

Infinite Sequences & Series

A sequence is an ordered set of numbers.

$\{a_n\} = a_1, a_2, a_3, \dots, a_n, \dots$ where a_n denotes the n^{th} term.

The sequence $\{a_n\}$ converges if $\lim_{n \rightarrow \infty} a_n = L$, otherwise the sequence diverges.

Thm 1: If $a_n \leq b_n \leq c_n$, $n \geq n_0$ and $\lim_{n \rightarrow \infty} a_n = L = \lim_{n \rightarrow \infty} c_n$ then $\lim_{n \rightarrow \infty} b_n = L$. [Sandwich thm]

Thm 2: If $\lim_{n \rightarrow \infty} |a_n| = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Thm 3: If $\lim_{x \rightarrow \infty} f(x) = L$ and $f(n) = a_n$, n an integer, then $\lim_{n \rightarrow \infty} a_n = L$. [Allows us to use l'Hospital rule]

Thm 4: $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r .

A sequence is increasing if $a_n \leq a_{n+1}$, $n \geq 1$ and decreasing if $a_n \geq a_{n+1}$, $n \geq 1$, and if the sequence is either increasing or decreasing the referred to as monotonic.

A sequence is bdd above if there is a number M st $a_n \leq M$, $n \geq 1$ and bounded below if there exists m st $m \leq a_n$, $n \geq 1$

If a sequence is bdd above and below, then said to be bdd.

Thm 5: Every bdd, monotonic sequence is convergent.

Series

Given a sequence $\{a_n\} = a_1, a_2, a_3, \dots, a_n, \dots$, if we add the terms we get a SERIES.

So $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots + a_n + \dots$ denotes an infinite series, where a_n is the n^{th} term.

Spse we have $\sum_{n=1}^{\infty} a_n$, let S_n represent the n^{th} partial sum,

$$S_n = \sum_{i=1}^n a_i = a_1 + a_2 + \dots + a_n,$$

these partial sums form a sequence, $\{S_n\}$.

If $\{S_n\}$ is convergent $\left(\lim_{n \rightarrow \infty} S_n = S \right)$ then $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} a_n = S$, and S is called the sum of the series, otherwise the series is divergent.

Although it is possible to find the sum of a series, in general this is very difficult and we will be satisfied to determine just convergence or divergence.

**** Geometric Series**** $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$

converges if $|r| < 1$ and $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$, $|r| < 1$.

**** Harmonic Series**** $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$, DIVERGES

Thm 6: If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \rightarrow \infty} a_n = 0$. Beware the converse is false!!!!!!!!!!!!!!

TEST 1: If $\lim_{n \rightarrow \infty} a_n \neq 0$ or DNE then $\sum_{n=1}^{\infty} a_n$ is divergent (Divergence Test)

Thm 7: If $\sum a_n$ and $\sum b_n$ are convergent, then so are

- (i) $\sum ca_n = c \sum a_n$
- (ii) $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n$

TESTS FOR CONVERGENCE

Here we are concerned with the convergence or divergence of an infinite series. If we can also determine the sum of the series then fine.

Divergence test: If $\lim_{n \rightarrow \infty} a_n \neq 0$ or DNE, then $\sum_{n=1}^{\infty} a_n$ diverges.

Integral test: Spse f is continuous, positive, and decreasing on $[1, \infty)$, let $a_n = f(n)$.

Then $\sum a_n$ is convergent if and only if $\int_1^{\infty} f(x) dx$ is convergent.

****P-series**** The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$ and divergent for $p \leq 1$.

Comparison test: Spse $\sum a_n, \sum b_n$ are series with positive terms.

i) If $\sum b_n$ is convergent and $a_n \leq b_n, \forall n$ then $\sum a_n$ is also convergent.

ii) If $\sum b_n$ is divergent and $a_n \geq b_n, \forall n$ then $\sum a_n$ is divergent.

Common series used as comparisons are geometric, harmonic, and p-series.

Limit Comparison test: Spse $\sum a_n, \sum b_n$ are series with positive terms.

i) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$, then both series converge or diverge.

ii) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

iii) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Alternating Series test: If $\sum_{n=1}^{\infty} a_n = a_1 - a_2 + a_3 - a_4 + \dots$ ($a_n > 0$) satisfies:

i) $a_{n+1} \leq a_n, \forall n$

ii) $\lim_{n \rightarrow \infty} a_n = 0$

then the series is convergent.

The series $\sum a_n$ is absolutely convergent if the series of absolute values, $\sum |a_n|$ is convergent.

The series $\sum a_n$ is conditionally convergent if it is convergent but not absolutely convergent.

Thm 8: If a series is absolutely convergent, then it is convergent.

Ratio test:

i) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then $\sum a_n$ is absolutely convergent (and thus convergent).

ii) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or ∞ , then $\sum a_n$ is divergent.

Root test:

i) If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L < 1$, then the series is absolutely convergent.

ii) If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L > 1$ or ∞ , then the series is divergent.

Power Series

A power series is an expression of the form:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

More generally:

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1 (x-c) + a_2 (x-c)^2 + \dots$$

is a power series centered at "c" or a power series about "c".

One usually determines convergence or divergence of a power series by using the ratio test.

For a given power series, one of following must occur:

i) Convergence at $x = c$ only ($R = 0$)

ii) Convergence for all x ($R = \infty$)

iii) There is a positive number R st the series converges if $|x-c| < R$ and diverges for $|x-c| > R$.

R is the radius of convergence. When $x = c \pm R$ (the endpoints) each must be checked individually for convergence (usually by comparison).

Our goal is to find a power series representation for a given function. The trick is to be able to determine the coefficients, a_n .

So suppose we have a function $f(x)$, we want:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n, \quad |x-c| < R$$

$$\text{then } a_n = \frac{f^{(n)}(c)}{n!}.$$

Taylor Series:

$$\begin{aligned} f(x) &= \frac{f^{(n)}(c)}{n!} (x-c)^n \\ &= f(c) + \frac{f'(c)}{1!} (x-c) + \frac{f''(c)}{2!} (x-c)^2 + \dots \end{aligned}$$

When $c = 0$, we have the special Taylor series called the Maclaurin Series.

Maclaurin Series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots$$

Now, we have assumed that a given function has a Taylor series expansion. How do we know? When is it possible for a function to have a Taylor series expansion?

Taylor's Formula: If $f(x)$ has $n+1$ derivatives in an interval with $c \in I$, then for $x \in I$ there is a number "z", strictly between x and c st

$$f(x) = f(c) + \frac{f'(c)}{1!} (x-c) + \frac{f''(c)}{2!} (x-c)^2 + \dots + \frac{f^{(n)}(c)}{n!} (x-c)^n + R_n(x)$$

$$\text{where } R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} (x-c)^{n+1}.$$

$R_n(x)$ is called the remainder term.

$$\text{So } f(x) = T_n(x) + R_n(x).$$

Now $f(x)$ is equal to its Taylor Series expansion on $|x-c| < R$ if $\lim_{n \rightarrow \infty} R_n(x) = 0 \dots$

