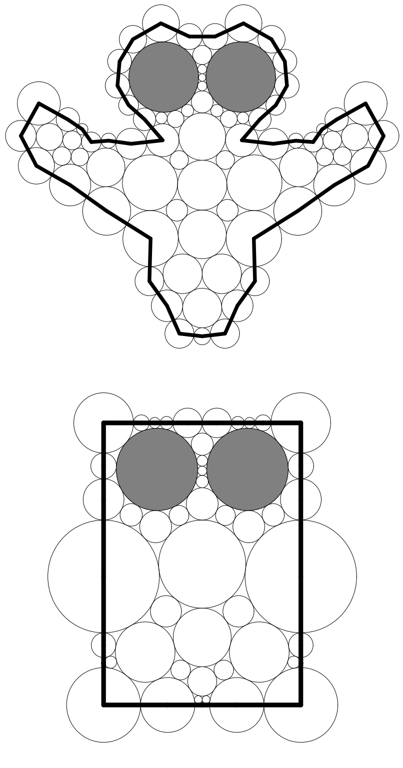


FIGURE 9. Owl abuse

branch set and prescribed boundary radii $g(w) = \infty, w \in \partial\mathcal{K}$. Moreover, in practice, these finite circle packings are computable by methods we discuss in Appendix A, so Theorem 3 is the basis for the practical implementation of circle packing. (One can also solve *boundary angle sum* problems, used, for example, in obtaining Figure 2(c).)

The global malleability of circle packings for $\mathcal{K}$ should not obscure the *local rigidity* implied by the packing condition at each vertex. The local-to-global linkage is moderated by the combinatorics of $\mathcal{K}$, and though we will shortly see a few additional methods for generating circle packings, many fascinating existence and computational issues remain open: existence and computation of branched spherical packings, computation of packings in prescribed regions, univalence criteria for packing labels, packings for infinite complexes, with issues of "ideal" boundary values, infinite branch sets, and so forth.

4. Discrete Analytic Functions

How can one impose some useful order on the potentially huge variety of packings for a given complex $\mathcal{K}$? I propose a *function theory* paradigm, wherein one associates each packing with the range of a mapping from the maximal packing $P_{\mathcal{K}}$. Let me begin with the braodest definition, which we will narrow and then justify through examples.

Definition. *A* ***discrete analytic function*** *is a map $f : Q \longrightarrow P$ between circle packings which preserves tangency and orientation. The corresponding* ***ratio function*** *$f^{\#}$ is defined by $f^{\#}(c) = radius(f(c))/radius(c), c \in Q$.*

There is no loss in restricting to cases where Q and P are packings for the same complex $\mathcal{K}$ and where for each vertex $v \in \mathcal{K}$, the circle for v in Q is mapped by f to the circle for v in P. Specifying further that Q be the maximal packing $P_{\mathcal{K}}$, gives us our

Organizing Principle: *associate each packing P for $\mathcal{K}$ with the discrete analytic function $f : P_{\mathcal{K}} \longrightarrow P$.*

Instead of studying packings *per se*, we are now studying functions, with the richness of analytic function theory as a guide. Moreover, since Q packs $\mathbb{S}^2, \mathbb{C}$, or $\mathbb{D}$, the standard classifications and terminology apply: rational, meromorphic, entire, univalent/multivalent, bounded/unbounded, branched, and so forth, have their usual meanings. Thus, if $\mathcal{K}$ is hyperbolic and P is a packing in $\mathbb{S}^2$, then $f : P_{\mathcal{K}} \longrightarrow P$ would be a *discrete meromorphic* function on the disc.

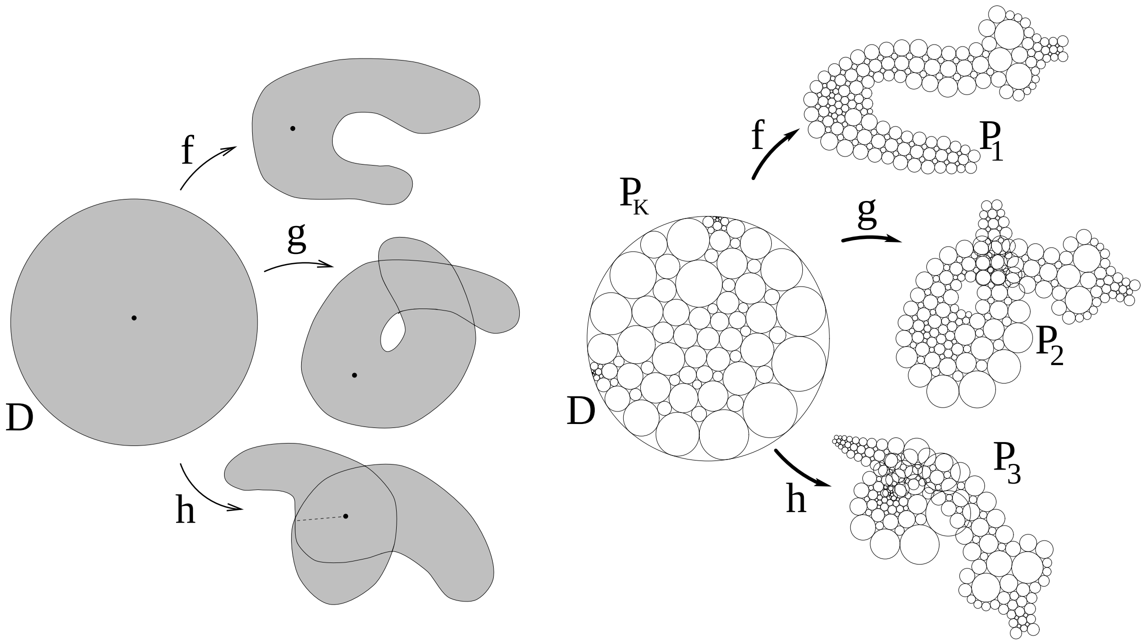


FIGURE 10. Discrete analytic functions on the disc

5. Examples of Discrete Functions

Figure 10 is a cartoon comparing the univalent, locally univalent, and branched discrete analytic functions to their classical models. Let's consider a range of more explicit examples.

Discrete Rational Functions: When $\mathcal{K}$ triangulates a sphere it is easy to show that any circle packing P for $\mathcal{K}$ must lie in $\mathbb{S}^2$. $f : P_{\mathcal{K}} \longrightarrow P$ clearly represents a **discrete rational function**.

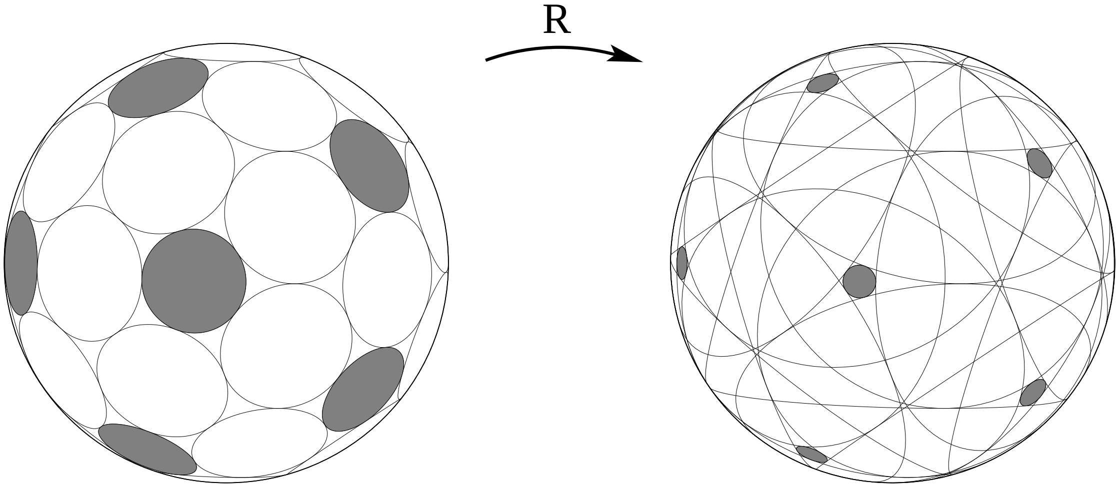


Figure 11. A discrete rational function

With no boundary to consider, packing issues revolve around the branching. An n-degree discrete rational map will, as usual, have $2n-2$ branch points (counting multiplicities). Packings with "polynomial" branching (half their branching at a single vertex) are easily generated, since they can be projected from the hyperbolic setting. However, existence and uniqueness for packings with general branching remain a challenging open questions. Figure 11 displays a 7-fold branched covering of $\mathbb{S}^2$ with 12 simple branch circles (shaded); construction relied heavily on "Schwarz triangles" (see [BoSt96]).

Discrete functions on $\mathbb{D}$: We say $f : P_{\mathcal{K}} \longrightarrow P$ is **defined on the disc** if $P_{\mathcal{K}}$ packs $\mathbb{D}$, that is, if $\mathcal{K}$ is simply connected and hyperbolic. We'll describe two of the many important classes of functions included.

When P is a univalent packing for $\mathcal{K}$ in the plane, as in Figure 12, f is termed a **discrete conformal mapping**. Thurston's 1985 conjecture revolved around such mappings, comparing them to the classical Riemann mapping $F : \mathbb{D} \longrightarrow \Omega$ (as we will see in §7).

The range packing in Figure 12 is simply cut from a regular hexagonal packing. In general, however, constructions of univalent packings are much more problematic,

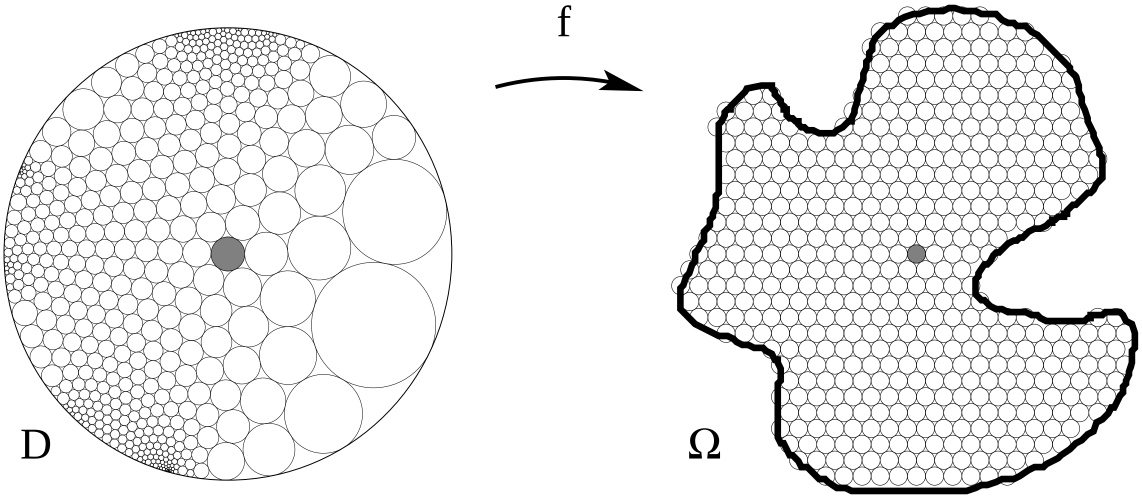


FIGURE 12. A discrete conformal mapping

and practice falls far short of theory. Here, for example, is a remarkable result due to He and Schramm [HeS95b]. Suppose $\mathcal{K}$ triangulates an open topological disc and is hyperbolic. Assume $P_{\mathcal{K}}$ has been normalized to center the circle for v at the origin and the circle for u on the positive x-axis.

