The Approximation of Conformal Structures *via* Circle Packing

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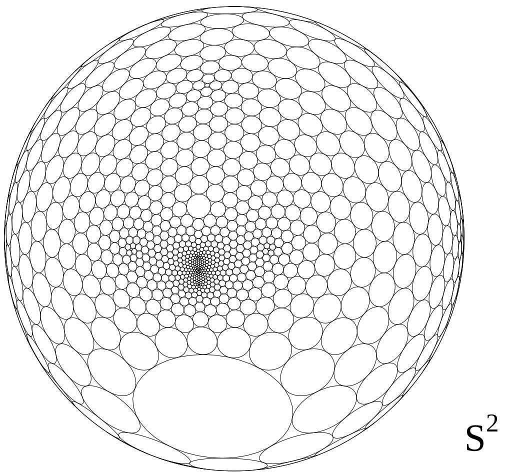
Abstract. This is a pictorial tour and survey of circle packing techniques in the approximation of classical conformal objects. It begins with numerical conformal mapping and the conjecture of Thurston which launched this topic, moves to approximation of more general analytic functions, and ends with recent work on the approximation of conformal tilings and conformal structures.

§1 Introduction

A *circle packing* is a configuration of circles with a specified pattern of tangencies. The regular hexagonal or "penny" packing in the plane — every circle tangent to six others — is certainly familiar to everyone, and the literature contains a smattering of other examples stretching back to the ancient Greeks. But I offer this as a first illustration of the type of packings we will discuss.



Circle packing has its beginning as a distinct topic with applications to 3-manifolds in Thurston's Notes [20]. Its connections to analytic functions, the subject of this survey, can be traced precisely to a conjecture of Thurston at the 1985 Bieberbach conference concerning the use of circle packings to approximate conformal mappings and to its subsequent proof by Rodin and Sullivan [16].

I must strongly emphasis right at the beginning that our topic has almost **no contact** with the more familiar "sphere packing" studies, which address density and coverage issues — how many ping pong balls fit in a boxcar. Look to our first figure: it is the **pattern of tangencies** and the triangular interstices which are central. These circles have had to assume widely varied sizes in order to fit together in a prescribed pattern — this packing and the source of its combinatorics will come up later.

When I outline the basic notions of circle packing in a moment, you will find that these objects present an intriguing blend of local geometric rigidity and global geometric flexibility. Mappings between one configuration of circles realizing a certain tangency pattern and another configuration realizing the same pattern turn out to behave in fundamental ways like analytic mappings and have become central to the theory. Two lines of development have emerged regarding these maps. The first, following the lines of Thurston's 1985 conjecture, concerns their use in the **approximation** of analytic functions — that is the subject of this survey. However, the second, which addresses the many **analogies** between these maps and analytic functions, also plays a role here since discoveries in that theory present us with several new targets for approximation.

This will be a decidedly pictoral survey, and I will rely heavily on the reader's geometric feeling for analytic functions. We start with traditional "conformal mapping", where circle packing has little to recommend it, in a numerical sense, over classical methods. However, we move on to more general analytic functions, where it begins to show some advantages, and finally reach quite new, unbroken ground in the approximation of conformal structures. In recent applications to Riemann surfaces and conformal tiling which I survey here, circle packing may stand as the only technique available.

My thanks to the organizers of Computational Methods in Function Theory '97 and to the University of Cyprus for their hospitality. At the end of the paper I point the reader to sources for a circle packing bibliography and for the software package `CirclePack` used for the numerical experiments illustrated here.

1.1 Classical Conformal Geometry

Our view of analytic functions will be almost exclusively geometric — you will see no power series or integral formulae here, forget complex arithmetic. Think in terms of conformality (angle preserving), curvature and metrics, surfaces and coverings, branching and projections, 'mappings' not 'functions' — the rubber-sheet imagery suggested by Figure 1.

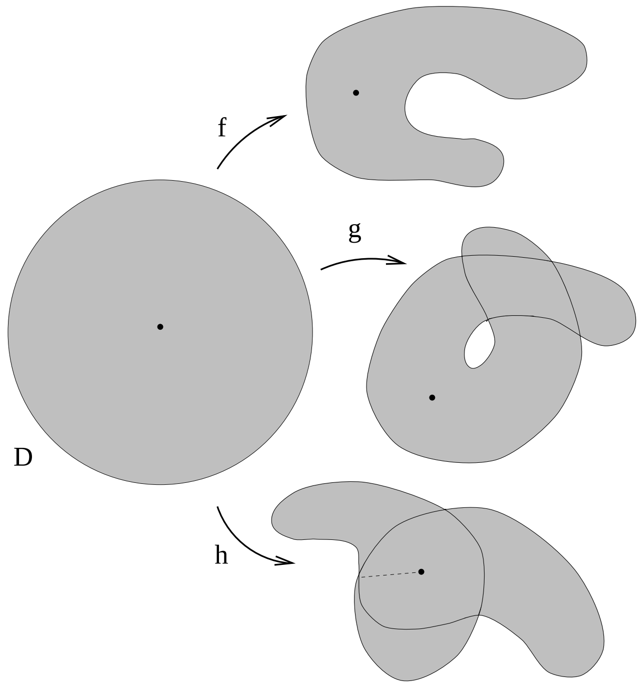


Figure 1. Geometry of classical analytic functions on the disc

In asking you to rely on geometric themes and intuition, it might help to recall a saying which goes to the heart of conformal geometry: namely, *analytic functions f map infinitesimal circles to infinitesimal circles.* Moreover, $|f'|$ is the stretching of these infinitesimal circles. Taking this image seriously and "zooming" in on points z and $f(z)$, one can imagine f mapping a pattern of infinitesimal circles in the domain to the pattern of infinitesimal image circles in the range. If you can get your mind around this image, you will not be far from the truth. In the discrete theory, we simply choose to work with actual circles!

1.2 Circle Packing Basics

The theory began on the sphere with this theorem, proven successively by Koebe, Andreev, and Thurston: *Let K be an abstract triangulation of a topological sphere. There exists a circle packing P on the Riemann sphere $\mathbb{S}^2$ having the combinatorics of K and P is unique up to Möbius transformations and inversions of* $\mathbb{S}^2$. The conditions on P are that each vertex of $v \in K$ is represented by a circle $c_v \in P$ and that if vertices v and w are connected by an edge in K, then c_v and c_w are tangent.

The richness of circle packings may be traced to their dual natures: *combinatoric* in the pattern of prescribed tangencies and *geometric* in the realization by actual circles. Until §2.4, our circle packings will lie in the sphere $\mathbb{S}^2$, the plane $\mathbb{C}$, or the hyperbolic plane, represented here as $\mathbb{D}$ with the Poincaré metric. Working with them requires a rather modest system of bookkeeping:

• *Complex:* K denotes an abstract simplicial 2-complex triangulating a topological surface $\mathcal{D}$.

• *Label:* R is a collection $\{R(v)\}$ of positive numbers associated with vertices v of K. If we are working in a metric space $\mathcal{D}$, these represent putative radii. We restrict attention to settings with Riemannian metrics of constant curvature $0, 1$, or -1, described as *euclidean*, *spherical*, or *hyperbolic*, respectively.

• *Angle Sums:* A label R determines an angle sum $\theta_R(v)$ at each vertex of K. In particular, for each face $\langle v, u, w\rangle \in K$, let $\alpha_R(v; u, w)$ denote the angle at c_v in a triangle formed by the centers of a mutually tangent triple $\langle c_v, c_u, c_w\rangle$ of circles of radii $\langle R(v), R(u), R(w)\rangle$; the law of cosines appropriate to the geometry of R lets us compute α. Then $\theta_R(v) = \sum_{\langle v,u,w\rangle} \alpha_R(v; u, w)$, where the sum is over all faces $\langle v, u, w\rangle \in K$ containing v.

• *Packing:* P is a configuration of circles in the metric space $\mathcal{D}$ satisfying the tangency pattern of K. If the radii are listed in the label R, we would write $P \leftrightarrow K(R)$.

• *Carrier:* $\operatorname{carr}(P)$ denotes the concrete geometric complex in $\mathcal{D}$ formed by connecting the centers of tangent circles of P with geodesic segments; this is simplicially equivalent to the abstract complex K.

Figure 2 illustrates these basics with a simple finite complex labeled K; (b) is a rather generic hyperbolic packing for K in $\mathbb{D}$, (c) a more deliberate euclidean packing with rectangular carrier, (d) is the maximal packing (in $\mathbb{D}$) to be discussed shortly. The carriers are shown for reference. Note that the *radii* and even the geometry change — it is the **combinatorics** which these packings share.

