Superprocess over a stochastic flow
with superprocess catalyst

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Abstract

In this paper, we study the catalytic superprocesses under a stochastic
flow where the catalyst itself is a superprocess under the same flow. Compar-
ing with the study of the superprocess under a stochastic flow with
deterministic catalyst, here we encounter a serious adaptability problem
caused by this common stochastic flow in proving the uniqueness by using
the usual conditional log-Laplace transform approach. To overcome this
difficulty, we find a limiting moment dual and show that the moments
increase not too fast so that the moments determine the distribution. We
also prove the state property by the moment method.

\textbf{Keywords:} Catalytic superprocesses, random medium, branching par-
ticle system

\textbf{AMS 2000 subject classifications:} Primary 60J80; Secondary 60K25.

1 Introduction

Superprocess over a stochastic flow was studied by Skoulakis and Adler [15]
using a moment duality argument. It is conjectured in [15] that the conditional
log-Laplace transform of the superprocess should be the unique solution to a
nonlinear stochastic partial differential equation (SPDE). This conjecture was
proved by Xiong [19]. As the first application of the conditional log-Laplace
approach, Xiong [20] studied the long-term behavior of the superprocesses over a stochastic flow.

A related model has been studied by Wang in two earlier papers ([16], [17]) where the random medium is given by a space-time color-white noise using the moment duality argument. It was proved in Wang ([16], [18]) that the process is absolutely continuous for the uniformly elliptic case and purely-atomic for the degenerate case. For the first case, a SPDE is derived for the density by Dawson et al [7]. This model was later generalized by Dawson et al [6]. For this generalized model, Li et al ([12], [13], [14]) studied the conditional log-Laplace equation and various properties of the process, e.g., the scaling limit, the long-term behavior and the excursion representation.

In this paper, we study a catalyst-reactant pair of superprocesses over a stochastic flow. Without the stochastic flow, this problem has been studied by many authors (see the survey papers of Dawson and Fleischmann [4] and Klenke [11]) since the early work of Dawson and Fleischmann [5].

Now let us introduce our model in details. First, we introduce the catalyst. At independent exponential times (with exponent $\gamma > 0$), the catalyst particles either die or split into 2 with equal probabilities. Between branching times, the motion of the $i$th particle is governed by an individual Brownian motion $B_{c_i}(t)$ and a common Brownian motion $W(t)$ which applies to all particles in the system:

$$
d\eta_i^t = b_c(\eta_i^t)dt + \tilde{\sigma}_c(\eta_i^t)dB_{c_i}(t) + \sigma_c(\eta_i^t)dW(t)
$$

where $b_c$, $\tilde{\sigma}_c$, $\sigma_c : \mathbb{R} \to \mathbb{R}$ are continuous maps, $W$, $B_{c_1}^i$, $B_{c_2}^i$, $\cdots$ are independent (standard) one dimensional Brownian motions, $\eta_i^t$ is the position of the $i$th catalyst particle at time $t$. Skoulakis and Adler [15] proved that the high-density limit $\rho_t$ exists as an $\mathcal{M}_F(\mathbb{R})$-valued process, where $\mathcal{M}_F(\mathbb{R})$ denotes the collection of all finite Borel measures on $\mathbb{R}$. By a conditional log-Laplace argument, Xiong [19] proved that $\rho_t$ is the unique solution to the following conditional martingale problem (CMP); $\forall \phi \in C^2_b(\mathbb{R}),$

$$
M_t^{\phi} = \langle \rho_t, \phi \rangle - \langle \rho_0, \phi \rangle - \int_0^t \langle \rho_s, a_c\phi'' + b_c\phi' \rangle ds
$$

$$
- \int_0^t \langle \rho_s, \sigma_c\phi' \rangle dW_s
$$

is a continuous $P^W$-martingale with quadratic variation process

$$\langle M^{\phi} \rangle_t = \int_0^t \langle \rho_s, \gamma \phi'^2 \rangle ds$$

where $a_c = \frac{1}{2} \left( \sigma_c^2 + \tilde{\sigma}_c^2 \right)$, $P^W$ denotes the conditional probability with the whole path of $W$ as given.