Theorem 4. *Let Ω be any proper, open, simply connected subset of the plane, $0 \in \Omega$. Then there exists a unique univalent circle packing P for $\mathcal{K}$ so that c_v is centered at the origin and c_u is centered on the positive x-axis.*

In other words, there is a unique discrete conformal mapping $f_{\mathcal{K}} : \mathbb{D} \longrightarrow \Omega$ with the standard normalization $f(0) = 0$ and "$f'(0) > 0$". This is quite amazing: for example, given *any* Ω, the 7-degree packing of Figure 4 can in theory be repacked to fill Ω! Unfortunately, I can't show you any notrivial examples, since to my knowledge there is no algorithm for *computing* such packings. Even for finite complexes $\mathcal{K}$, there are as yet no *univalence criteria* to tell whether a given packing label will lead, when the circles are laid out, to a univalent packing P.

Foregoing univalence and allowing branching, Theorem 3 provides a huge and computationally accessible variety of discrete analytic functions on the disc. The extreme opposite, in some sense, to the univalent mappings are the proper self-mappings of $\mathbb{D}$, classically the finite Blaschke products. Figure 13 illustrates a **discrete finite Blaschke product** obtained by setting boundary labels to infinity and prescribing 3 simple branch circles (shaded in Figure 13(a)). The image packing in Figure 13(b) covers the disc with multiplicity 4, so I've shown only the chain of boundary circles (wrapping 4 times around $\partial\mathbb{D}$), along with three extremely small dots near the origin, the branched circles.

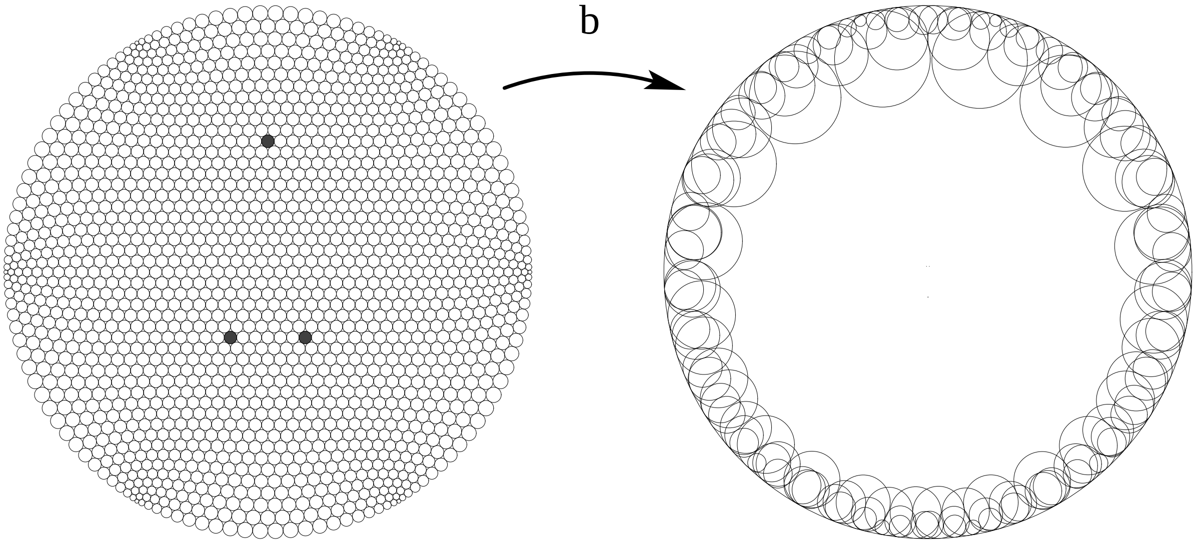


FIGURE 13. A 4-fold discrete Blaschke product

Discrete Entire Functions: When $\mathcal{K}$ is simply connected and parabolic, $P_{\mathcal{K}}$ fills $\mathbb{C}$; a discrete analytic function $f : P_{\mathcal{K}} \longrightarrow P$ would be called **discrete entire** if P lies in $\mathbb{C}$ or **discrete meromorphic** if P lies in $\mathbb{S}^2$. Here are some examples.

Among the most pleasing infinite packings are the **Doyle spirals**, Figure 1(d) being but one of a two-parameter family of such spirals. Based on the hexagonal complex H, these packings exist by virtue of the symmetries of H and a special scaling noted by Peter Doyle. A discrete entire function from P_H to the Doyle spiral P of Figure 1(d) should clearly be considered as a **discrete exponential**, since one can observe familiar properties — periodicity (note: every circle of P has infinitely many preimages in P_H), nonvanishing, local univalence, growth of $f^{\#}$, and so forth. Creating infinite packings P — or even proving existence — is a significant challenge, even for the hexagonal complex H. Open question: *do there exist any locally univalent packings of H other than P_H and the Doyle spirals?*

Exploiting symmetry in a different way leads to a **discrete sine** function. A regular hexagonal packing of a half plane, Figure 14(b), has been repacked to fill a half-infinite strip in Figure 14(a). Reflections in and identifications along the edges in (a) lead to a univalent packing of the plane, *a fortiori* a maximal packing $P_{\mathcal{K}}$. The natural identification of (a) with (b) can be extended *a la* "Schwartz reflection" to a discrete entire function with domain $P_{\mathcal{K}}$. This map will have precisely the geometric mapping properties of $z \mapsto \sin(z)$; in particular, note that the construction gives 2π periodicity, with the circles at points $\pi/2 + n\pi$ in (a) becoming simple branch circles with image circles at ± 1.

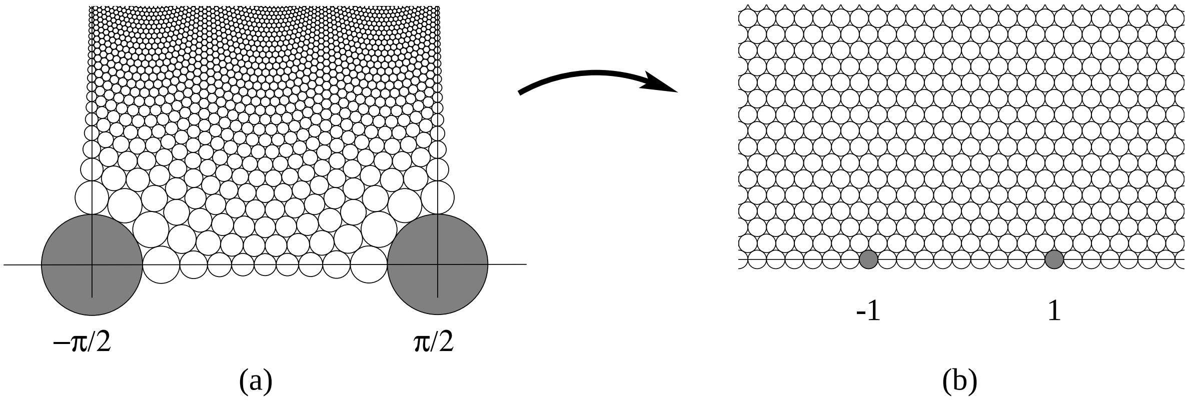


Figure 14. A discrete sine function

Only a few more general construction techniques are known; Figure 15 illustrates two approaches to the construction of **discrete polynomials**. Dubejko [Dub97d] specifies branch points for packings of the hexagonal complex H. A finite stage in constructing the analogue of $P : z \mapsto z(z^2 - 1)$ is shown in the top of Figure 15; note the shaded branch circles on the left. Valence considerations are key to proving that the finite stage packings converge to a packing for H itself.

Alternately, one can construct discrete polynomials with prescribed branch *values* by directly constructing their image packings, a capability not available in the classical setting. Figure 15 shows a finite stage in constructing a polynomial Q of degree 3. The image surface is built from three (truncated) copies of the regular hexagonal packing which have been slit and cross-connected along curves from circles over the desired branch values. With the branched image packing comes a new simply connected complex whose maximal packing (appropriately scaled) becomes the domain. Q is the composition of ψ with the projection π.

Discrete Conformal Structures: By the Circle Packing Theorem, combinatorics determines conformal structure. For instance, Figure 16(a), with the indicated side pairings, represents a genus 2 surface S. The additional markings define what is known as a "dessin" and lead in a cannonical way to a triangulation $\mathcal{K}$ of S. There exists a unique conformal structure on S which supports (in its intrinsic metric) a circle packing $P_{\mathcal{K}}$ for $\mathcal{K}$; that Riemann surface is laid out as a fundamental domain in $\mathbb{D}$ in Figure 16(b), with the carrier of the packing included for reference. Note that this represents a classical conformal structure — a unique point in Teichmüller space $\mathbf{T}_2$; we use the adjective "discrete" simply to emphasis that it is determined by the abstract combinatorics of $\mathcal{K}$.