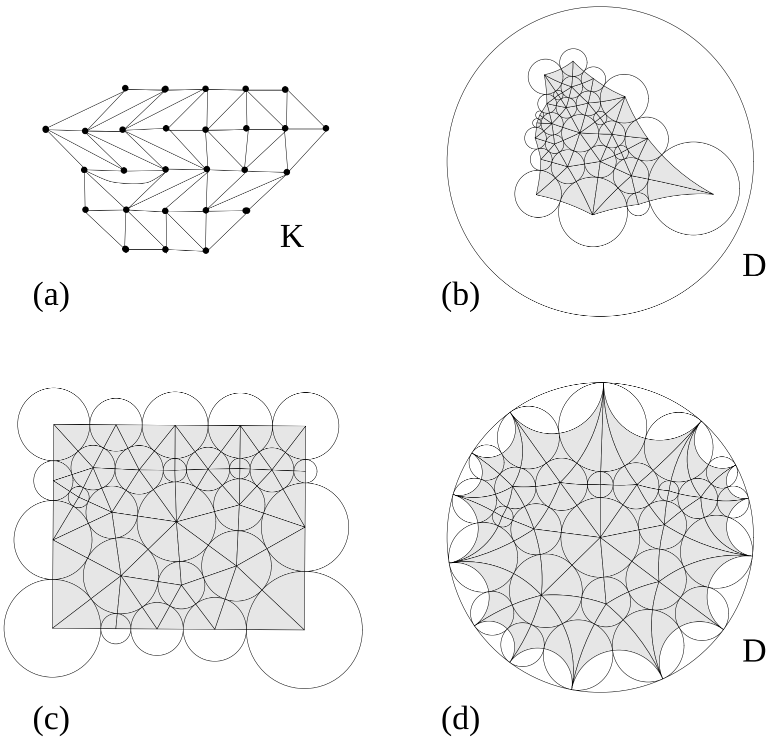


Figure 2. Circle packings for a simple complex K

The surprise, of course, is that for a given complex K there exist *any* packings whatsoever, much less a potentially huge variety of packings. Computationally, the labels are the key.

Definition 1. *A label R is a packing label for K if the angle sum $\theta_R(v)$ is an integral multiple of 2π for every interior vertex v.*

This is clearly a necessary condition for R to represent the radii of a packing P, since $\theta_R(v) = 2\pi n, n \geq 1$, simply reflects the fact that the circles c_{v_i} for the neighbors of v must reach precisely n times around c_v. It is also a sufficient condition if there are no topological obstructions:

Lemma 1. *If K triangulates a simply connected surface, then a label R for K represents the radii for a circle packing of K if and only if R is a packing label.*

We say little about circle centers because they play a secondary role: once a packing label R is found, the circles are laid out in $\mathcal{D}$ ($\mathbb{S}^2, \mathbb{C}$, or $\mathbb{D}$, as appropriate) using a process akin to analytic continuation. The resulting circle packing $P \leftrightarrow K(R)$ is *essentially unique* — that is, it is unique up to a conformal automorphism of $\mathcal{D}$.

A bit of terminology: A packing is *univalent* if its circles have mutually disjoint interiors. Univalence fails locally at a *branch circle* (or *vertex*), that is, at an interior circle whose neighbors wrap two or more times around it (angle sum $4\pi, 6\pi, \cdots$). Figure 3 illustrates a branch circle; the same nine circles which wrap once about the central circle in Figure 3(a) wrap exactly twice around the smaller central circle of Figure 3(b). The *branch set* for a packing P, denoted $br(P)$,

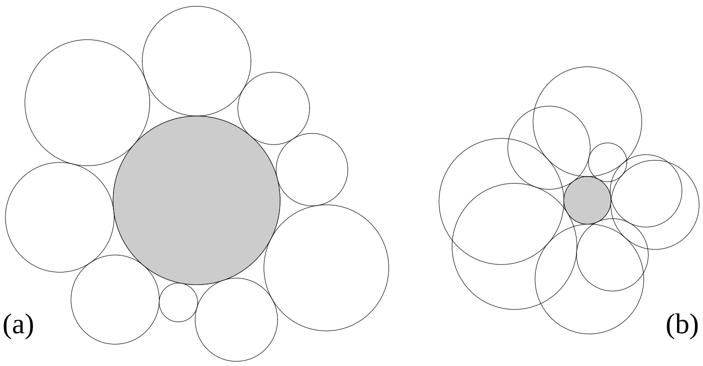


Figure 3. Local univalence *versus* branching

is the set of branch circles, repeated according to multiplicities. If all interior angle sums are 2π, then $br(P) = \emptyset$ and the packing is called *locally univalent*. Univalence can fail, even without branch circles, when circles from one part of a packing overlap those from another.

Packing Variety: Given a complex K, there are numerous results on the existence and variety of packings for K; we need two main types. Let's begin with the *extreme rigidity* displayed by maximal packings, proven by Koebe in the finite case [14], Beardon and

Stephenson in the bounded degree case [2], and He and Schramm in general [11]:

Theorem 1. *Let K triangulate a simply connected surface. Then for $\mathcal{D}$ equal to precisely one of the spaces $\mathbb{S}^2, \mathbb{C}$, or $\mathbb{D}$, there exists a circle packing $\mathcal{P}_K$ for K in $\mathcal{D}$ with the property that $\mathcal{P}_K$ is univalent and has a carrier filling $\mathcal{D}$. Furthermore, $\mathcal{P}_K$ is essentially unique.*

This circle packing $\mathcal{P}_K$ is called a *maximal packing* and K is said to be *spherical*, *parabolic*, or *hyperbolic* as $\mathcal{P}_K$ lies in $\mathbb{S}^2, \mathbb{C}$, or $\mathbb{D}$, respectively.

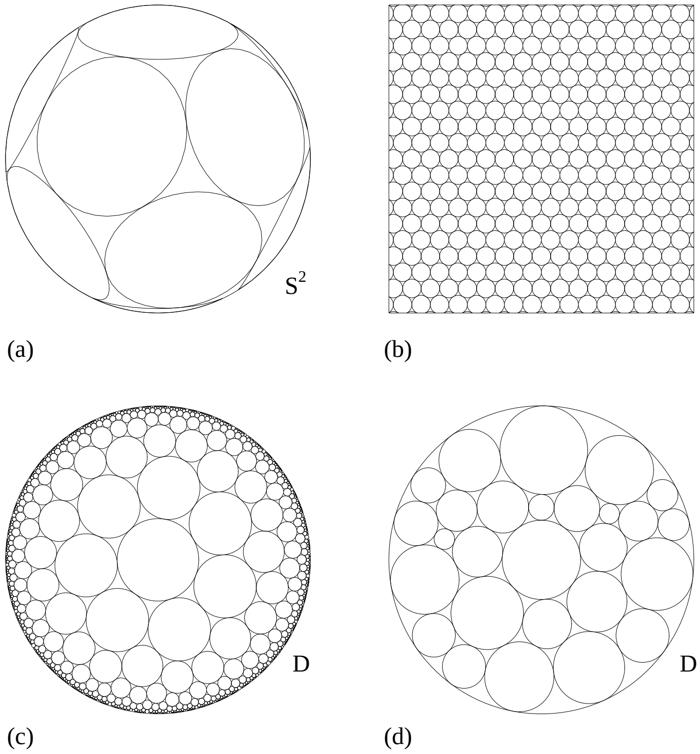


Figure 4. Maximal Packings

Figure 4 illustrates the four basic situations. A complex K is spherical iff it triangulates a topological sphere; in (a) we see the regular pentagonal packing (every circle with five neighbors). The familiar regular hexagonal or "penny" packing (b) is parabolic, the regular "septagonal" packing (c) is hyperbolic. Computationally, packings such as that of Figure 4(d) are the most important. Here K triangulates a closed topological disc (in fact, this is the maximal packing for the complex of Figure 2). Note that $\mathrm{carr}(\mathcal{P}_K)$ "fills" $\mathbb{D}$ in the sense that its boundary circles are *horocycles* (euclidean circles internally tangent to $\partial\mathbb{D}$) and the boundary vertices have been pushed to the ideal boundary. Spherical packings are computed by puncturing at a vertex, packing the rest of the complex in $\mathbb{D}$, then projecting back to $\mathbb{S}^2$. Infinite packings are handled as appropriately rescaled limits of packings in $\mathbb{D}$ for finite subcomplexes.

In contrast to the rigidity of maximal packings, let's move to the *variety* available when we construct packings meeting prescribed boundary value and branching conditions.

Let K triangulate a closed topological disc. Let $\beta = \{b_1, \cdots, b_k\}$ be a legal branch set for K; this is a (perhaps empty) list of interior vertices of K, with possible repetitions. (Necessary/sufficient combinatorial conditions on β have been proven by Dubejko [9] and Bowers [4].) ∂K denotes the boundary vertices of K.