Next, we introduce the reactant branching particle system. Let $\mathcal{A} = \{\alpha = n_1 \cdots n_{\ell(\alpha)} : n_i \in \mathbb{N}\}$ denote the collection of particles. We define an arboreal
order in $\mathcal{A}$ as follows: $m_1 \cdots m_p < n_1 \cdots n_q$ iff $p \leq q$ and $m_i = n_i$, $i = 1, 2, \ldots, p$. Let \{\(B_\alpha(t) : t \geq 0, \alpha \in \mathcal{A}\)\} be a family of independent one-dimensional Brownian motions. Let $S_\alpha$ be i.i.d. with common exponential distribution with parameter $\theta$. For $t \geq 0$ and $x \in \mathbb{R}$, let $\eta_\alpha(t, x), \alpha \in \mathcal{A}$, be i.i.d. $\mathbb{Z}^+$-valued random variables such that

\[
\mathbb{E}^\eta_\alpha(t, x) = 1 \quad \text{and} \quad \mathbb{E}^\eta_\alpha(t, x)^2 = \rho_t(x)
\]

where $\rho_t$ is the Brownian semigroup.

For $\alpha \in \mathcal{A}$, let $\beta_\alpha$ and $\zeta_\alpha$ ($= \beta_\alpha + S_\alpha$) be the birth and death times of the particle $\alpha$. Then $\beta_\alpha = 0$ if $\ell(\alpha) = 1$; $\beta_\alpha = \zeta_\alpha - 1$ if $\ell(\alpha) > 1$ where $\alpha - 1 = n_1 \cdots n_{\ell(\alpha)} - 1$ is the father of $\alpha$. The trajectory $\{x_\alpha(t) : \beta_\alpha \leq t \leq \zeta_\alpha\}$ is given by

\[
x_\alpha(\beta_\alpha + t) = x_\alpha(\beta_\alpha) + \int_{\beta_\alpha}^{\beta_\alpha + t} b_r(x_\alpha(s))ds
\]

where $x_\alpha(\beta_\alpha) = x_{\alpha-1}(\zeta_{\alpha-1})$, $b_r, \tilde{\sigma}_r, \sigma_r : \mathbb{R} \to \mathbb{R}$ are continuous maps. Let

\[
\langle X^\theta_t, \phi \rangle = \theta^{-1} \sum_{\alpha \in \mathcal{A}} \phi(x_\alpha(t))1_{[\beta_\alpha, \zeta_\alpha)}(t).
\]

In this paper, we study the limit of $X^\theta$.

**Theorem 1.** Suppose that $\theta \to \infty$, $\epsilon \to 0$ and $\epsilon^{-2} \theta^{-1} \to 0$. Then $\{X^\theta\}$ is tight in $C([0, \infty), \mathcal{M}_F(\mathbb{R}))$ and the limit $X$ solves the following CMP:

\[
\forall \phi \in C^2_b(\mathbb{R}), \quad \mathbb{E}^{\nu^\theta, \epsilon}(X^\theta_t, \phi) = \langle X^\theta_t, \phi \rangle - \langle X^\theta_0, \phi \rangle - \int_0^t \langle X^\theta, a_r \phi'' + b_r \phi' \rangle ds
\]

\[
- \int_0^t \langle X^\theta, \sigma_r \phi' \rangle dW_s.
\]

is a continuous $\mathbb{P}^{\nu^\theta, \epsilon}$-martingale with quadratic variation process

\[
\langle M^{\nu^\theta, \epsilon} \rangle_t = \langle L_{[X^\theta]}(t), \phi^2 \rangle
\]

where $a_r = \frac{1}{2} \left( \sigma_r^2 + \tilde{\sigma}_r^2 \right)$.