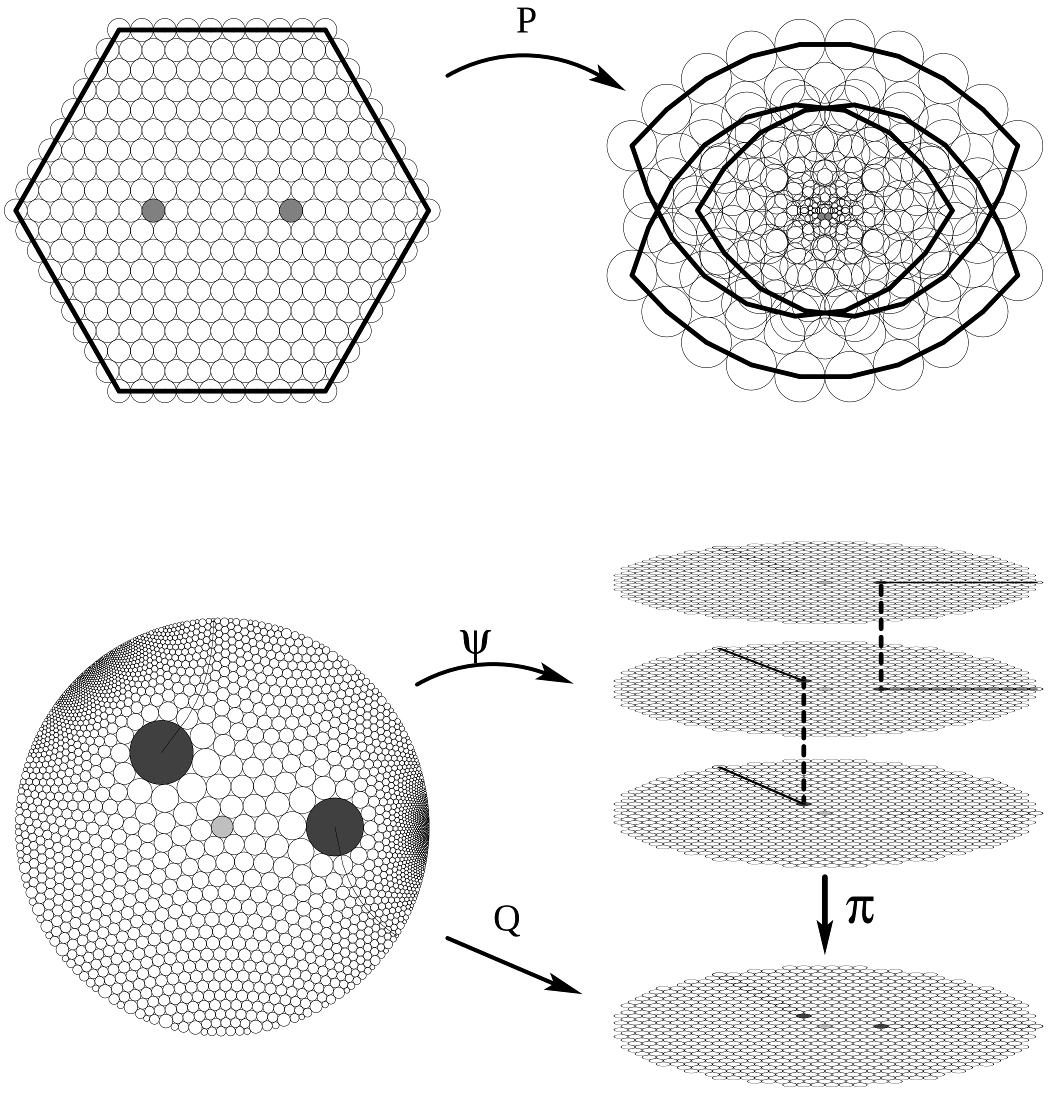


FIGURE 15. Discrete polynomials

6. EXAMPLES OF DISCRETE THEORY

Were this topic mere mimickry, it would not deserve our long term attention. To understand the deeper connections, let's be a little more explicit about discrete analyticicty. First, as to the intrinsic structure on domains implied by the Circle Packin Theorem, we might paraphrase the fundamental analogy:

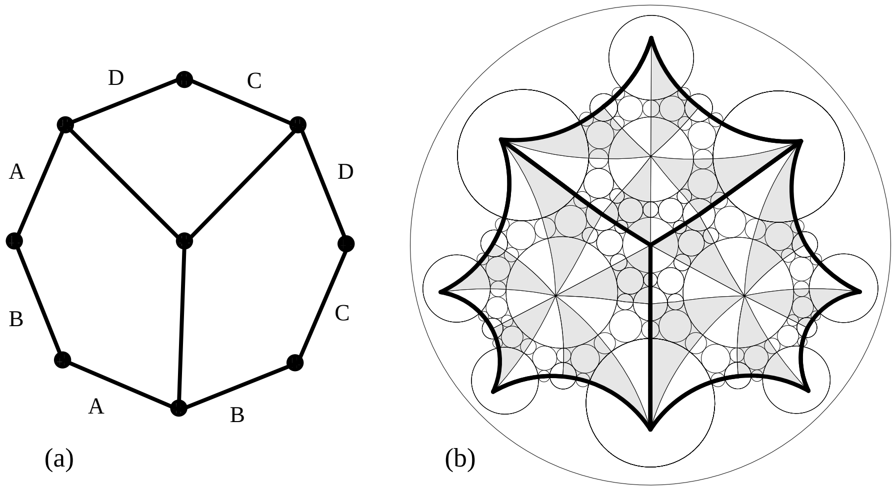


FIGURE 16. A discrete conformal structure

"*A Riemann surface S has a* ***conformal structure*** *which determines an* ***infinitesimal metric*** *of constant curvature, while a complex $\mathcal{K}$ has a* ***combinatorial structure*** *which determines a* ***discrete metric*** *of constant curvature.*"

Likewise, the notion of discrete analyticity of mappings is not so very far from our experience: a familiar saying has it that an analytic function is one which "maps infinitesimal circles to infinitesimal circles". The discrete versions simply operate instead on *real* circles; where $|f'|$ measures the stretching or shrinking of infinitesimal circles, the ratio function $f^{\#}$ measures the stretching/shrinking of *real* circles. The reader can not go far wrong by using this "infinitesimal *versus* real" analogy to transfer classical geometric intuition to the discrete setting.

In this spirit, the Circle Packing Theorem of §3 applied to simply connected complexes clearly qualifies as the **Discrete Uniformization Theorem**, while for non-simply connected $\mathcal{K}$, it serves as the **Discrete Covering Theorem**. Within the proof, one can isolate two particularly fundamental results.

The Schwarz-Pick Lemma lies behind a huge portion of the classical theory of functions, and one might anticipate an equally important role for its discrete version.

Discrete Schwarz-Pick Lemma. *If $f : \mathbb{D} \longrightarrow \mathbb{D}$ is a discrete analytic function, then f is a contraction in the hyperbolic metric. In particular, if $f(0) = 0$, then $f^{\#}(0) \leq 1$, with equality if and only if f is a Möbius transformation.*

Let me interpret: For some simply connected hyperbolic complex $\mathcal{K}$, f maps $P_{\mathcal{K}} \subset \mathbb{D}$ to a packing $P \subset \mathbb{D}$ for $\mathcal{K}$. $f(0) = 0$ indicates that a circle c_v is centered at the origin in $P_{\mathcal{K}}$ and $f(c_v)$ is centered at the origin in P. Assuming $P_{\mathcal{K}} \leftrightarrow \mathcal{K}(R_{\mathcal{K}})$ and $P \leftrightarrow \mathcal{K}(R)$, we observed earlier that $R \leq R_{\mathcal{K}}$. Applied to v, that immediately gives $f^{\#}(c_v) \leq 1$ (i.e., $f^{\#}(0) \leq 1$), the *Schwarz* portion of the result. More subtly, if $d(\cdot,\cdot)$ denotes the distance between circle centers, then it is the case that for any circles c_u, c_w of $P_{\mathcal{K}}$, $d(f(c_u), f(c_w)) \leq d(c_u, c_w)$. This is the *Pick*, or *hyperbolic contraction* portion of the result.

When $\mathcal{K}$ is simply connected and parabolic, the criterion for the hyperbolic/parabolic dichotomy shows that there can be no non-trivial hyperbolic packing label for $\mathcal{K}$ — that is, there can be no circle packing for $\mathcal{K}$ lying in $\mathbb{D}$. This gives

Discrete Liouville Theorem. *There are no bounded discrete entire functions.*

(In case the reader anticipated the adjective "nonconstant" in the statement, note that there seems to be no natural notion for constant function in the discrete theory.) An alternate statement, proven by Dubejko [Dub97e] with probabilistic methods under the assumption that $\mathcal{K}$ has bounded degree, is that if $f : P_{\mathcal{K}} \longrightarrow P$ is entire and $f^{\#}$ is bounded, then $f^{\#}$ is constant; that is, a discrete entire function with bounded derivative is linear.

There are several other geometric results, standard in classical analytic function theory, which have substantially parallel discrete versions. Formal statements involve more details than we have room for here, so I'll simply mention the **Discrete Picard Theorem** [CR93], a **Discrete Distortion Lemma** [DSt95a], companion to the Schwarz-Pick result, **Discrete Maximum Principles** for f and $f^{\#}$, and a **Discrete Löwner Theorem** for self-maps of the disc. There are also valuable random walk methods, the analogue of Brownian motion, which we discuss later, and the powerful discrete winding number techniques of He and Schramm [HeS93].

7. Comment on Approximation

If anything could lend more substance to the analogies we have been discussing, it is the companion approximation results. Under appropriate refinement of the circle packings involved, virtually all of our discrete analytic objects converge to their classical models. Though we are concentrating on analogy, I would be remiss if I did not give at least the flavor of *approximation*, so let me illustrate Thurston's original conjecture.

Return to the situation pictured in Figure 12, where we used a Jordan region Ω to cut out a finite piece of a regular hexagonal packing of circles. Supposing these circles have radii 2^{-n}, we will write P_n for this finite packing in Ω, $\mathcal{K}_n$ for its complex, and $f_n : P_{\mathcal{K}_n} \longrightarrow P_n$ for the associated discrete conformal mapping from $\mathbb{D}$ to Ω (with appropriate normalizations). Figure 17 repeats the construction for two successively larger values of n — a typical "refinement" process.