Theorem 2. *Let K, β, and the boundary label $g : \partial K \longrightarrow (0,\infty)$ be given. Then there exists a unique euclidean packing label R for K with branch set β and satisfying $R(w) = g(w), w \in \partial K$. The same result holds in hyperbolic geometry, where g is also allowed to assume the value $+\infty$.*

In other words, there exists an essentially unique circle packing having a prescribed branch set and prescribed boundary radii. In hyperbolic geometry, infinite radii correspond to horocycles. Thus the maximal packing for K is the packing in $\mathbb{D}$ having prescribing branch set $\beta = \emptyset$ and prescribed boundary radii $g(w) = \infty, w \in \partial K$.

Don't let this global malleability of circle packings obscure the *local rigidity* implied by the packing condition at each vertex. The local-to-global linkage is moderated by the combinatorics of K.

Discrete Analytic Functions: We can attempt to impose some order on the numerous packings for a given complex K by using a *function theory* paradigm. Define *discrete analytic functions* as maps between circle packings which preserve tangency and orientation. Various packings of K may now be associated with discrete analytic functions $f : \mathcal{P}_K \longrightarrow P$. Compare the univalent, locally univalent, and branched discrete analytic functions of Figure 5 to their classical models in Figure 1.

As classical analytic functions map infinitesimal circles to infinitesimal circles, so our discrete versions operate with *real* circles.

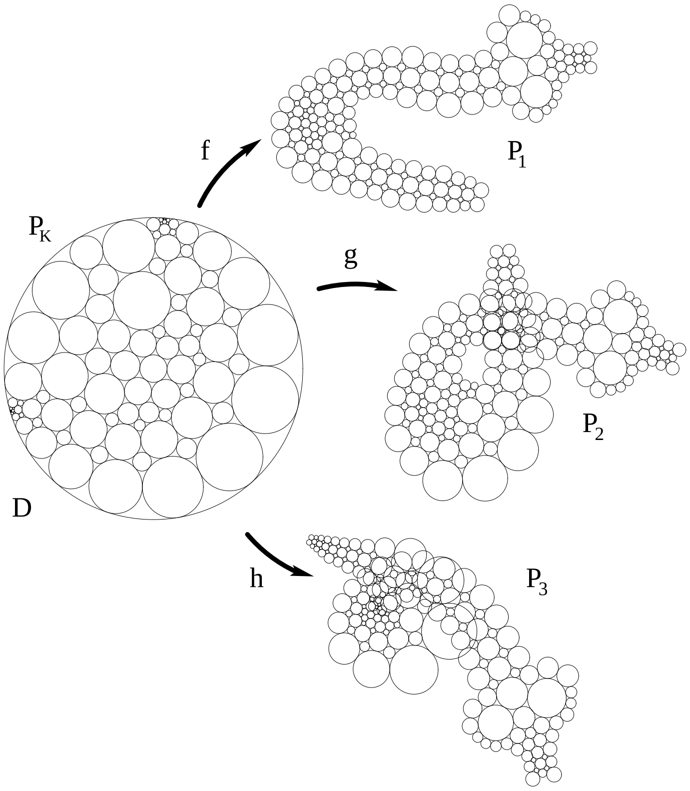


Figure 5. Discrete analytic functions on the disc

Fundamental behavior is surprisingly similar. There is a *ratio* function $f^{\#}$ defined on vertices of K by $f^{\#}(v) = \text{radius}(f(c_v))/\text{radius}(c_v)$; this reflects the amount that f stretches or shrinks c_v and is the analog of $|f'|$. Hidden in P_3 is a branch circle like that of Figure 3(b); the topological behavior of the mapping is precisely that at a branch point of an analytic function.

Computation: Finally, let me point out that the computational problem centers on finding packing labels. Thurston suggested an iterative algorithm which works beautifully in hyperbolic and euclidean geometry to solve the boundary value problems described above. In this survey, we will take the algorithm for granted; see [8] for a description and an efficient implementation.

§2 Approximation

Let's proceed now to the principal approximation issues. I will break my survey into four topics: the original conformal mapping setting of Thurston's Conjecture, generalizations to other analytic functions, then on to new uses in conformal tiling and conformal structures on surfaces.

2.1 Conformal Mapping

Figure 6 illustrates Thurston's 1985 conjecture. Given the Jordan domain Ω, cut out a finite piece of a regular hexagonal packing of circles having radii $1/n$. Write P_n for this finite circle packing and let K_n encode its combinatorics. If P_{K_n} is the maximal packing for K_n, then $f_n : P_{K_n} \longrightarrow P_n$ represents for us a discrete conformal mapping from $\mathbb{D}$ to Ω. The illustration captures domains and ranges for three such mappings (appropriate normalizations are assumed).

Thurston conjectured and Rodin and Sullivan proved that the discrete conformal maps f_n converge uniformly on compact subsets of $\mathbb{D}$ to the classical conformal mapping $F : \mathbb{D} \longrightarrow \Omega$ as $n \to \infty$. From that convergence one can also show that the ratio functions $f_n^{\#}$ converge uniformly on compact sets to $|F'|$.

I will not go into the proof of Thurston's conjecture, but I would like to highlight the two key ingredients, which reappear in one guise or another in every setting we consider: namely, *refinement* and *distortion control*. "Refinement" refers to the process of conveniently generating sequences of successively finer circle packings (that is, more numerous and smaller circles) appropriate to a setting — hexagonal packings with successively smaller radii serve that purpose in Figure 6. "Distortion" concerns the comparison of a face in (the carrier of) one packing with the corresponding face in another packing of the same complex, normally quantified in terms of quasiconformal distortion. Less distortion of carrier faces converts to "more conformal" mapping behavior. In Figure 7, observe the distortion of various triangles on the right compared to the corresponding equilateral triangles on the left. Distortion near the center is smaller.

The important Hexagonal Packing Lemma of Rodin and Sullivan and its descendants imply, roughly speaking, that in comparing two univalent packings the amount of distortion that a given face can manifest decreases as its combinatorial distance from the boundary increases. The Rodin/Sullivan Theorem has been considerably extended to accommodate more general combinatorics and to quantify the convergence. The deepest and most general results are due to Z-X. He and Oded Schramm [12, 13]; see also [19, 18].