\[
\langle L_{[X^\theta]}(t), \phi^2 \rangle = \lim_{\delta \to 0} \int_0^t ds \frac{1}{\delta} \int_0^\delta du \int X^\theta(dx) \int \rho_s(dy) \int \rho_z(dz)
\]

\[
\mathbb{E}^{\nu^\theta, \epsilon}(u, (x, y), (z, z))\phi^2(z)
\]
is the collision local time between $X$ and $\rho$, $\mathbb{P}^{rc}$ is the transition probability density of the 2-dimensional diffusion consists of one catalyst particle and one reactant particle, i.e., the diffusion with generator

$$L^{cr}f(x_1, x_2) = a_c(x_1)\partial_{x_1}^2 f + b_c(x_1)\partial_{x_1} f + a_r(x_2)\partial_{x_2}^2 f + b_r(x_2)\partial_{x_2} f$$

$$+ \sigma_c(x_1)\sigma_r(x_2)\partial_{x_1,x_2} f$$

for $f \in C^2_b(\mathbb{R}^2)$.

**Remark 2.** Usually, the collision local time is defined as (cf. Dawson et al [3] which generalizes that of Barlow et al [1])

$$\lim_{\delta \to 0} \frac{1}{\delta} \int_0^\delta \int_0^t ds \int \mathbb{R} dx p_{\xi}(x) X_s(x)p_{\xi} \cdot \rho_s(x)\phi(x).$$

(5)

If $X$ and $\rho$ have continuous density, these two definitions coincide and is given by

$$\langle L_{[X,\rho]}(t), \phi^2 \rangle = \int_0^t \langle X_s, \rho_s \phi^2 \rangle \, ds.$$

However, they are different in general. For example, take

$$X(dx) = x^{-1/2}1_{x>0}dx, \quad \rho(dx) = (-x)^{-1/2}1_{x<0}dx.$$

Then $L_{[X,\rho]}(t) = ct\delta_0$ where the constant depends on the approximate procedure.

Also, note that by [8] (Theorem 6.4.5, P141),

$$\mathbb{P}^{rc}(u,(x,y),(z,z)) \leq c p_u(x-z)p_u(y-z)$$

and hence,

$$\langle L_{[X,\rho]}(t), \phi^2 \rangle \leq c \lim_{\delta \to 0} \frac{1}{\delta} \int_0^\delta \int_0^t ds \int \mathbb{R} dx p_{\xi}(x) X_s(x)p_{\xi} \cdot \rho_s(x)\phi^2(x)$$

where $p_{\xi}$ is the heat kernel.

In the special case that $\sigma_r = \sigma_c = 1$, $b_r = b_c = \bar{\sigma}_r = \bar{\sigma}_c = 0$, we see that our definition of the collision local time coincides with (5).

Throughout this paper, we make the following assumptions:

(BC) $\sigma_c$, $\bar{\sigma}_c$, $b_c$ and $\sigma_r$, $\bar{\sigma}_r$, $b_r$ are bounded Lipschitz continuous functions. Further, $\bar{\sigma}_r$ and $\bar{\sigma}_c$ are bounded below away from 0.

(IC) $X_0$, $\rho_0 \in \mathcal{M}_F(\mathbb{R})$.

In this paper, we first prove in Section 2 the tightness of $X^\theta$ and show that each limit point will be a solution to the CMP (3-4).

As indicated by [15] in the study of a related model, it is natural to use the conditional log-Laplace transform of $X_t$ to derive the uniqueness for the
solution to the CMP (3-4). For deterministic branching rate case, this has been studied by [19] and [13]. It has been demonstrated in [20] and [13] that such a log-Laplace transform is useful in deriving properties of the process. Mimic [19], we guess that

$$E_{\mu} W \exp(-\lambda X_t, \phi) = e^{-\lambda (\mu, \gamma t)}$$

where $y_{s,t}$ is governed by the following stochastic partial differential equation (SPDE):

$$y_{s,t}(x) = \phi(x) + \int_s^t \left[ (a_r(x) \partial_x^2 y_{u,t}(x) + b_r(x) \partial_x y_{u,t}(x) - \rho_u(x) y_{u,t}(x)^2) du + \int_s^t \sigma_r(x) \partial_x y_{u,t}(x) d\tilde{W}_u, \right]$$

