

CIRCLE PACKING BIBLIOGRAPHY AS OF SEPTEMBER 2005

MAINTAINED BY KENNETH STEPHENSON

The term **circle packing** here refers to configurations of circles with specified patterns of tangency, intersection, or separation. The central issues concern connections between the combinatorics of packings and their geometries, the variety among packings sharing combinatorics, computational methods, and connections with analytic function theory and conformal geometry. (Note that this topic is distinct from classical “sphere packing”, which is concerned with density and coverings using spheres of given radii.)

The software package *CirclePack* for creating, manipulating, and displaying circle packings under X-Windows is available at

<http://www.math.utk.edu/~kens>

This site also contains a few pictures and some background material on circle packing.

Please send e-mail if you have corrections, updates, or new items for the bibliography.

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