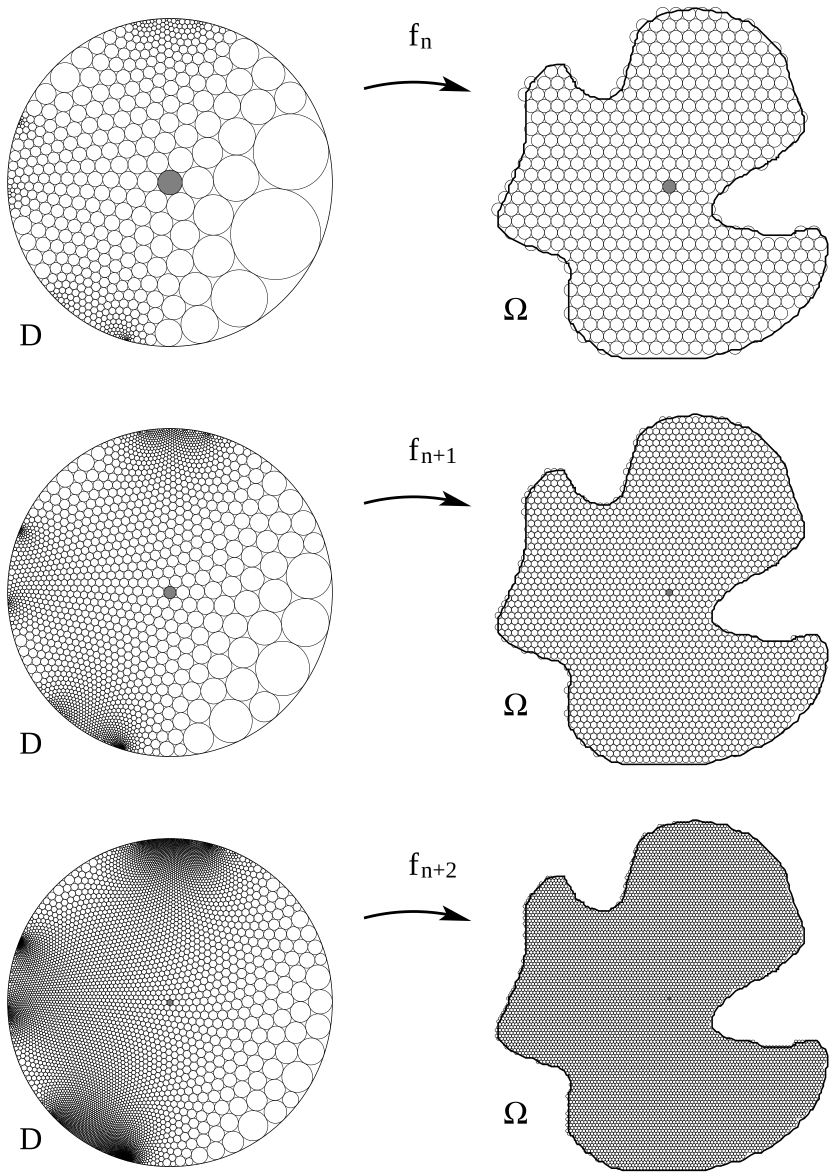


Figure 6. Illustrating Thurston's Conjecture

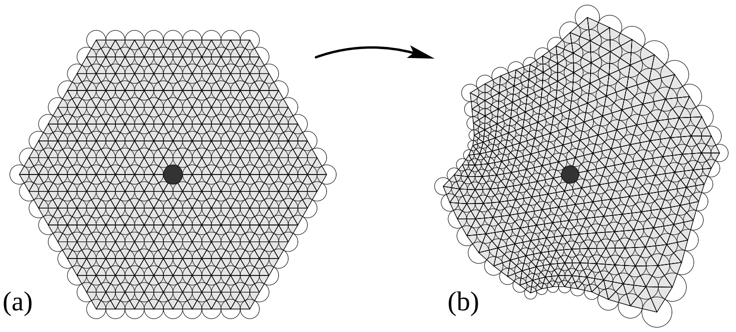


Figure 7. The Hexagonal Packing Lemma

Numerical Methods: On the more practical side, for those involved in numerical conformal mapping circle packing may appear to be a disappointment: the "packing" process itself — e.g., computing the radii for P_{K_n} — turns out to be rather slow, data sets are bulky, and the process of laying out the packing itself introduces additional error. Circle packing certainly cannot compete with the classical numerical methods for speed and accuracy.

Circle packing does have some strengths. For instance, it quite easily handles multiply connected regions. Figure 8(b) illustrates a generic 4-connected region Ω and the packing P it cuts from a regular hex packing. Augment the complex underlying P with an "ideal" ver-

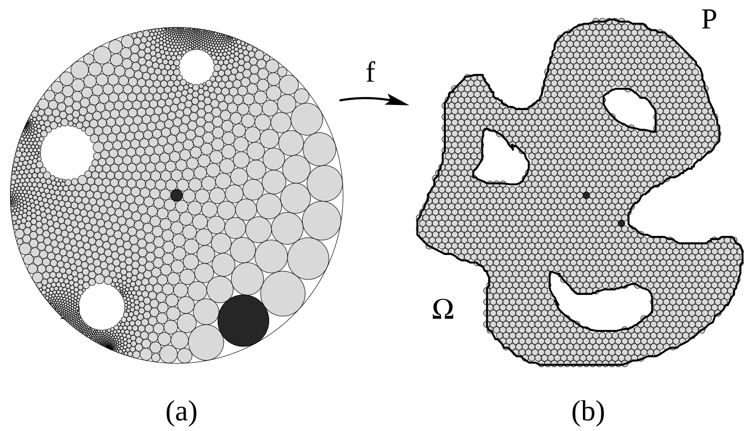


Figure 8. A multiply connected example

tex for each hole; that is, a common neighbor for the circles bounding that hole. The augmented complex is simply connected, and displaying its maximal packing with the circles corresponding to the ideal vertices removed leaves the "circle domain" on the left in Figure 8. This then is (discretely) conformally equivalent to the original domain Ω. (Z-X. He and oded Schramm in [11] applied circle packing techniques in a major advance on Koebe's Conjecture regarding infinitely connected *circle domains*; their work is far deeper than our simple numerical example here.)

Prototyping: In the settings of traditional conformal mapping, circle packing may be helpful at what I will call *prototyping*. Suppose one plans to apply Schwarz-Christoffel to approximate a conformal map of $\mathbb{D}$ to the polygon R of Figure 9(b). One might start

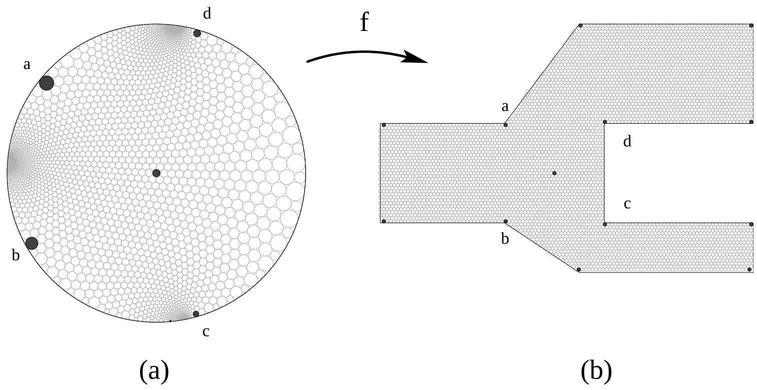


Figure 9. Prototyping a Schwarz-Christoffel Map

with a discrete conformal map f involving a modest number of circles (just over 5000 in this instance). I've marked only the four most visible circles associated with the indicated corners of the polygon; these and the other corner preimages can serve as the initial guesses for the Schwarz-Christoffel parameters. Prototyping could be valuable in other situations as well, such as the multiply connected setting discussed earlier.

2.2 Analytic Functions

Classical analytic function theory involves more than just conformal maps. So it is, too, with the discrete theory. The numerous parallels are not the aim in this survey, but in passing note that the discrete theory involves fairly general functions on $\mathbb{D}$, polynomials and a smattering of entire and meromorphic functions on the plane, rational functions on the sphere, and discrete maps on Riemann surfaces. The theory has its own Schwarz-Pick, Uniformization, Liouville and Picard theorems, and its own notions of covering theory, extremal length, harmonic measure, and Brownian motion (random walks). It is approaching something like a "full service" function theory, though there remain many gaps.

I have chosen two examples to illustrate the computational and approximation side of this more general discrete theory: one involves a function in the disc algebra, the other a finite Blaschke product.

A Disc Algebra Example: Consider the closed curve γ of Figure 10, with a finite number of transverse self-intersections. Using winding numbers alone, one can verify that γ is the image of the unit circle under a disc algebra function (continuous on $\overline{\mathbb{D}}$ and analytic on $\mathbb{D}$); one can also determine the distribution of branch values (images of branch points) among the components of $\mathbb{C} - \gamma$. (See [15].)

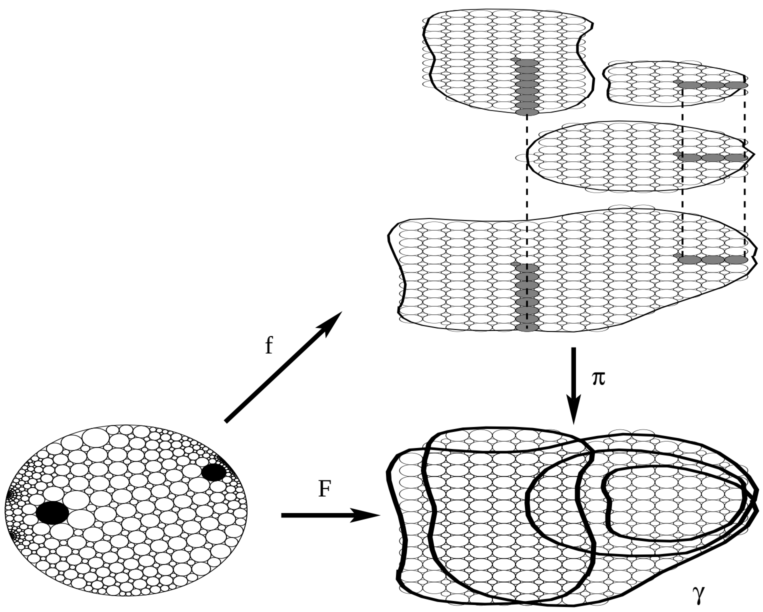


Figure 10. Approximating a disc algebra function

Actual construction of a disc algebra solution is another matter; Schwarz-Christoffel methods apply, but I'm not aware of implementations. We can construct the analogous discrete solutions by introducing branch points to our earlier univalent methods.

Our aim is construction of a multi-sheeted image packing; we use a "ball bearing" rather than a hexagonal pattern as our starting material and overlay the region defined by γ as in the univalent case. However, several "sheets" (up to three deep inside the smallest loop) may be inferred from γ and the branch structure, so appropriate pieces are cut out and attached as in the upper right of Figure 10. The resulting packing P has a complex K which is simply connected. Our discrete disc algebra function F results from mapping $\mathcal{P}_K$ to P with f and projecting with π to a packing in $\mathbb{C}$. The pasting resulted in two branch points, one simple and one of order 2. These are the shaded circles in $\mathcal{P}_K$; the former has 8 neighbors which map 2-to-1 onto the 4 neighbors of its projected image circle, the latter has 12 neighbors mapped 3-to-1 onto the 4 neighbors of its projected image circle.