Thurston conjectured and Rodin and Sullivan proved that the discrete conformal maps f_n converge uniformly on compact subsets of $\mathbb{D}$ to the classical conformal mapping $F : \mathbb{D} \longrightarrow \Omega$ as $n \to \infty$. In recent work, He and Schramm have removed quasiconformal

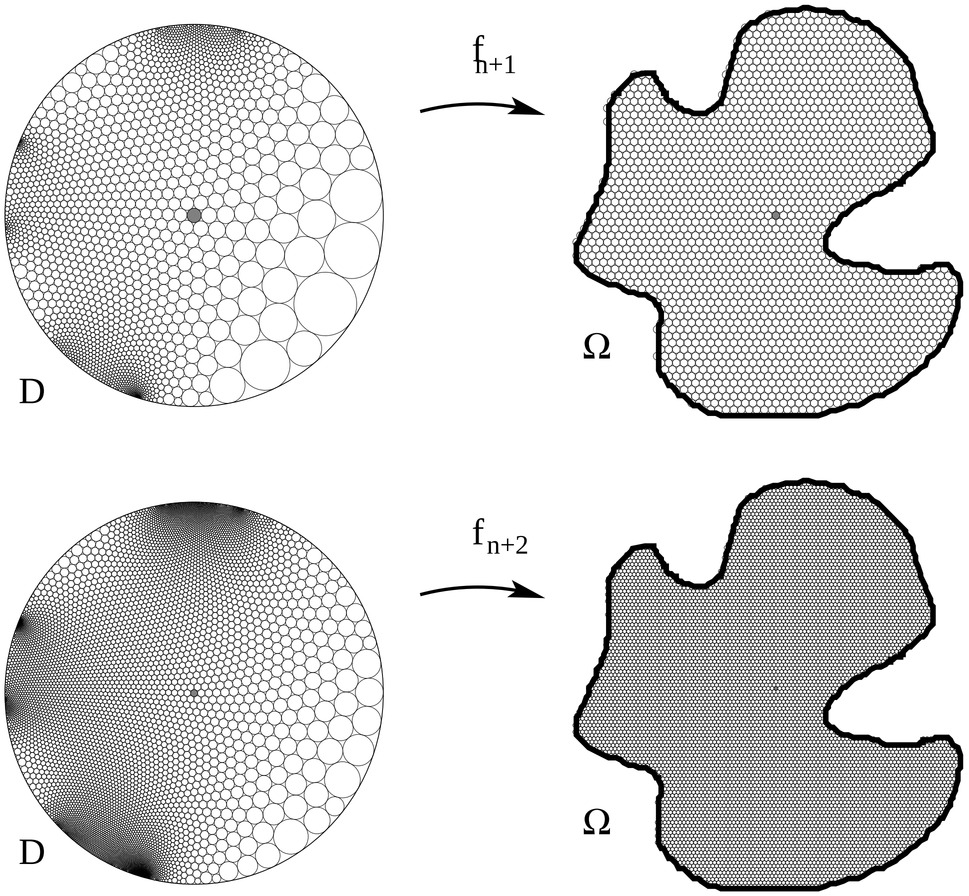


FIGURE 17. Illustrating Thurston's Conjecture

notions in certain key arguments, so the proof of Thurston's Conjecture now provides an independent *proof* of the Riemann Mapping Theorem!

Approximation results have been extended far beyond the conformal mapping setting, including approximation not only of analytic functions but also of conformal structures. See Appendix A for brief remarks on computational aspects of circle packing, and the survey [St99] for more details on approximation.

Part II: Discrete Conformal Geometry

At this juncture, I hope the reader has successfully transferred some geometric intuition from the classical setting to our new discrete world. I'll rely on that now as we investigate various standard geometric topics and see how they are manifest in the setting of circle packings.

1. The "Type" Problem

The classical "type" problem presents us with an open and simply connected Riemann surface, defined in some fairly explicit form, and asks whether it is conformally equivalent to (is the *type* of) the plane or the unit disc.

Discrete Type Problem. *Given a specific triangulation $\mathcal{K}$ of an open topological disc, is $\mathcal{K}$ parabolic or hyperbolic?*

The constant m-degree comlexes $\mathcal{K}^m$ are instructive here. Cases $m = 5, 6$, and 7 are shown in Figure 4. The reader can confirm the local geometry on a flat table by contemplating construction of an m-flower from $m + 1$ equal sized discs. The familiar 6-petal flower will lie perfectly flat, whereas the petals of a 5-petal flower must be drawn up if they are to form a closed chain, leading to positive curvature, while the petals of a 7-petal flower must be drawn alternately up and down, giving the saddle shape characteristic of negative curvature. As a rule of thumb, *degree 6 vertices tend to be flat (euclidean), degree* 5 *and less tend to be spherical, and degree* 7 *and more, hyperbolic.*

Global effects when the degrees are mixed are much more subtle. Various type conditions have been established, many reminiscent of those encountered elsewhere, e.g., in discrete potential and graph theory. Perhaps one of the cleanest is due to Gareth McCaughan [McC98]:

Theorem 5. *Let $\mathcal{K}$ be an infinite simply connected complex with a uniform upper bound d on the degree of its vertices and consider the simple random walk on the 1-skeleton of $\mathcal{K}$. Then $\mathcal{K}$ is parabolic if the random walk is recurrent and hyperbolic if the random walk is transient.*

This is a discrete version of the classical Kakutani type condition: *A simply connected Riemann surface R is parabolic if Brownian motion on R is recurrent, and hyperbolic if Brownian motion is transient.* In general terms, a complex tends to be hyperbolic if it has lots of "room" at infinity, for instance if the number of vertices n generations from some fixed vertex v_0 grows rapidly with n or if the average degree of the vertices exceeds six or if areas grow slowly compared to their perimeters.

Motivated by the dichotomy between packings of constant 6 and 7-degree complexes, let us arrange a mixture. Attach a half-plane of 6-degree combinatorics to a half-plane of 7-degree combinatorics to get the complex $\mathcal{K}$ suggested by Figure 18(a). The maximal packing lies in $\mathbb{D}$; the circles on the dividing line between the 6 and 7-degree regions are shaded in the figure, and one sees that the 6-degree circles are forced into a horocyclic region. Applying McCaughan's criterion, transience is clearly winning out: a random walker stumbling into the degree 7 side risks getting lost and never returning to its starting point, so the walk is transient.

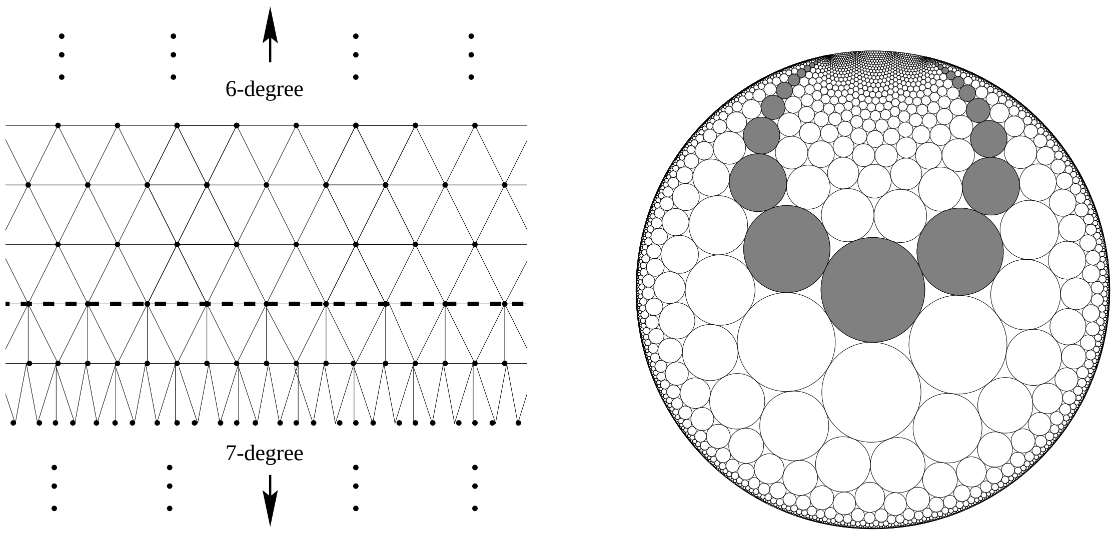


FIGURE 18. A split 6/7-degree complex

2. Harmonic Measure

Of the last example, one might well observe that the "flat half" of $\mathcal{K}$ has a null *harmonic boundary* — full harmonic measure on the "ideal boundary" of $\mathcal{K}$ resides on the boundary of the 7-degree portion, as suggested by the maximal packing. Boundary influences, mediated by combinatorics, are rife in this topic, and though not yet quantified, they strongly resemble the classical influence of "harmonic measure". Careful examination of Figure 12, for example, would show that circles "closer" to the origin in Ω become (euclideanly) larger in the maximal packing in $\mathbb{D}$ compared to circles which in Ω are further from or obscured from the origin. Probability theory holds some hope of quantifying these effects

Given a Jordan region Λ, classical harmonic measure μ (with respect to the $0 \in \Lambda$) is defined for Borel subsets E of $\partial\Lambda$ as the probability that a Brownian particle starting at the origin will first exit Λ at a point of E. To illustrate a discrete version, look to Λ in Figure 1(f), with packing P, complex $\mathcal{K}$, and interior circle c_v centered at the origin. Given a random walk $\mathcal{W}$ on the circles of P (e.g., on the embedding of the 1-skeleton of $\mathcal{K}$), there is an associated hitting probability $\mu_{\mathcal{W}}$ on the boundary circles analogous to μ. The general questions: *How closely are $\mu_{\mathcal{W}}$ and μ related? Can $\mu_{\mathcal{W}}$ play the roles in discrete function theory that μ plays in the classical?*

A first step, of course, is to decide *which* random walk $\mathcal{W}$ to consider — a walk is determined by local "transition probabilities", yet we would like it to reflect global behavior. In fact, there are walks closely attuned to packing geometry.

In a euclidean packing P, consider an edge $\langle u, v\rangle$, as pictured in Figure 19. Each of the triples of circles to which c_u and c_v belong determines a mutually orthogonal "dual" circle. Let r_u, r_v be the radii of the circles c_u, c_v and let a, b be the radii of the dual circles. Define the **conductance** of the edge $\langle u, v\rangle$ as

$$C_{u,v} = \frac{a+b}{r_u + r_v}.$$

Conductances on all the edges determine, in a standard way, a random walk on P, and we'll refer to the walk resulting from these particular conductances as the **tailored random walk** $\mathcal{W}_P$ on P.

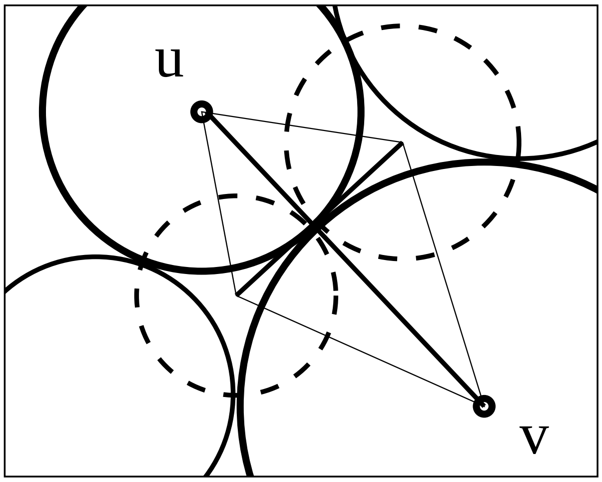


Figure 19. Geometry of conductances

I think tailored random walks may encode probabilistically the local-to-global transition that packings themselves seem to accomplish. In fact, these walks were inspired by observations of the packing process itself. Developed first in the hyperbolic setting, they model the flow of hyperbolic angles and area due to perturbations of boundary labels and were used in a probabilistic proof of Thurston's Conjecture [St96]. The euclidean tailored walks were developed and applied by Dubejko [Dub97a], who discovered the formulation given here.

