Here is a trigonometric substitution example. Study carefully, there is a lot here of interest.

Integrate: $\int \frac{\sqrt{9-x^2}}{x}\,dx$. This is an example of the form $\sqrt{a^2-x^2}$ so we will use the substitution x = 3sinθ, here we use the domain of the inverse sine function, that is $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Let x = 3sinθ $\Rightarrow$ $\sin\theta = \frac{x}{3}$

dx = 3cosθ dθ

$$\int \frac{\sqrt{9-x^2}}{x}\,dx = \int \frac{\sqrt{9-9\sin^2\theta}}{3\sin\theta}\left(3\cos\theta\,d\theta\right) = \int \frac{\sqrt{9\left(1-\sin^2\theta\right)}}{3\sin\theta}\left(3\cos\theta\right)d\theta =$$

$$3\int \frac{\sqrt{\cos^2\theta}}{\sin\theta}\cos\theta\,d\theta = 3\int \frac{\cos^2\theta}{\sin\theta}\,d\theta = 3\int \frac{1-\sin^2\theta}{\sin\theta}d\theta = 3\int\left(\csc\theta - \sin\theta\right)d\theta =$$

$$3\left\{-\ln\left|\csc\theta + \cot\theta\right| + \cos\theta\right\}.$$

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Note: above we have $\sqrt{\cos^2\theta} = \cos\theta$ this is valid since we know $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

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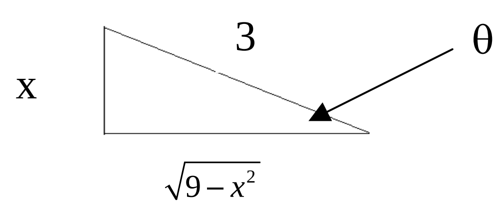
Now draw the triangle



So substituting from the triangle into $3\left\{-\ln\left|\csc\theta + \cot\theta\right| + \cos\theta\right\}$ we have

$$\int \frac{\sqrt{9-x^2}}{x}\,dx = -3\ln\left|\frac{3}{x} + \frac{\sqrt{9-x^2}}{x}\right| + \sqrt{9-x^2} + C$$