As in the conformal setting, a sequence of refined packings constructed in this manner will (with suitable normalization) converge uniformly on compact subsets of the disc to the classical disc algebra solution. We can routinely generate the necessary refinements from this initial coarse construction, as we will see later, so this is in fact a fairly practical procedure.

Finite Blaschke Products: Classical finite Blaschke products arise not only in function theory on the disc and its applications, but also in studies of rational functions, iteration theory, and topological covering theory. An n-fold Blaschke product is typically represented as a product of n Möbius factors, one for each zero, multiplied by a unimodular constant. It assumes each value in the disc n times, counting multiplicities, and maps the unit circle n times around itself.

Lacking arithmetic, we need a more geometric description for the discrete construction. The finite Blaschke products are precisely the "proper" analytic maps of the disc to itself, and the argument principle in turn implies that an n-fold Blaschke product has $n-1$ branch points, counting multiplicities. These geometric features are enough for our construction, as I illustrate with a 4-fold example.

Take as the domain a maximal packing $\mathcal{P}_K$ in the unit disc, shown on the left in Figure 11. I have identified three shaded interior circles for branching. We repack the complex in hyperbolic geometry, specifying that: (1) boundary circles get infinite hyperbolic radii; (2) designated branch circles get angle sum 4π rather than the usual 2π. The resulting packing label allows us to lay out the range packing. Displaying all its circles on the right would lead to visual meltdown since points are covered four times, so I have drawn only the chain of boundary horocycles and the 3 branch circles (they appear as extremely

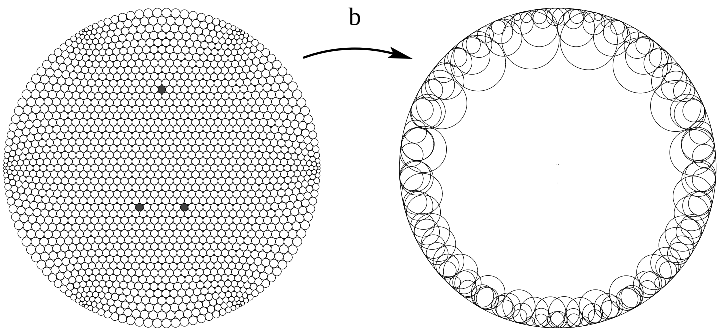


Figure 11. Boundary circles for 4-fold discrete Blaschke product

small dots towards the middle) in the image packing. A careful survey would confirm that those boundary circles wrap 4 times around the unit circle. The construction is complete, aside from applying a Möbius transformation of $\mathbb{D}$ for the sake of normalization.

To approximate a particular classical 4-fold Blaschke product B, we would first need to know its branch points. Then we would run this construction with successively finer complexes K_n, with branch circles approximating the desired branch points, and with appropriate normalizations. The sequence $\{b_n\}$ of discrete finite Blaschke products would converge uniformly on compact subsets of $\mathbb{D}$ to B.

The simplicity of the discrete construction is quite impressive; the repacking process converts the local geometric conditions on boundary radii and branch points to the inevitable global behavior. Classical methods would have difficulty coping with prescribed branch points. On the other hand, constructing a discrete finite Blaschke product with specified *zeros* is tough.

2.3 Conformal Tilings

Conformal tilings, introduced by Phil Bowers and me in [7], involve tiles with specified "conformal" rather than euclidean shapes. Though classical objects, they were inspired by experiments with circle packings and circle packings remain the only avenue for approximating their true conformal shapes.

I'll illustrate with a finite example based on the simple schematic pattern of Figure 12(a). Pasting together seven equilateral triangles in the prescribed pattern, as suggested in Figure 12(b), yields a topological disc with a piecewise affine structure; there are three "cone" points, each of angle $4\pi/3$. There is a standard (and very classical)

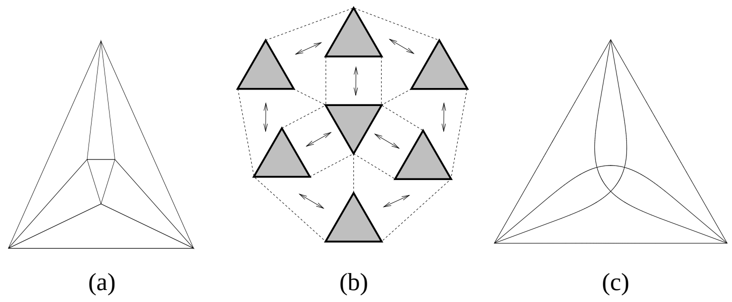


Figure 12. Combinatorial, affine, and conformal stuctures

procedure for defining a conformal structure compatible with such an affine structure; its atlas involves appropriate power maps $z \mapsto z^{3/2}$ at the cone points (see, e.g., Beardon [1]). We arrive, then, at a Riemann surface, and if we map it conformally onto an equilateral triangle, carrying along the markings of the original seven triangles, we arrive at the triangle T of Figure 12(c).

The curves marked on T are uniquely determined by the original combinatorics and by what Bowers and I call a *reflective* structure — namely, any two of the seven triangular pieces having a common edge are anticonformal reflections of one another across that edge. How can one approximate these curves in T? I am not aware of any classical numerical methods which apply. However, circle packing provides *discrete* curves which do the trick.

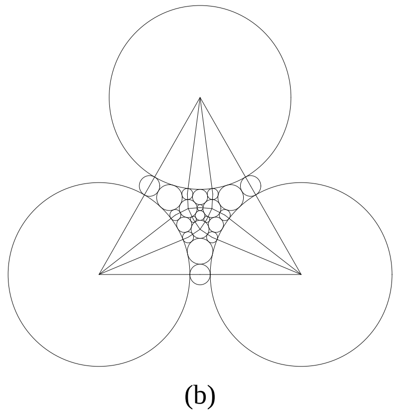
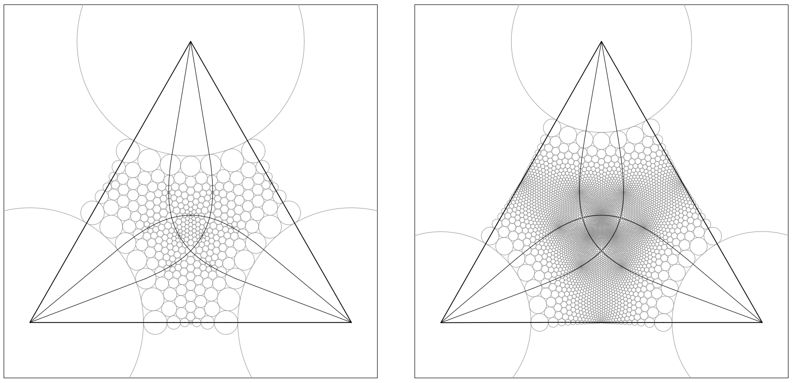


Figure 13. The discrete construction

The discrete construction mimics the original. Conceptually, we take seven copies of the packing of Figure 13(a) and attach them to one another by identifying circles along appropriate edges, just as in Figure 12(b). (In fact, only the resulting combinatorics are important.) When the results are repacked to form an equilateral carrier, the repacking process forces the circle sizes to adjust to flatten out the cone points, as in Figure 13(b).

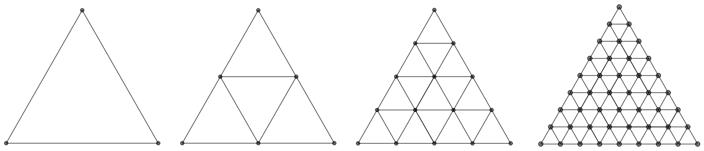
Figure 14 shows the same process carried out with two successively more refined packings; one can see the marked curves approaching limit shapes.



stage-2 refinement stage-4 refinement

Figure 14. Circle packings at two additional refinement levels

I want to take a moment to comment on the refinement process used here. It is called "hex refinement" and operates as shown in Figure 15. Hex refinement can be applied to any triangulation and



original stage-1 stage-2 stage-3

Figure 15. Hex refinement process

has several valuable features: • New combinatoric data is very easy to generate. • The original vertices of a triangulation remain, with their degrees unchanged. • New (interior) vertices are degree six. This means that under successive hex refinement, one is in good position to deploy the Hex Packing Lemma. • "Refinement" is actually a combinatorial process — one must always compute new radii for the refined complex. The self-scaling nature of hex refinements may provide *a priori* estimates which will save effort in the repacking computation.

Infinite Tilings: The processes involved in our simple example, both the classical construction and its discrete analogue, work with only minor adjustment in much more general circumstances. We will look at some infinite patterns here and move to patterns on compact surfaces in the next section.