3. Potential Theory

There is a considerable literature on discrete harmonic functions, and in the presence of these harmonic influences, one might anticipate some notion of discrete harmonic function. For example, every random walk is associated with a **discrete Laplacian** and corresponding **discrete harmonic functions**. To show that $\mathcal{W}_P$ has some claim to being the geometrically "correct" random walk for P, consider the "center function" $Z_P : \mathcal{K}^{(0)} \longrightarrow \mathbb{C}$. That is, for each vertex v of $\mathcal{K}$, $Z_P(v)$ is the center of the circle c_v of P.

Theorem 6. *Let P be a euclidean circle packing, Z_P its center function, and $\mathcal{W}_P$ its tailored random walk. Then Z_P is a harmonic function for $\mathcal{W}_P$.*

This is a very modest beginning. Blending harmonic and analytic notions in this discrete setting seems a highly challenging business. Is there, for example, some sense in which the real and imaginary parts of discrete analytic functions are "harmonic"?

4. Extremal Length

A conformal quadrilateral Q is a conformal disc with four distinguished boundary points; extremal length $\mathrm{EL}(A, B; Q)$ measures in a conformally invariant way the distance between opposite ends of Q. In analogy, a combinatorial quadrilateral $\mathcal{Q} = \{\mathcal{K}; a, b, c, d\}$ is a triangulation $\mathcal{K}$ of a closed disc with four distinguished boundary vertices; an example is shown in Figure 20(a), with ends A, B. The reader will recognize the familiar quantities going into extremal length in this combinatorial version due to Duffin:

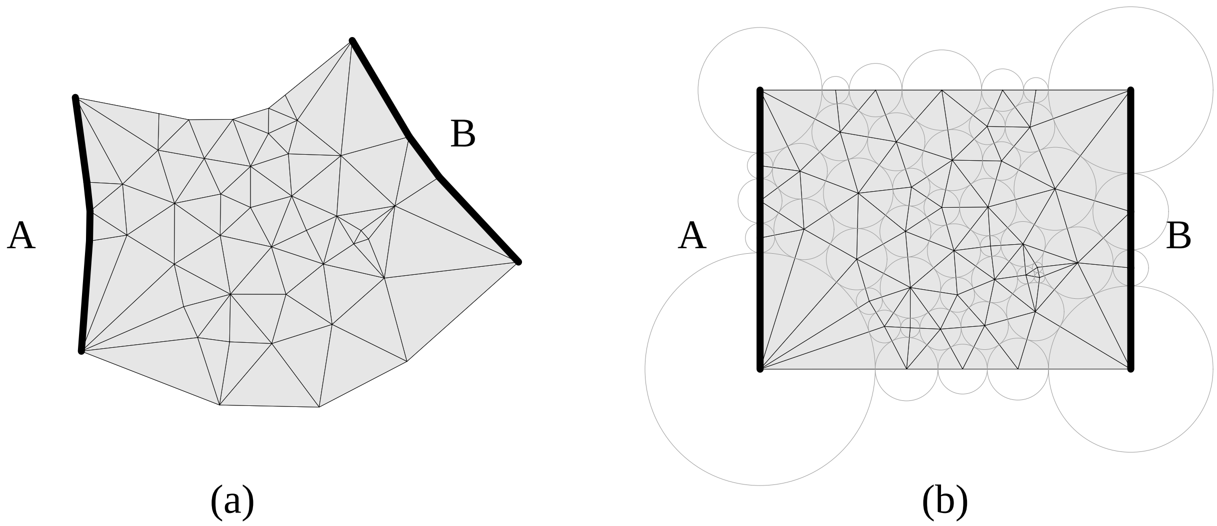


Figure 20. A combinatorial and a packed quadrilateral

Definition. *Let Γ denote the edge-paths γ of $\mathcal{Q}$ connecting A to B. For any set $W = \{w_j\}$ of nonnegative "weights" assigned to the vertices v_j of $\mathcal{K}$, let $L_W(\Gamma)$ denote the* *length of* Γ, $L_W = \inf_\gamma \sum_{v_j \in \gamma} w_j$, *and let* $Area(W) = \sum_j w_j^2$. *Then the* ***vertex extremal length*** *of* Γ *is defined by*

$$VEL(A, B; \mathcal{Q}) = \sup_W \Big(\frac{L_W^2(\Gamma)}{Area(W)}\Big).$$

It is pleasing to discover concrete geometry here. For instance, $\mathcal{Q}$ can be represented as a "squared rectangle", a rectangle packed with squares having the extremal weights w_j as edge lengths (see [CFP94, Sch93]). Of course, our inclination would be to pack $\mathcal{Q}$ instead. Prescribing angle sums $\pi/2$ at the vertices a, b, c, d and π at the remaining boundary vertices leads to a euclidean circle packing P for $\mathcal{K}$ having a rectangular carrier $R_{\mathcal{Q}}$ with corners a, b, c, d, as in Figure 20(b). The rectangle $R_{\mathcal{Q}}$ is determined up to similarity by combinatorics, allowing us to define the **packing extremal length** of $\mathcal{Q}$ as the aspect ratio

$$\mathrm{PEL}(A, B; \mathcal{Q}) = \frac{\mathrm{length}(R_{\mathcal{Q}})}{\mathrm{height}(R_{\mathcal{Q}})}.$$

Typically $\mathrm{VEL}(A, B; \mathcal{Q}) \sim \mathrm{PEL}(A, B; \mathcal{Q})$ (meaning that these discrete extremal lengths are "comparable" within constants depending only on the degree of $\mathcal{Q}$.) However, PEL, besides being more computationally accessible, also parallels the classical geometry associated with EL.

To see what I mean, let's return to Figure 20. Let Q denote the conformal quadrilateral on the left; in fact, Q is the carrier of a circle packing P_1 (not shown). Classical considerations begin with a conformal mapping $F : Q \longrightarrow R_Q$, where R_Q is a euclidean quadrilateral. Likewise, we have the discrete conformal mapping $f : P_1 \longrightarrow P$, treated as a map of $Q = \mathrm{carr}\, P_1$ to the euclidean quadrilateral $R_{\mathcal{Q}} = \mathrm{carr}\, P$. Classically, $EL(A, B; \Omega)$ is the aspect ratio of R_Q and the extremal density on Q is $|F'|$. Discretely, $\mathrm{PEL}(A, B; \mathcal{Q})$ is the aspect ratio of $R_{\mathcal{Q}}$ and the discrete extremal density (what you multiply radii in P_1 by to get the radii of P) is precisely $f^{\#}$. Furthermore, due to the Rodin/Sullivan Ring Lemma, $\mathrm{EL}(A, B; \Omega) \sim \mathrm{PEL}(A, B; \mathcal{Q})$. In other words, PEL brings with it discrete "conformal invariance", the true hallmark of extremal length. Similar considerations apply to discrete extremal length in the settings of annuli, tori, and so forth.

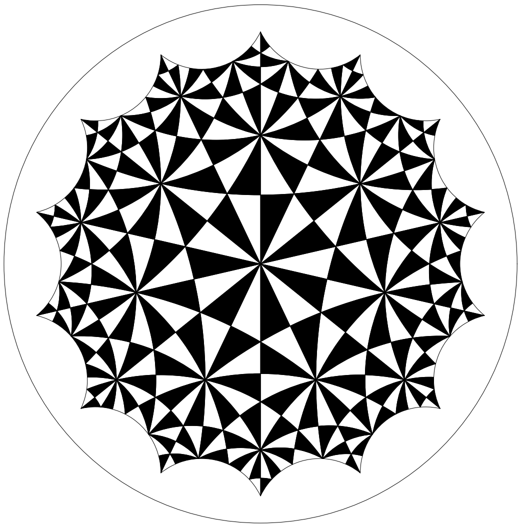
5. "Packable" Surfaces and Dessins

According to the Circle Packing Theorem, each complex $\mathcal{K}$ "chooses" its geometry; that is, there is a unique conformal structure on $\mathcal{K}$ so that the resulting Riemann surface $\mathcal{X}$ supports $P_{\mathcal{K}}$. We'll say in this case that $\mathcal{X}$ is **packable**.

Since the number of complexes $\mathcal{K}$ is countable, there exist at most countably many packable surfaces. Which ones are they? This is largely a mystery, with a few results but mostly intriguing open questions. By results of Brooks [Br86] and Bowers and Stephenson [BoSt92, BoSt93], we have

Theorem 7. *The packable Riemann surfaces are dense in Teichmüller space for surfaces of finite topological type $T_{g,n,m}$.*

Let's illustrate two classical surfaces of genus 3, the Klein and Picard surfaces. These have intrinsic triangulations which can be circle packed to give the fundamental domains shown in Figure 21 on the left and right, respectively (side-pairings are not shown). The ubiquitous symmetries of the Klein surface imply that this discrete version is in fact exact and the Klein surface is packable — indeed, our picture duplicates a century-old image from Fricke/Klein [FK97]. Our discrete Picard surface, with the circles shown for reference, is most likely only an approximation and I don't know if the Picard surface itself is packable.



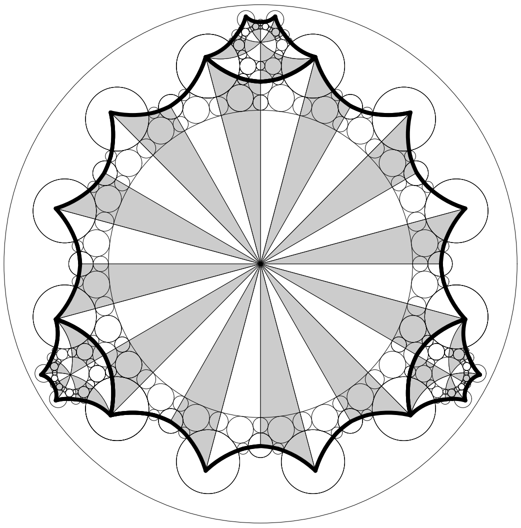


FIGURE 21. Discrete Klein and Picard surfaces

These issues are of more than casual interest. Conformal structures are central to Grothendieck's theory of *dessins d'enfants*, which establishes connections among "drawings", triangulations, conformal structures, Belyĭ maps (meromorphic functions branching over at most three points), and number fields. The parallel discrete theory based on circle packing is developed in [BoSt97]. Like the packable surfaces, the "dessin" surfaces are dense in Teichmüller space, but the latter have an algebraic characterization. Is there an analogous characterization for packable surfaces? McCaughan [McC96] has shown that a packable 1-torus will have an algebraic modulus, while a packable n-torus, $n \geq 2$, will have a Möbius covering group which is conjugate to one having (complex) algebraic entries. The converse statements remain open.

Figure 22 illustrates a genus 0 dessin (the dark graph), the associated triangulation (shaded/unshaded), and the discrete Belyĭ map (i.e., discrete rational function). The triangulation would converge to the conformally correct one if we were to refine the circle packings.