where $d\tilde{W}_u$ is the backward Itô integral. Note that $\rho_u$ is $\mathcal{F}_t$-measurable and for the backward Itô integral, it requires the integrand to be $\mathcal{F}_{[s,t]}$-measurable. However, it is not clear whether $y_{u,t}(x)$ is $\mathcal{F}_{[s,t]}$-measurable. Therefore, we encounter serious adaptivity problem in solving the SPDE (6). To prove the uniqueness of the solution to the CMP (3-4), we construct a limit moment dual of the process. The following theorem will be proved in Section 3. Recall that $p_s$ is the heat kernel.

**Theorem 3.** Suppose that

$$\sup_{s>0} \int_{\mathbb{R}^2} p_s(x-y)X_0(dx)\rho_0(dy) < \infty. \quad (7)$$

Then the CMP (3-4) has a unique solution.

Finally, in Section 4, we derive a property of the process based on the moment formula.

**Theorem 4.** Suppose that Conditions (BC), (IC) and (7) hold, then $X_t$ is absolutely continuous with respect to Lebesgue measure for almost all $t > 0$ and $\mathbb{P}_\mu$-almost surely.

## 2 Existence of a solution

In this section, we establish the existence of a solution to the CMP (3-4).

**Lemma 5.**

$$\langle X^0_t, \phi \rangle = \langle X^0_0, \phi \rangle + M^0_t(\phi) + N^0_t(\phi)$$

$$+ \int_0^t \langle X^0_s, \sigma_r \phi' \rangle dW_s + \int_0^t \langle X^0_s, a_r \phi'' + b_r \phi' \rangle ds \quad (8)$$
where $M_t^\theta$ and $N_t^\theta$ are orthogonal $\mathcal{P}^{\rho,W}$-martingales with quadratic variation processes

$$\langle M^\theta(\phi) \rangle_t = (2\theta)^{-1} \int_0^t \langle X_s^\theta, \bar{a}_r(\phi')^2 \rangle ds$$

(9)

and

$$\langle N^\theta(\phi) \rangle_t = \int_0^t \langle X_s^\theta, (P_r\rho_s)^2 \rangle ds$$

(10)

where $\bar{a}_r = \tilde{a}_r^2$.

Proof: By Itô’s formula, we see that (8) holds with

$$M_t^\theta(\phi) = \sum_{a \in \mathcal{A}} \int_0^t \theta^{-1} \bar{a}_r \phi'(x_a(s)) \mathbf{1}_{[\theta_0, \zeta_\alpha]}(s) dB_a(s)$$

and

$$N_t^\theta(\phi) = \sum_{a \in \mathcal{A}} (\eta_a(\zeta_\alpha, x_a(\zeta_\alpha -)) - 1) \theta^{-1} \phi(x_a(\zeta_\alpha -)) \mathbf{1}_{(0, t]}(\zeta_\alpha).$$

It is clear that $M_t^\theta(\phi)$ is a $\mathcal{P}^{\rho,W}$-martingale with quadratic variation process given by (9). Similar to Li et al [13], we can prove that $N_t^\theta(\phi)$ is a $\mathcal{P}^{\rho,W}$-martingale, orthogonal to $M_t^\theta(\psi)$, with quadratic variation process given by (10).

To consider the limit of $X^\theta$ as $\theta \to \infty$ and $\epsilon = \epsilon(\theta) \to 0$, we need some estimates on the moments. To derive formulas for the moments of $X_t^\theta$, we consider the following backward SPDE

$$p^W_{s,t}(x) = \phi(x) + \int_s^t \left( a_r \bar{a}_r^2 p^W_{u,t}(x) + b_r \partial_x p^W_{u,t}(x) \right) du + \int_s^t \sigma_r \partial_x p^W_{u,t}(x) dW_u$$