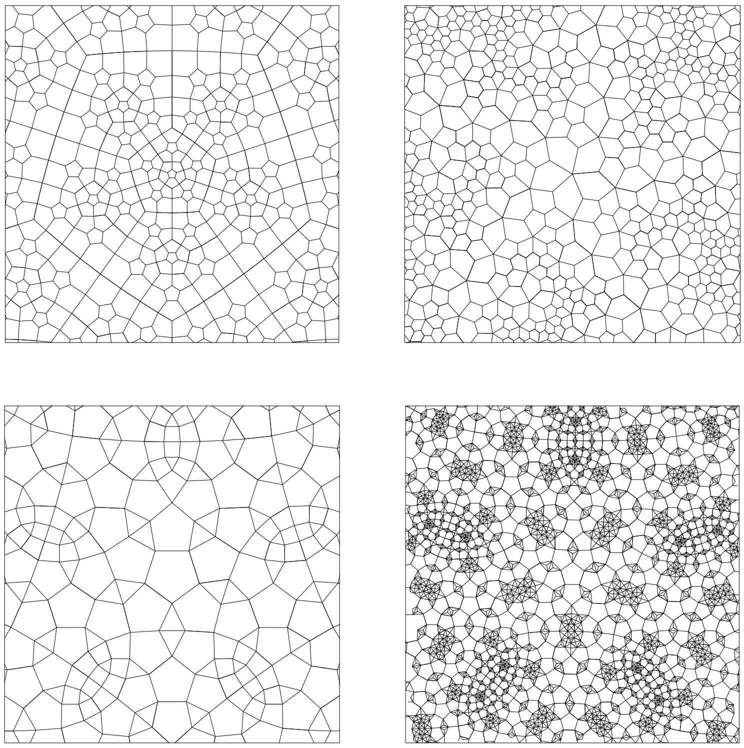
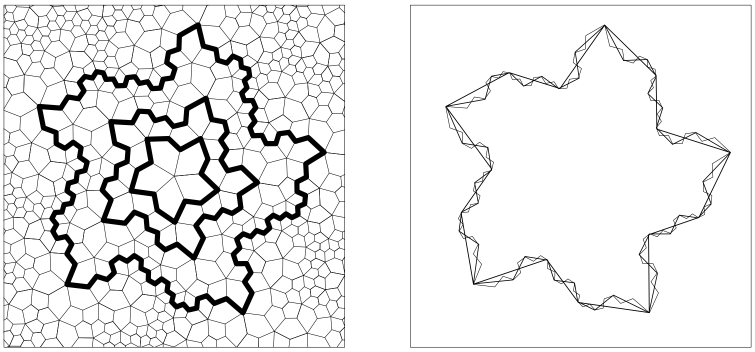


Figure 16. Four examples of infinite conformal tilings of the plane

Figure 16 illustrates four infinite conformally regular tilings of $\mathbb{C}$. Each tile can be mapped conformally onto a regular euclidean n-gon, corners going to corners (and there are additional conditions which we won't go into). These particular patterns are quite special; they arise from "subdivision rules" being applied by Jim Cannon, William Floyd, and Walter Parry to the study of Thurston's Geometrization Conjecture. (See [7] and references therein.) My thanks to Bill Floyd for the data needed to generate these tilings.

What is our interest in tilings? All of these pictures were generated *via* circle packing — the circles themselves aren't shown, but only certain edges of their carriers. Packings bring an experimental component to this research and they provide approximations (like these pictures) of the true conformal shapes of the tiles — capabilities not otherwise available.

I have space to investigate only one example, so let's look at the "twisted pentagonal" tiling in the upper right of Figure 16. In Figure 17(a), denote the three dark curves as $\gamma_0, \gamma_1, \gamma_2$, from inner to outer. These mark central "aggregate tiles" associated with the first three stages of the subdivision rule which generated this pattern. Stare at the picture hard enough and you might guess at some scaling relationship among the γ_i's. In fact, using $\lambda \approx 0.55e^{-0.23\pi}$, I have



(a) (b)

Figure 17. Aggregates in the twisted pentagonal tiling

overlaid $\lambda^2 \cdot \gamma_2$, $\lambda \cdot \gamma_1$, and γ_0 in Figure 17(b). Our hunch was right, the vertices $\lambda \cdot \gamma_i$ coincide almost perfectly with vertices of γ_{i-1}. You can probably anticipate what the curves $\gamma_3, \gamma_4, \cdots$ look like; it appears that $\lim_{n\to\infty}(\lambda^n \cdot \gamma_n)$ converges to some fractal curve Γ, that is, the aggregate tiles converge to a fractal tile. The same holds for other tiles in the pattern (not marked), leading to a fractal pentagonal tiling of the plane. Motivated by these very experiments, Walter Parry has used classical methods to prove this scaling property, though the precise value of λ is not known.

Central issues in conformal tiling include the "type" problem — does a pattern tile the plane or the hyperbolic plane — questions about tile shapes, sizes, distortions, and questions about symmetries and self-similarities within the overall pattern. But this is a new topic, and new phenomena appear in nearly every experiment. Numerical computation and display are currently the pivotal tools.

2.4 Grothendieck Dessins

Let me begin this section by introducing a certain type of Riemann surface construction. Suppose S denotes a compact, orientable topological surface and K is a triangulation of S. One can paste euclidean equilateral triangles together in the pattern of K to impose a piecewise affine structure on S. Precisely as described in the seven-triangle example of Figure 13, that affine structure determines a conformal structure on S, making it into a Riemann surface. Surfaces constructed this way will be called *equilateral* surfaces.

We quote this surprising result of G. Belyĭ [3], for which we need one other bit of terminology. A nonconstant meromorphic function on a compact Riemann surface which branches only over points $\{0, 1, \infty\}$ is termed a *Belyĭ map*.

Theorem 3. *For a Riemann surface R of positive genus, the following statements are equivalent:*

(a) R is equilateral.

(b) R supports a Belyĭ map.

(c) There exists a defining equation for R whose coefficients lie in an algebraic number field F over the rationals.

In other words, in the Teichmüller space Teich(S) for surfaces of genus $g > 0$ there is a set of points which simultaneously enjoys combinatorial, function theoretic, and algebraic characterizations. By (a) this set is countable, and by (c) it is dense in Teich(S).

Let's turn now to the discrete parallels. We find that S and K determine a point in Teichmüller space in a second way by this result (see [20, 2]).

Theorem 4. *Let K be a triangulation of a compact oriented topological surface S. Then there exists a unique Riemann surface R homeomorphic to S that supports a univalent circle packing $\mathcal{P}_K$ with the combinatorics of K.*

In other words, the combinatorics prescibed for a circle packing will determine the geometry of the space in which the packing can be realized. (The metric for this packing $\mathcal{P}_K$ is the *intrinsic metric* on R; that is, the metric of constant curvature 0 or ± 1 which R inherits from its universal covering surface.) When S has positive genus, the surface

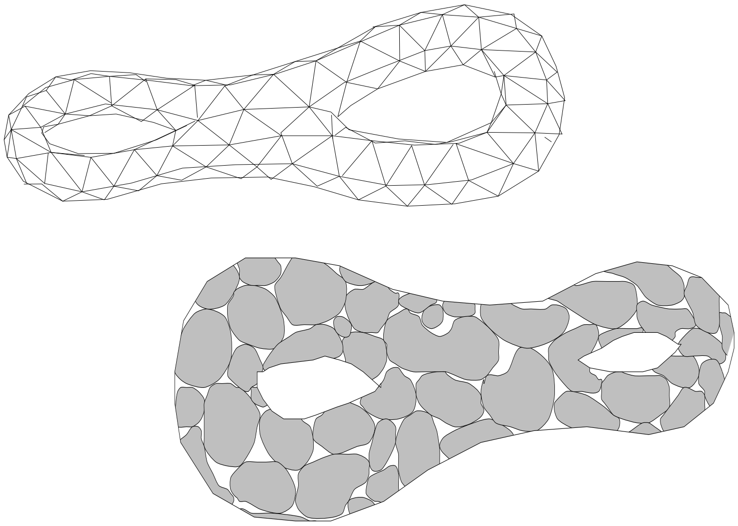


Figure 18. "Equilateral" *versus* "packable" 2-tori

R which supports the packing for K is called a *packable* surface of Teich(S). The packable surfaces are clearly countable, and by a result of Bowers and me [6], also dense. With a certain technical condition that we may assume for K, one can also define a discrete meromorphic map from $\mathcal{P}_K$ to a packing in $\mathbb{S}^2$ which branches only over $\{0, 1, \infty\}$; a *discrete* Belyĭ map. In other words, we have discrete parallels for the classical objects described earlier.