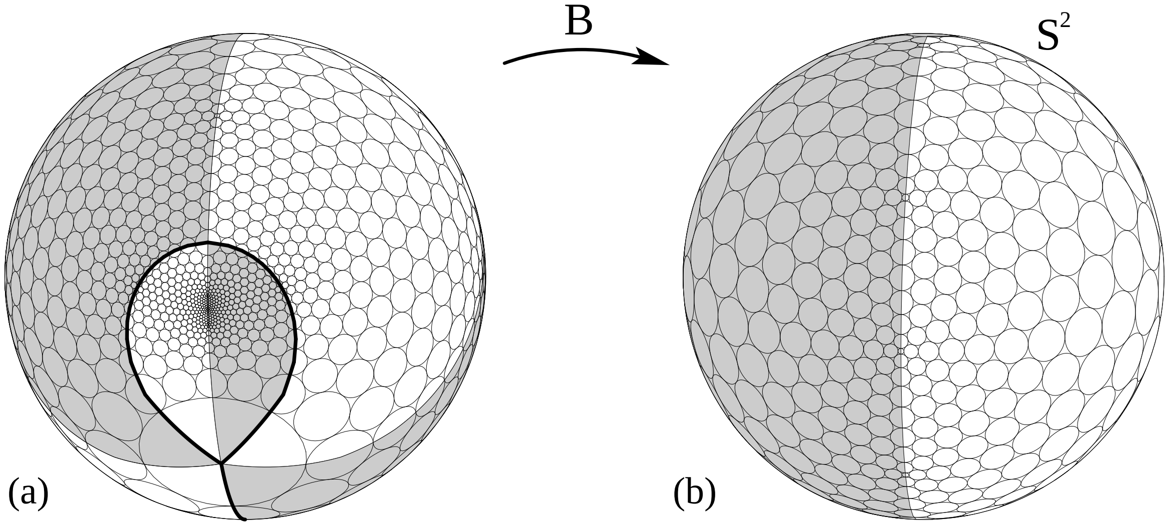


FIGURE 22. The Belyĭ map for a genus 0 dessin

6. Conformal Welding

Let $\phi : \partial\mathbb{D} \longrightarrow \partial\mathbb{D}$ be an orientation preserving homeomorphism. Attach two copies of $\mathbb{D}$, say $\mathbb{D}_1$ and $\mathbb{D}_2$, along their boundaries by identifying $z \in \partial\mathbb{D}_1$ with $w = \phi(z) \in \partial\mathbb{D}_2$. The result is a topological sphere. Pump it full of air to form a sphere and see where the seam Γ between the two discs settles. ϕ is called a "welding homeomorphism", and the Jordan curve Γ is its "welding curve". The well-studied correspondence $\phi \leftrightarrow \Gamma$ is (essentially) a bijection between the class of quasisymmetries ϕ of $\partial\mathbb{D}$ and quasicircles Γ.

We can see the analogous theory, as formulated by Williams [Wil99b], in the pictures of Figure 23. In (a), a triangulation $\mathcal{K}$ of a disc has been doubled across its boundary and the result packed in the sphere (the Koebe-Andreev-Thurston Theorem). This symmetric packing is now broken at the equator in (b), giving the two triangulations $\mathcal{K}_1$ and $\mathcal{K}_2$, both isomorphic to $\mathcal{K}$. $\mathcal{K}_1$ and $\mathcal{K}_2$ are re-identified along their boundaries in (c), but now using (a discretization of) ϕ, interpolating extra boundary vertices as necessary. The resulting triangulation of a topological sphere is packed in $\mathbb{S}^2$ (Koebe-Andreev-Thurston again), giving (d). The common boundary Γ of the two triangulations in their new locations — the **discrete welding curve** — is the heavy sinuous line in (d). Maps f and g carrying circles in the northern and southern hemispheres of (a) to the regions above and below Γ in (d), respectively, are discrete conformal mappings. The reader familiar with classical conformal welding will find precise parallels here: If ϕ is a k-quasisymmetry, then Γ is a $\hat{k}$-quasicircle ($\hat{k}$ depending on k) and under refinement such discrete welding curves converge to the classical welding curve for ϕ, while the discrete conformal maps f and g of the hemispheres converge to the classical conformal maps F and G satisfying $F \equiv G \circ \phi$ on $\mathbb{D}$.

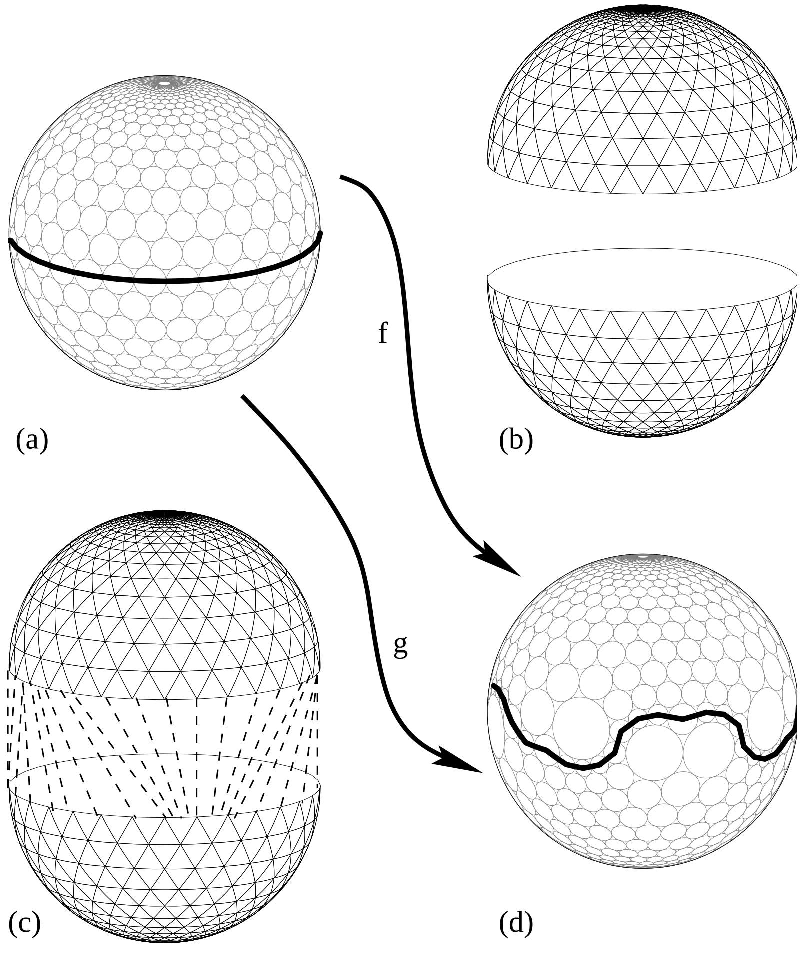


FIGURE 23. A discrete conformal welding

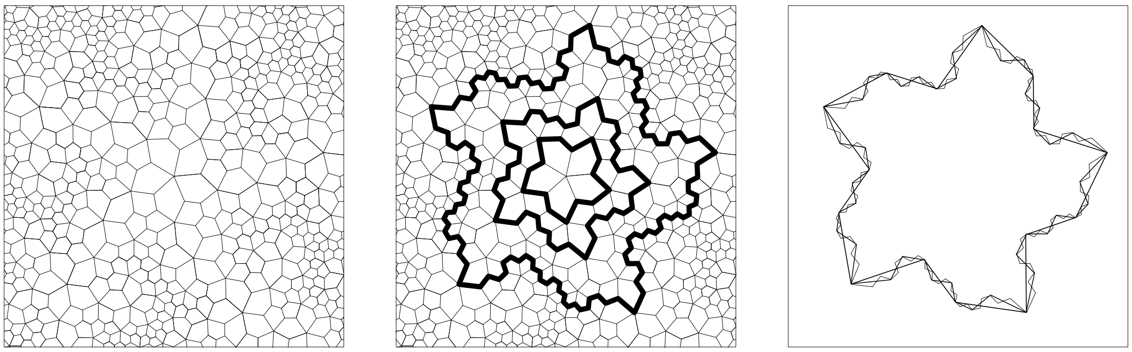
7. Classical Analysis Issues

We've seen several instances now of the close parallels between the discrete objects of circle packing and the familiar objects of classical analysis and conformal geometry. I would like to conclude this survey by emphasizing that the connections are a two-way street.

Circle packing brings to the classical theory a significant experimental capability, new methods of approximation, and a flexible visualization tool. It also has the potential to suggest new ideas, as it apparently did for He and Schramm, who in [HeS93] took a major step towards solving Koebe's Uniformization Conjecture using key ideas from their circle packing work.

In the other direction, circle packing theory has obviously benefited by following the rich classical model. That in turn has allowed it to carry notions from complex analysis into new topics. Graph embedding is a good example: any locally planar graph can be endowed with an intrinsic geometry *via* circle packing. In addition to ease of computation and visual appeal, these embeddings are often useful precisely because of their discrete conformal natures, for example in obtaining estimates for graph "separators" in [MTh90], in estimating graph "resolution" in [MP94, AS97], and in proving existence of finite dirichlet functions in [BSc96].

Let's wrap up with a final example illustrating these mutual influences. The topic is "conformal tiling", a notion introduced by Bowers and me in [BoSt97]. The original motivation came from the abstract combinatorial patterns of Jim Cannon, Bill Floyd, and Walter Parry (see [CFP96]) arising in their investigation of Thurston's Geometrization Conjecture. The pentagonal tiling of Figure 24(a) is one example.



(a) (b) (c)

Figure 24. Structure in a conformal tiling

Circle packing began simply as a convenient embedding device for intricate graphs. Once concrete images were in hand, however, notions of discrete conformality quickly pointed to a classical formulation. The requisite theory was in fact available a century ago, but now the visual and experimental capabilities of circle packing kicked in — only with circle packing could one study specific examples in depth and develop essential insights. Unexpected structural features were observed and conjectures made; some have been confirmed using classical theory, others remain tantalizingly open.

Consider, for instance, the tiling of Figure 24(a). This is associated with a "twisted pentagonal" subdivision rule and though the circles themselves are not shown, this embedding is obtained from a circle packing. Relevant "aggregate" tiles arising in the combinatorics have been outlined in (b). Close observation suggests an underlying self-scaling, which seems to be confirmed when the outlines are rotated, scaled by $\lambda \approx 1.82$, and overlayed in (c) — the corners of each aggregate tile fall directly on corners at the next aggregation level. Motivated by these experimental images, Cannon, Floyd, and Parry were lead first to a proof of scaling and then, with Rick Kenyon, to connections with the dynamics of rational functions leading to the exact scaling factor $\lambda = |(25/16(5 + 3\sqrt{-15})|^{1/5} \approx 1.8162516$.

This example concludes our survey and perhaps shows best the potential for synergy between the familiar classical notions of analytic function theory and this new discrete realization in terms of circle packing.