(11)

where $dW$ denotes the backward Itô integral. For convenience, we denote the solution of (11) formally by

$$p^W_{s,t}(x) = \int p^W(s, x; t, du) \phi(u).$$

Lemma 6.

$$\mathbb{E}^{\rho,W} \langle X^\theta_t, \phi \rangle = \langle X_0, p^W_{0,t} \rangle = \int X_0(dx) \int p^W(0, x; t, du) \phi(u).$$

(12)

Proof: Take expectations on both sides of (8), we have

$$\mathbb{E}^{\rho,W} X^\theta_t, \phi \rangle = \langle X_0, \phi \rangle + \int_0^t \langle \mathbb{E}^{\rho,W} X^\theta_s, \sigma_r \phi' \rangle dW_s$$

$$+ \int_0^t \langle \mathbb{E}^{\rho,W} X^\theta_s, a_r \phi'' + b_r \phi' \rangle ds.$$

(13)
(12) follows from similar arguments as in Lemma 5.3 in Li et al [13] (take 
Z = 0, σ = m = 0 there, and replace (5.1) and (5.2) there by current (13) and 
(11)).

For f ∈ C^2_b(R^2), let
\[ Lf(x_1, x_2) = a_r(x_1)\partial_{x_1}^2 f + b_r(x_1)\partial_{x_1} f + a_r(x_2)\partial_{x_2}^2 f + b_r(x_2)\partial_{x_2} f \]
+ \sigma_r(x_1)\sigma_r(x_2)\partial_{x_1,x_2} f \]

and
\[ Gf(x_1, x_2) = \sigma_r(x_1)\partial_{x_1} f + \sigma_r(x_2)\partial_{x_2} f. \]

Let \( P^W_{s,t}(x_1, x_2) \) be the unique solution to the following linear SPDE:
\[ P^W_{s,t} = f + \int_s^t L P^W_{r,t} dr + \int_s^t G P^W_{r,t} dW_t. \]
(14)

Again, for convenience, we denote formally
\[ P^W_{s,t}(x_1, x_2) = \int P^W(s, (x_1, x_2); t, d(u_1, u_2)) f(u_1, u_2). \]

Lemma 7.
\[ \mathbb{E}^W \langle (X_t^\theta)^{\otimes 2}, f \rangle \]
\[ = \int \int X_0(dx_1)X_0(dx_2) \int P^W(0, (x_1, x_2); t, d(u_1, u_2)) f(u_1, u_2) \]
+ \[ \int_0^t ds \int X_0(dx) \int P^W(0, x; s, dy) (P_t \rho_s)(y) \]
\[ \times \int P^W(s, (y, y); t, d(u_1, u_2)) f(u_1, u_2) \]
+ \[ \int_0^t ds \int X_0(dx) \int P^W(0, x; s, dy) \theta^{-1} a_r(y) \]
\[ \times \partial_{y_1,y_2}^2 \bigg|_{y_1 = y_2 = y} \int P^W(s, (y_1, y_2); t, d(u_1, u_2)) f(u_1, u_2). \]