Dessins: Equilateral surfaces are at the core of a program initiated by Grothendieck to study the absolute Galois group Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$ and to address what is known as the Inverse Galois Problem. This amazing theory begins with a "child's drawing" D on S and ends with an algebraic number field. Briefly, the drawing leads to a triangulation of S, the triangulation leads to an equilateral Riemann surface S_D and a Belyĭ map B_D on S_D, giving the so-called "Belyĭ pair" (S_D, B_D). Schematically,

$$D \longrightarrow K \longrightarrow (S_D, B_D).$$

Of course this is all quite abstract and in only limited special circumstances are any concrete details about the surface and the Belyĭ map known. (See [17].)

On the other hand, the parallel discrete development can be carried out at various refinement stages and has the advantage of being computable. The idea is to apply n-stages of hex refinement to K, giving a complex $K^{(n)}$, then to construct the associated *discrete Belyĭ pair* $(s_D^{(n)}, b_D^{(n)})$ consisting of $s_D^{(n)} \in \text{Teich}(S)$ and discrete Belyĭ map $b_D^{(n)} : P_{K^{(n)}} \longrightarrow \mathbb{S}^2$. Schematically,

$$D \longrightarrow K \longrightarrow K^{(n)} \longrightarrow (s_D^{(n)}, b_D^{(n)}).$$

Phil Bowers and I have proven the following result [5]:

Theorem 5. *Let D be a drawing on a compact orientable surface S. Then the discrete Belyĭ pairs $(s_D^{(n)}, b_D^{(n)})$ converge to the classical Belyĭ pair (S_D, B_D) as $n \to \infty$. In other words, $s_D^{(n)}$ converges to S_D in the Teichmm̈uller metric of Teich(S) and the functions $b_D^{(n)}$ converge uniformly to B_D on S.*

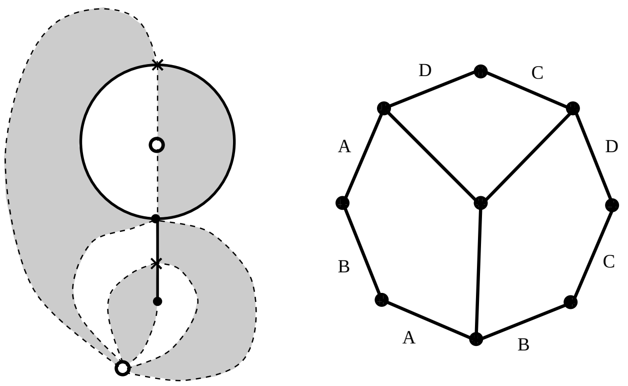


Figure 19. Sample dessins of genus 0 and genus 2

As examples, let me discuss the two dessins of Figure 19. The first, a very simple genus 0 drawing D on the sphere, is shown in Figure 19(a) along with its canonical triangulation, four shaded and four unshaded faces. Here $s_D^{(n)} = S_D$ for all n since there is only one conformal structure on $\mathbb{S}^2$. Nevertheless, the convergence of $b_D^{(n)}$ to B_D allows us to approximate the conformal shapes of the faces of K. The stage $n = 3$ construction is shown in Figure 20; the packing is the one shown at the beginning of the paper, but the faces of K are marked in its carrier as the shaded and unshaded regions, and the drawing D itself is shown as the heavy curve. The discrete Belyĭ map $b_D^{(3)}$ maps this packing to the spherical packing on the right with multiplicity 4-to-1, the shaded faces mapping to the shaded hemisphere.

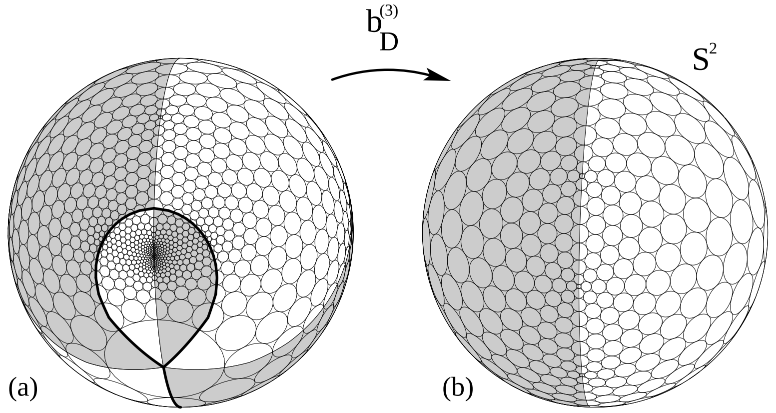


Figure 20. A genus 0 discrete Belyĭ map

Our second example is the dessin D of Figure 19(b); the indicated side-pairings show this to be a genus 2 case. I have packed this at the original "coarse" stage and at a stage-3 hex refinement and display the results in Figure 21. The packings are laid out in $\mathbb{D}$ as fundamental domains for the universal covering maps from $\mathbb{D}$ onto $s_D^{(0)}$ and $s_D^{(3)}$, respectively. The heavy curve in each case is D and the faces of K are alternately shaded and unshaded.

The packing computations are carried out directly in hyperbolic geometry with the combinatorics of the 2-torus in tact. The results are then laid out as a fundamental domain where numerical information, such as side-pairing maps, can be read off. (Note the side-pairings designated on the upper packing.) The fundamental domains converge as $n \to \infty$ to the fundamental domain of the universal covering of S_D. One thing to note is the surprising accuracy of even the coarse packing — its eight boundary cusps differ by less than 1/500 from those of the stage-3 refinement.

By the way, have our discrete objects lost their connection to number fields? In the original sense, yes; but in fact, number fields reenter in a new and intriguing way as entries in the associated covering maps.

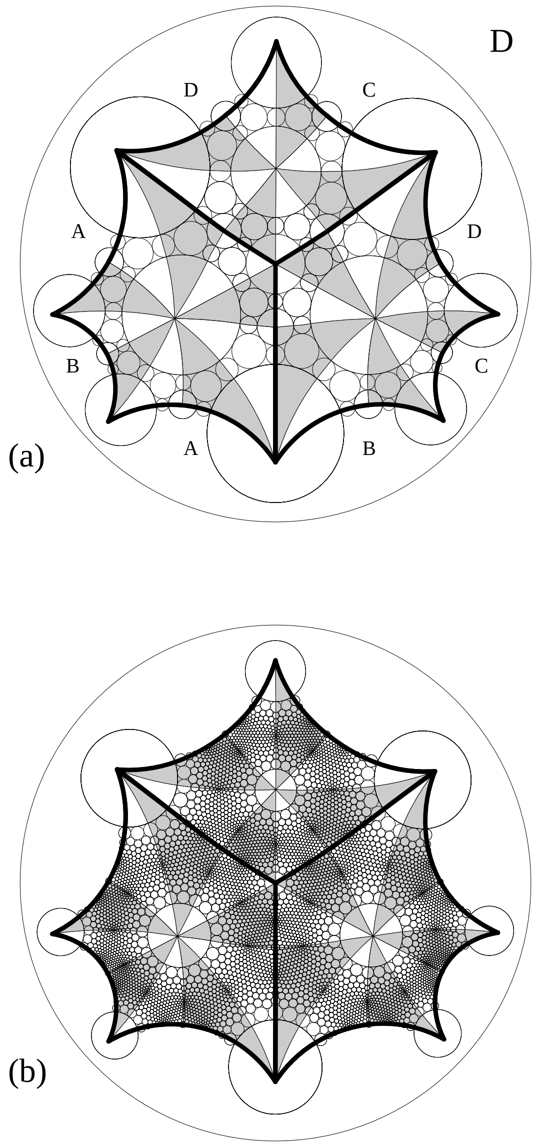


Figure 21. Fundamental domains for genus 2 dessins

Classical Surfaces: Let's wrap up our look at surfaces with two classical examples which (in hindsight) are associated with dessins. The quite famous image in Figure 22 is Klein's surface. He was able to publish the original image more than a century ago because of the surface's ubiquitous symmetries (an order 168 symmetry group) — the triangles are all geodesic. The image here, on the other, generated *via* circle packing, took approximately two seconds for ten digit accuracy starting from purely combinatorial data.