8. Appendix A: Computational notes

The effective computability of circle packings and the availability of software for creating, manipulating, and displaying them have been of central importance both in the theory and the application of circle packing. Here are some brief remarks on computational issues.

The Algorithm: The computational task in circle packing involves approximation of packing labels. Thurston suggested an iterative scheme: start with an initial (perhaps even random) label R_0, repeatedly revisit individual vertices of $\mathcal{K}$, on each visit adjusting the label of that vertex so that it is compatible with its neighbors — that is, so its angle sum takes the intended value (typically 2π or a multiple of 2π at branch points). This scheme has been implemented and refined and works remarkably well in the euclidean and hyperbolic settings (monotonicity is absent in the spherical setting, and no one yet has a spherical algorithm). A generic boundary value problem involving, say, 5000 circles, will typically "pack" in a few seconds on a personal computer, while packings of over 200,000 circles may take a few hours, but are routine. See [] for a discussion of the algorithm and its implementation. Packings of multiply connected complexes $\mathcal{K}$, e.g. Figures 16, 21, are done intrinsically and then automatically laid out as fundamental domains in $\mathbb{D}$ or $\mathbb{C}$, as appropriate. These computations are often remarkably fast.

Software: The packing algorithm is implemented in my comprehensive software package called `CirclePack`. This package is freely available over the web, but currently operates only under X-Windows on Unix machines (e.g., under *linux*). It allows for the creation, storage, manipulation, analysis, display, and printing of circle packings. Many parts of the theory have come directly from live experiments, from observing unanticipated phenomena and searching for explanations, often motivated by the emerging connections with classical analytic function theory and conformal geometry.

Circle Packing Maps: The definition of a discrete analytic function $f : Q \longrightarrow P$ as a map between collections of circles is rather abstract. When more concrete point mappings are required, e.g., with approximations, one can formulate $f : \operatorname{carr} Q \longrightarrow \operatorname{carr} P$ as a point mapping. Define f by mapping the center of each circle c_v to the center of the circle $f(c_v)$ and extending to the edges and faces. From a practical standpoint, piecewise-affine or barycentric coordinates are easiest, but some of the deepest work in the topic depends on extensions introduced by He [zxH91], based on Möbius transformations between packing interstices.

Approximation: In this survey I have alluded to various "approximation" aspects of circle packing, both in Thurston's original conjecture on conformal mapping (*a la* Figure 17) and in more general settings. There are two key ingredients which reappear in one guise or another in all the approximation results: namely, *refinement* and *distortion control*. "Refinement" refers to the process of conveniently generating sequences of successively finer circle packings (that is, more numerous and smaller circles) appropriate to a setting — hexagonal packings with successively smaller radii serve that purpose in Figure 17. "Distortion" concerns the behavior of discrete analytic functions (interpreted as point mappings); distortion is normally quantified in terms of *quasiconformal dilatation*, so dilatation converging to 1 means the mappings converge to conformal ones. It seems that the objects of the discrete theory invariably approximate their classical counterparts, often with surprising precision, though errors and rates of convergence are as yet poorly quantified. The interested reader should consult [St99] for details.

9. Appendix B: The Literature

William Thurston coined the term "circle packing" in [Thur], where he introduced circle packings for the construction of certain orbifolds. He found that his proof reinterpreted results on reflection groups due to E. M. Andreev [And70a, And70b]. More recently it was found that both these author's were preceeded by P. Koebe [Koe36]. Hence the name Koebe-Andreev-Thurston Theorem for our fundamental existence and uniqueness result. Brooks [Br85, Br86] made early use of circle packings to parameterize Schottky groups. However, circle packing came to prominence with analysts only after Thurston's conjecture [Th85] on their use in approximating conformal maps and its proof by Rodin and Sullivan [RS87]. As this survey has suggested, circle packing has developed in several different directions; here is an attempt to place the literature in some rough categories.

Existence/Uniqueness: [Koe36, And70a, And70b, Thur, Mor84, Riv86, BSt90, MaR90, BSt91b, Sch91a, Sch91b, Rod91, Hod92, Bow93, Mat93, Sch93, Mat94, Dub95, HeS95b, BoSt96, Sch96, Dub97d, zxH99]. ***Rodin/Sullivan and Extensions:*** [Th85, RS87, Car89, St90, zxH91, HeR93, DHR94a, Ah94, CdVM96, DHR94b, CdVM94, HeS96, Mat96, Sch96, St96]. (Note: Early work centered on hexagonal combinatorics, later work successively removed combinatorial restrictions.) ***Function Theory:*** [Rod87, Rod89, BSt90, zxH90, Ah91, BSt91b, MR91, CaR92, CR93, HeS93, HeS94, vE94, Dub95, DSt95a, DSt95b, HeS95c, HeS95b, BoSt97, Dub97e, Dub97d, Dub97a, Dub97b, Sch97, Dub97c, Wil99b, Wil99a]. ***Type Problems:*** [BSt91a, HeS95a, McC96, Bow97, McC98, Sid98]. ***Riemann Surfaces:*** [Br85, Br86, MR91, BoSt92, Gar92, Nim92, BoSt93, BoSt97]. ***Probability:*** [St90, BSc96, Dub96, Mat96, St96, Dub97e, McC98]. ***Approximation:*** [CdVM94, DSt95a, DSt95b, BoSt97, BoSt97, HeS98, St99, Wil99b, Wil99a]. ***Graph embedding:*** [RK89, MTh90, MTV91, MP94, Moh93, zxH94, PaA95, BSc96, Bow97, BoSt97, HBS99]. ***Packing Algorithms:*** [Th85, CdV89, CdV91, Bra92, Moh93,]. (Caution: authors are not always careful in distinguishing between convergence of "packing algorithms" and convergence of packings to analytic functions.) ***Geometry:*** [Sod36, Br85, Riv86, Han88, Ah97, AS97, Ah91, Ber92, Bra93, BDS94, Sa94, BoSt97, Wil99b].

References

[Ah91] Dov Aharonov, *The Descartes circle theorem and applications to function theory*, Technion Preprint (1991).

[Ah94] ______, *The hexagonal packing lemma and the Rodin Sullivan Conjecture*, Transactions Amer. Math. Soc. **343** (1994), 157–167.

[Ah97] ______, *The sharp constant in the ring lemma*, Complex Variables Theory Appl. **33** (1997), 27–31.

[AS97] Dov Aharonov and Kenneth Stephenson, *Geometric sequences of discs in the Apollonian packing*, Algebra i Analiz, dedicated to Goluzin **9** (1997), no. 3, 104–140.

[And70a] E. M. Andreev, *Convex polyhedra in Lobacevskii space*, Math. USSR Sbornik **10** (1970), 413–440 (English).

[And70b] ______, *Convex polyhedra of finite volume in Lobacevskii space*, Math. USSR Sbornik **12** (1970), 255–259 (English).

[BDS94] Alan F. Beardon, Tomasz Dubejko, and Kenneth Stephenson, *Spiral hexagonal circle packings in the plane*, Geometriae Dedicata **49** (1994), 39–70.

[BSt90] Alan F. Beardon and Kenneth Stephenson, *The uniformization theorem for circle packings*, Indiana Univ. Math. J. **39** (1990), 1383–1425.

[BSt91a] ______, *Circle packings in different geometries*, Tohoku Math. J. **43** (1991), 27–36.

[BSt91b] ______, *The Schwarz-Pick lemma for circle packings*, Ill. J. Math. **35** (1991), 577–606.

[BSc96] I. Benjamini and O. Schramm, *Harmonic functions on planar and almost planar graphs and manifolds, via circle packings*, Invent. Math. **126** (1996), 565–587.

[Ber92] Marcel Berger, *Les placements de cercles*, Pour la Science (French Scientific American) **176** (1992).

[Bow93] Philip L. Bowers, *The upper Perron method for labelled complexes with applications to circle packings*, Proc. Camb. Phil. Soc. **114** (1993), 321–345.

[Bow97] ______, *Negatively curved graph and planar metrics with applications to type*, Mich. Math. J. **45** (1998), 31–53.

[BoSt97] Philip L. Bowers and Kenneth Stephenson, *Uniformizing dessins and Belyĭ maps via circle packing*, submitted.

[BoSt92] ______, *The set of circle packing points in the Teichmüller space of a surface of finite conformal type is dense*, Math. Proc. Camb. Phil. Soc. **111** (1992), 487–513.

[BoSt93] ______, *Circle packings in surfaces of finite type: An in situ approach with applications to moduli*, Topology **32** (1993), 157–183.

[BoSt96] ______, *A branched Andreev-Thurston theorem for circle packings of the sphere*, Proc. London Math. Soc. (3) **73** (1996), 185–215.

[BoSt97] ______, *A "regular" pentagonal tiling of the plane*, Conformal Geometry and Dynamics **1** (1997), 58–86.

[Bra92] Walter Brägger, *Kreispackungen und Triangulierungen*, L'Enseignement Math. **38** (1992), 201–217.

[Bra93] ______, *Disk packings and volume of hyperbolic polyhedra*, (1993), preprint.

[Br85] Robert Brooks, *On the deformation theory of classical Schottky groups*, Duke Math. J. **52** (1985), 1009–1024.

[Br86] ______, *Circle packings and co-compact extensions of Kleinian groups*, Inventiones Mathematicae **86** (1986), 461–469.

[CR93] Kevin Callahan and Burt Rodin, *Circle packing immersions from regularly exhaustible surfaces*, Complex Variables, Theory and Appl. **21** (1993), 171–177.

[CFP96] J. W. Cannon, W. J. Floyd, and Walter Parry, *On the conformal invariance of tiling-systems*, preprint, 1997.

[CFP94] ______, *Squaring rectangles: the finite Riemann Mapping Theorem*, vol. 169, 1994, pp. 133–212.

[Car89] Ithiel Carter, *Circle packing and conformal mapping*, Ph.D. thesis, UCSD (advisor: Burt Rodin), 1989.

[CaR92] Ithiel Carter and Burt Rodin, *An inverse problem for circle packing and conformal mapping*, Transactions Amer. Math. Soc. **334** (1992), 861–875.

Chuck Collins and Kenneth Stephenson, *A circle packing algorithm*, preprint 1997.

[CdV89] Yves Colin de Verdière, *Empilements de cercles: Convergence d'une methode de point fixe*, Forum Mathematicum **1** (1989), 395–402.

[CdV91] ______, *Une principe variationnel pour les empilements de cercles*, Inventiones Mathematicae **104** (1991), 655–669.

[CdVM94] Yves Colin de Verdière and Frederic Matheus, *Empilements de cercles et approximations conformes*, Comptes rendus de la Table Ronde de Geometrie Rieman-nienne en l'honneur de Marcel Berger, Seminaires et Congres, S.M.F., 1994.