Proof: Apply Itô’s formula to (8), we have
\[ d \langle X_t^\theta, \phi_1 \rangle \langle X_t^\theta, \phi_2 \rangle \]
\[ = d(P^\theta-W\text{-mart.}) + \langle X_t^\theta, \phi_1 \rangle \left( \langle X_t^\theta, a_r \phi_1'' + b_r \phi_1' \rangle dt + \langle X_t^\theta, \sigma_r \phi_1' \rangle dW_t \right) \]
+ \[ \langle X_t^\theta, \phi_2 \rangle \left( \langle X_t^\theta, a_r \phi_2'' + b_r \phi_2' \rangle dt + \langle X_t^\theta, \sigma_r \phi_2' \rangle dW_t \right) \]
+ \[ \left( \langle X_t^\theta, \sigma_r \phi_1' \rangle \langle X_t^\theta, \sigma_r \phi_2 \rangle + \langle X_t^\theta, (P_t \rho_t) \phi_1 \phi_2 \rangle + \theta^{-1} \langle X_t^\theta, a_r \phi_1' \phi_2 \rangle \right) dt. \]
Therefore
\[
\begin{align*}
d \left( \langle X_t^\theta \rangle^{\otimes 2}, f \right) &= \langle X_t^\theta \rangle^{\otimes 2}, \mathbb{L}_t f \rangle dt + \langle X_t^\theta \rangle^{\otimes 2}, \mathbb{G}_t f \rangle dW_t \\
&+ \langle X_t^\theta, (P_t \rho_t) f(x, x) + \theta^{-1} \bar{a}_r \partial^2_{x_1 x_2} \mid x_1 = x_2 = x \rangle f(x_1, x_2) \rangle dt \\
&+ d(P_t^{\rho, W}\text{-}\text{mart.})_t
\end{align*}
\]
holds for \( f = \phi_1 \otimes \phi_2 \). Take expectations on both sides of (16), we have
\[
\begin{align*}
\mathbb{E}^{\rho, W} \langle X_0^\theta, f \rangle &= \int_0^t \langle \mathbb{E}^{\rho, W} \langle X_s^\theta \rangle^{\otimes 2}, \mathbb{L}_s f \rangle ds \\
&+ \int_0^t \langle \mathbb{E}^{\rho, W} \langle X_s^\theta \rangle^{\otimes 2}, \mathbb{G}_s f \rangle dW_s \\
&+ \int_0^t ds \int X_0(dx)p^W(0, x; s, du) \\
&\times \left( (P_s \rho_s)(u) f(u, u) + \theta^{-1} \bar{a}_r(u) \partial^2_{u_1 u_2} \mid_{u_1 = u_2 = u} f(u_1, u_2) \right)
\end{align*}
\]
holds for \( f = \phi_1 \otimes \phi_2 \). By approximation, it is easy to see that (17) holds for all \( f \in C^2_b(\mathbb{R}^2) \). Similar to Lemma 5.3 in [13] again (take \( Z = 0, \sigma = 0 \),
\[m = \int X_0(dx)p^W(0, x; s, du) \]
\[
\left( (P_s \rho_s)(u) f(u, u) + \theta^{-1} \bar{a}_r(u) \partial^2_{u_1 u_2} \mid_{u_1 = u_2 = u} f(u_1, u_2) \right)
\]
there, and replace (5.1) and (5.2) there by current (17) and (14)), we have (15) holds.

The following corollary follows from the same arguments as (12) and (15).

**Corollary 8.** Let \( q_{s,t}^W \) and \( Q_{s,t}^W \) be defined similar to \( p_{s,t}^W \) and \( P_{s,t}^W \) (with \( r \) replaced by \( c \)) in (11) and (14). Then
\[
\mathbb{E}^{W} \langle \rho_s, \phi \mid \rho_s \rangle = \int \rho_s(dx) \int q^W(s, x; s', du) \phi(u)
\]
and
\[
\mathbb{E}^{W} \langle \rho_t^{\otimes 2}, f \mid \rho_s \rangle
\]
\[
= \int \int \rho_s(dx_1) \rho_s(dx_2) \int Q^W(s, (x_1, x_2); t, d(u_1, u_2)) f(u_1, u_2)
\]
\[
+ \int_0^s ds' \int \rho_s(dx) \int q^W(s, x; s', dy) \int Q^W(s', (y, y); t, d(u_1, u_2)) f(u_1, u_2).
\]
Here is the key estimate in proving the tightness.