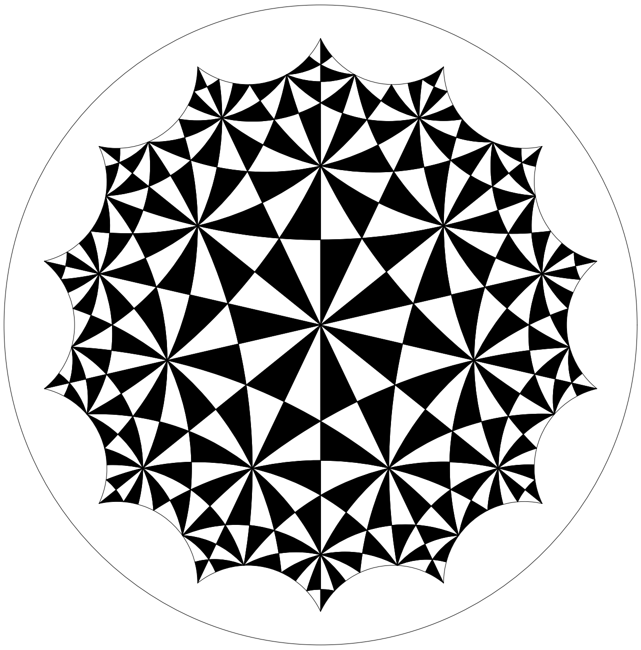


Figure 22. Klein's surface

Another famous classical example is Picard's curve $y^3 = x^4 - 1$, a genus three surface. It is actually associated with two distinct dessins. One is shown in Figure 23(a); this is a fundamental domain for a first stage coarse approximation and the circles have been left in for reference. This classical surface also has many symmetries, but the edges of its faces are not all geodesic. As far as I know, aside from circle packing methods, there would be no way to approximate the surface or to create these images.

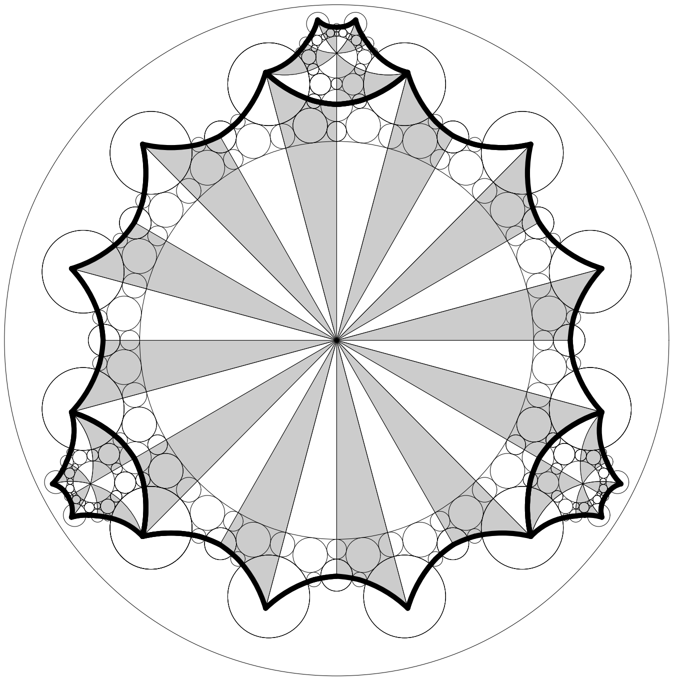


Figure 23. Picard's surface

§3 Concluding Remarks

The survey examples suggest a general paradigm: given a classical object of interest — a function, a Riemann surface, or a conformal shape — proceed as follows:

• Construct a combinatorial abstraction — that is, a triangulation and (possibly) a branch structure.

• Endow the triangulation with an appropriate affine structure. Currently we need an "equilateral" model, but certain recent developments are expected to allow more general decompositions.

• Circle pack the result. This is the local-to-global step; nature endows the abstraction with a global discrete conformal geometry compatible with the local packing conditions.

• Apply a process of repeated refinement and repacking to construct a sequence of discrete conformal objects.

If the local geometry is faithfully encoded, then faithful global consequences appear to be inevitable and the discrete conformal objects converge to their classical counterpart. The methods clearly have broad applicability, and the equilateral models, hex refinement, and the hexagonal packing lemma fit so well together that one can contemplate a rather comprehensive and fairly automated "analytic engine" (to steal a phrase) for approximating classical surfaces and functions in a variety of situations.

The methods are also geometrically honest: the discrete objects are generally very faithful to their classical models, so one's intuition remains in tact; the computations take place within the intrinsic geometry; and the packings provide a natural means of visualization.

3.1 Numerical Issues

Of course, the devil is in the details. Standard issues of accuracy and rates of convergence remain almost totally open for circle packing methods. These may be quite difficult; the original discrete model, the computed packing labels, and the laying-out process, all introduce errors. A beginning has been made with estimates of the quasiconformal distortion associated with discrete analytic functions, primarily in the hexagonal setting; for the deepest results, see the work of Z.X. He and Oded Schramm [13]. The packing algorithm has some beautiful geometry and can be modeled probabilistically, so there is hope of eventual convergence estimates [18].

Nonetheless, we seem to be stuck with the heuristic approach for now — experiments, checks for internal consistency, and, when possible, comparisons to classical examples. In situations where circle packing is the only available tool, we may be forgiven for using it in experiments without first settling the various details of convergence.

3.2 Technical Issues

Visit my homepage at *www.math.utk.edu/∼kens* for a circle packing bibliography, further packing examples, and directions for acquiring the software package `CirclePack`, which runs under X-Windows.

I have based my circle packing algorithms on an iterative scheme suggested by Thurston; recent improvements are detailed in [8]. Moderate packings, say 5,000–10,000 circles, typically require less than a minute; packings of 100,000+ circles are routine, though data sets become bulky. Hex refinement and "multi-grid" methods open the way for parallelization and potentially significant efficiency improvements. See [10] for discriptions of several experiments and [5] for sample timings.

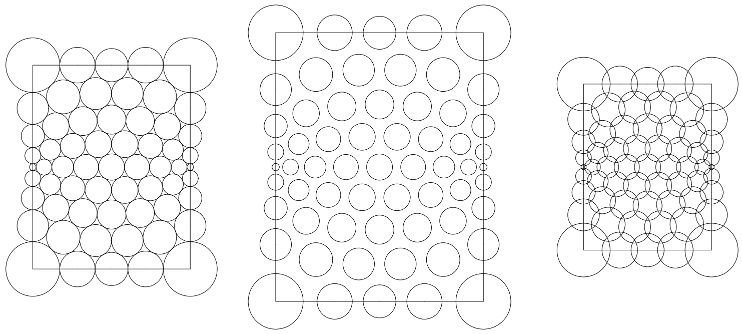


Figure 24. Packing options: tangency, inversive distances, overlaps

3.3 Conclusion

Circle packing is a new and, I think, very exciting topic. It comes with substantial and interesting problems, new capabilities, some available for the first time, an experimental and visual environment, and many directions for further development. As merely one example, Figure 24 illustrates *inversive distance* and *overlap* packings; these generalizations of "tangency" bode even greater flexibility for discrete conformal methods in the future. Moreover, I have left out emerging uses of this geometry inspired by circle packing in areas such as graph embedding and modeling of various physical phenomena.

The traditional applications of conformal mapping — areas which have motivated some truly beautiful mathematics over the years — are increasingly falling to numerical PDE and other methods. At the same time, *conformal geometry* in the broader sense seems to be growing in importance, both in theory and applications. We are confronted with substantial opportunities for numerical approximation, but with flexibility, convenience, visualization, and the intuition gained from real-time experiments taking precedence over traditional speed and precision. Incongruous as it may seem, "circle packing" might provide just the thing in this new approximation environment.

Finally, I admit to loving these pictures, so I want to leave with this last one: a "dodecahedral" tiling from a division rule of Jim Cannon, data from Bill Floyd. None of the circles actually used in the construction are shown, but an intricate pattern of "circles" seems to be emerging from the edge pattern itself!

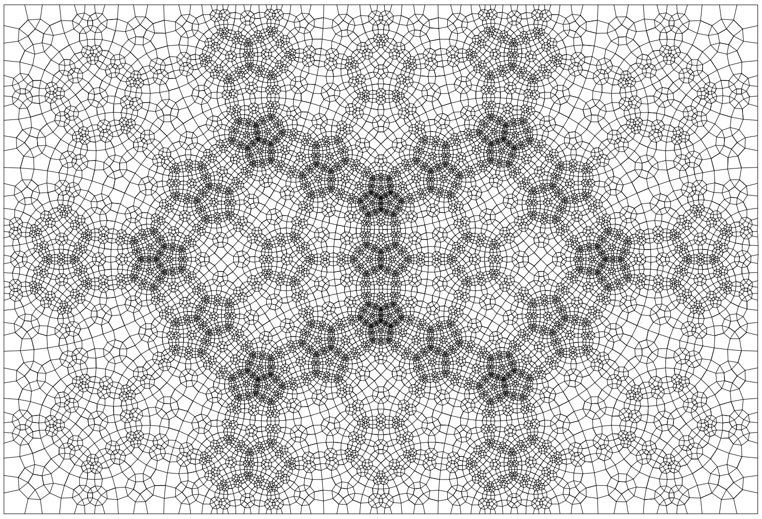


Figure 25. Cannon's dodecahedral pattern

Acknowledgement. The author gratefully acknowledges support of the National Science Foundation and the Tennessee Science Alliance.

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