[CdVM96] ______, *Empilements de cercles et approximations conformes*, Les Actes de la Table Ronde de Geometrie Riemannienne en l'honneur de Marcel Berger (Lumigny, 13-17 juillet 1992), S.M.F., 1996, Seminaires et Congres, No. 1.

[DHR94a] P. Doyle, Zheng-Xu He, and Burt Rodin, *The asymptotical value of the circle packing constants s_n*, Discrete and Computational Geometry **12** (1994), 105–116.

[DHR94b] ______, *Second derivatives and estimates for hexagonal circle packings*, Discrete and Computational Geom. **11** (1994), 35–49.

[Dub95] Tomasz Dubejko, *Branched circle packings and discrete Blaschke products*, Trans. Amer. Math. Soc. **347** (1995), no. 10, 4073–4103.

[Dub96] ______, *Circle-packing connections with random walks and a finite volume method*, Sémin. Théor. Spectr. Géom. **15** (1996–97), 153–161.

[Dub97a] ______, *Approximation of analytic functions with prescribed boundary conditions by circle packing maps*, Discrete Computat. Geom. **17** (1997), 67–77.

[Dub97b] ______, *Circle packings in the unit disc*, Complex Variables Theory Appl. **32** (1997), 29–50.

[Dub97c] ______, *Discrete solutions of Dirichlet problems, finite volumes, and circle packings*, (1997), preprint, 6/97, submitted to Disc. and Comput. Geom.

[Dub97d] ______, *Infinite branched packings and discrete complex polynomials*, J. London Math. Soc. **56** (1997), 347–368.

[Dub97e] ______, *Recurrent random walks, Liouville's theorem, and circle packings*, Math. Proc. Cambridge Philos. Soc. **121** (1997), no. 3, 531–546.

[DSt95a] Tomasz Dubejko and Kenneth Stephenson, *The branched Schwarz lemma: a classical result via circle packing*, Mich. Math. J. **42** (1995), 211–234.

[DSt95b] ______, *Circle packing: Experiments in discrete analytic function theory*, Experimental Mathematics **4** (1995), no. 4, 307–348.

[FK97] Robert Fricke and Felix Klein, *Vorlesungen über die Theorie der automorphen Functionen*, vol. I and II, B. G. Teubner, 1897-1912.

[Gar92] B. T. Garrett, *Circle packings and polyhedral surfaces*, Disc. Comput. Geom. **8** (1992), 429–440.

[Han88] Lowell J. Hansen, *On the Rodin and Sullivan ring lemma*, Complex Variables, Theory and Appl. **10** (1988), 23–30.

[zxH90] Zheng-Xu He, *Solving Beltrami equations by circle packing*, Transactions Amer. Math. Soc. **322** (1990), 657–670.

[zxH91] ______, *An estimate for hexagonal circle packings*, J. Differential Geometry **33** (1991), 395–412.

[zxH94] ______, *Coarsely quasi-homogeneous circle packings*, Mich. Math. J. **41** (1994), 175–180.

[zxH99] ______, *Rigidity of infinite disk patterns*, Ann. of Math. **149** (1999), 1–33.

[HeR93] Zheng-Xu He and Burt Rodin, *Convergence of circle packings of finite valence to Riemann mappings*, Comm. in Analysis and Geometry **1** (1993), 31–41.

[HeS93] Zheng-Xu He and Oded Schramm, *Fixed points, Koebe uniformization and circle packings*, Ann. of Math. **137** (1993), 369–406.

[HeS94] ______, *Rigidity of circle domains whose boundary has σ-finite linear measure*, Invent. Math. **115** (1994), 297–310.

[HeS95a] ______, *Hyperbolic and parabolic packings*, Discrete & Computational Geom. **14** (1995), 123–149.

[HeS95b] ______, *Inverse Riemann mapping theorem for relative circle domains*, Pac. J. Math **171** (1995), 157–165.

[HeS95c] ______, *Koebe uniformization for "almost circle domains"*, Amer. J. Math **117** (1995), 653–667.

[HeS96] ______, *On the convergence of circle packings to the Riemann map*, Invent. Math. **125** (1996), 285–305.

[HeS98] ______, *The c^∞-convergence of hexagonal disk packings to the Riemann map*, Acta Math. **180** (1998), 219–245.

[Hod92] Craig D. Hodgson, *Deduction of Andreev's theorem from Rivin's characterization of convex hyperbolic polyhedra*, Topology 90, Proceedings of Research Semester in Low Dimensional Topology at O. S. U., de Gruyter Verlag, New York, 1992.

[HBS99] Monica K. Hurdal, Philip L. Bowers, Ken Stephenson, and et al, *Quasi-conformally flat mapping the human cerebellum*, to appear.

[Koe36] Koebe, *Kontaktprobleme der Konformen Abbildung*, Ber. Sächs. Akad. Wiss. Leipzig, Math.-Phys. Kl. **88** (1936), 141–164.

[MP94] S. Malitz and A. Papakostas, *On the angular resolution of planar graphs*, SIAM J. Discrete Math. **7** (1994), 172–183.

[MaR90] Al Marden and Burt Rodin, *On Thurston's formulation and proof of Andreev's theorem*, Computational Methods and Function Theory, Proceeding, Valparaiso 1989, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1990, Lecture Notes in Mathematics, Vol. 1435, pp. 103–115.

[Mat93] Frédéric Mathéus, *Rigidite des empilements infinis immerges proprement dans le plan et dans le disque*, Seminaire de Theorie Spectrale et Geometrie **12** (1993-94), 69–85.

[Mat94] ______, *Empilements de cercles : rigidite, discretisation d'applications conformes*, Ph.D. thesis, Institut Fourier (Grenoble) (advisor: Colin de Verdière), 1994, These de Doctorat de l'Universite Grenoble I.

[Mat96] ______, *Empilements de cercles et approximations conformes: une nouvelle preuve du theoreme de rodin-sullivan*, L'Enseignement Mathematique **42** (1996), 125–152.

[McC96] G. J. McCaughan, *Some topics in circle packing*, Ph.D. thesis, University of Cambridge (advisor: Keith Carne), 1996.

[McC98] Gareth McCaughan, *A recurrence/transience result for circle packings*, Proc. Amer. Math. Soc. **126** (1998), 3647–3656.

[MTV91] Gary L. Miller, Shang-Hua Teng, and Stephen A. Vavasis, *A unified geometric approach to graph separators*, Proc. 32nd Symp. on Found. of Comp. Sci. (1991), 538–547.

[MTh90] Gary L. Miller and William Thurston, *Separators in two and three dimensions*, Proceedings of the 22nd Annual ACM Symposium on Theory of Computing, ACM, Baltimore, May 1990, pp. 300–309.

[MR91] David Minda and Burt Rodin, *Circle packing and Riemann surfaces*, J. d'Analyse Math. **57** (1991), 221–249.

[Moh93] Bojan Mohar, *A polynomial time circle packing algorithm*, Discrete Math. **117** (1993), 257–263.

[Mor84] John W. Morgan, *On Thurston's uniformization theorem for three-dimensional manifolds*, The Smith Conjecture (Hyman Bass and John W. Morgan, eds.), Academic Press, New York, 1984, pp. 37–126.

[Nim92] Barbara Nimershiem, *Isometry classes of flat 2-tori appearing as cusps of hyperbolic 3-manifolds are dense in the moduli space of the torus*, Low-Dimensional Topology (K. Johannson, ed.), International Press, Boston, 1992, Knoxville.

[PaA95] J/'anos Pach and Pankaj K. Agarwal, *Combinatorial Geometry*, Wiley-Interscience, New York, 1995.

[Riv86] I. Rivin, *On geometry of convex polyhedra in hyperbolic 3-space*, Ph.D. thesis, Princeton, 1986.

[Rod87] Burt Rodin, *Schwarz's lemma for circle packings*, Invent. Math. **89** (1987), 271–289.

[Rod89] ______, *Schwarz's lemma for circle packings II*, J. Differential Geometry **30** (1989), 539–554.

[Rod91] ______, *On a problem of A. Beardon and K. Stephenson*, Indiana Univ. Math. J. **40** (1991), 271–275.

[RS87] Burt Rodin and Dennis Sullivan, *The convergence of circle packings to the Riemann mapping*, J. Differential Geometry **26** (1987), 349–360.

[RK89] F. Rothen and A.-J. Koch, *Phyllotaxis or the properties of spiral lattices. II. packing of circles along logarithmic spirals*, J. Phys. France **50** (1989), 1603–1621.

[Sa94] Horst Sachs, *Coin graphs, polyhedra, and conformal mapping*, Discrete Math. **134** (1994), 133–138.

[Sch91a] Oded Schramm, *Existence and uniqueness of packings with specified combinatorics*, Israel J. Math. **73** (1991), 321–341.

[Sch91b] ______, *Rigidity of infinite (circle) packings*, Journal Amer. Math. Soc. **4** (1991), 127–149.

[Sch93] ______, *Square tilings with prescribed combinatorics*, Israel J. Math. **84** (1993), 97–118.

[Sch96] ______, *Conformal uniformization and packings*, Israel J. Math. **93** (1996), 399–428.

[Sch97] ______, *Circle patterns with the combinatorics of the square grid*, Duke Math. J. **86** (1997), 347–389.

[Sid98] Ryan Siders, *Layered circlepackings and the type problem*, Proc. Amer. Math. Soc. **126** (1998), no. 10, 3071–3074.

[Sod36] F. Soddy, *The Kiss Precise*, Nature (June 20, 1936), 1021, poem.

[St90] Kenneth Stephenson, *Circle packings in the approximation of conformal mappings*, Bulletin, Amer. Math. Soc. (Research Announcements) **23, no. 2** (1990), 407–415.

[St96] ______, *A probabilistic proof of Thurston's conjecture on circle packings*, Rendiconti del Seminario Mate. e Fisico di Milano **LXVI** (1996), 201–291.

[St99] ______, *Approximation of conformal structures via circle packing*, Computational Methods and Function Theory 1997, Proceedings of the Third CMFT Conference (N. Papamichael, St. Ruscheweyh, and E. B. Saff, eds.), vol. 11, World Scientific, 1999, pp. 551–582.

[Thur] William Thurston, *The geometry and topology of 3-manifolds*, Princeton University Notes, preprint.

[Th85] ______, *The finite Riemann mapping theorem*, 1985, Invited talk, An International Symposium at Purdue University on the occasion of the proof of the Bieberbach conjecture, March 1985.

[vE94] Jeff van Eeuwen, *The discrete Schwarz-Pick lemma for overlapping circles*, PAMS **121** (1994), 1087–1091.

[Wil99a] George Brock Williams, *Approximation of quasisymmetries using circle packings*, preprint, 1999.

[Wil99b] ______, *Discrete approximation of conformal weldings using circle packings*, preprint, 1999.

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