**Theorem 9.**

$$\sup_{\theta > 0, t \leq T} \mathbb{E} \langle X^\theta_t, P_t \rho_t \rangle^2 < \infty. \quad (19)$$

**Proof:** Take

$$f(x_1, x_2) = (P_t \rho_t)(x_1)(P_t \rho_t)(x_2)$$

in (15), we have

$$\mathbb{E}^W \langle (X^\theta_t)^2, f \rangle \equiv I_1 + I_2 + I_3$$

where

$$I_1 = \int \int X_0(dx_1) X_0(dx_2) \int P^W(0, (x_1, x_2); t, d(u_1, u_2))(P_t \rho_t)(u_1)(P_t \rho_t)(u_2),$$

$$I_2 = \int_0^t ds \int X_0(dx) \int P^W(0, x; s, dy)(P_s \rho_s)(y) \times \int P^W(s, (y, y); t, d(u_1, u_2))(P_t \rho_t)(u_1)(P_t \rho_t)(u_2)$$

and

$$I_3 = \int_0^t ds \int X_0(dx) \int P^W(0, x; s, dy)\theta^{-1} \tilde{a}_r(y) \times \tilde{\partial}^2_{y_1 y_2} |_{y_1 = y_2 = y} \int P^W(s, (y_1, y_2); t, d(u_1, u_2))(P_t \rho_t)(u_1)(P_t \rho_t)(u_2).$$

By Corollary 8, we have

$$\mathbb{E}^W I_3 = \mathbb{E}^W I_{31} + \mathbb{E}^W I_{32}$$

where

$$I_{31} = \int_0^t ds \int X_0(dx) \int P^W(0, x; s, dy)\theta^{-1} \tilde{a}_r(y) \times \tilde{\partial}^2_{y_1 y_2} |_{y_1 = y_2 = y} \int P^W(s, (y_1, y_2); t, d(u_1, u_2)) \int \int \rho_s(dx_1) \rho_s(dx_2)$$

$$\times \int Q^W(s, (x_1, x_2); t, d(z_1, z_2)) p_r(u_1 - z_1)p_r(u_2 - z_2)$$
and

\[ I_{32} = \int_0^t ds \int X_0(dx) \int p^W(0, x; s, dy) \theta^{-1} a_r(y) \]
\[ \times \partial^2_{y_1 y_2} |_{y_1 = y_2 = 0} \int p^W(s, (y_1, y_2); t, d(u_1, u_2)) \]
\[ \times \int \int \int ds' \int \rho_s(dx') \int q^W(s, x'; s', dy') \]
\[ \times \int Q^W(s', (y', y'); t, d(z_1, z_2)) p_e(u_1 - z_1)p_e(u_2 - z_2). \]

Note that

\[ E^W I_{31} = \int_0^t ds \int X_0(dx) \int p^W(0, x; s, dy) \theta^{-1} a_r(y) \]
\[ \times \partial^2_{y_1 y_2} |_{y_1 = y_2 = 0} \int p^W(s, (y_1, y_2); t, d(u_1, u_2)) \int \int \rho_0(dx_1) \rho_0(dx_2) \]
\[ \times \int Q^W(0, (y_1, y_2); s, d(x_1, x_2)) \int Q^W(s, (x_1, x_2); t, d(z_1, z_2)) \]
\[ \times p_e(u_1 - z_1)p_e(u_2 - z_2) \]
\[ + \int_0^t ds \int X_0(dx) \int p^W(0, x; s, dy) \theta^{-1} a_r(y) \]
\[ \times \partial^2_{y_1 y_2} |_{y_1 = y_2 = 0} \int p^W(s, (y_1, y_2); t, d(u_1, u_2)) \int_0^s ds' \int \rho_0(dx_1) \]
\[ \times \int q^W(0, x_1; s', d(v); s, d(v_1, v_2)) \int Q^W(s', (v, v); s, d(v_1, v_2)) \]
\[ \times \int Q^W(s, (v_1, v_2); t, d(z_1, z_2)) p_e(u_1 - z_1)p_e(u_2 - z_2) \]
\[ = I_{311} + I_{312}. \]

Then

\[ E I_{312} = \int_0^t ds \int_0^s ds' \int X_0(dx) \int \int p^W(0, x; s', dz) p^W(s', z; s, dy) \theta^{-1} a_r(y) \]
\[ \times \partial^2_{y_1 y_2} |_{y_1 = y_2 = 0} \int p^W(s, (y_1, y_2); t, d(u_1, u_2)) \int \rho_0(dx_1) \]
\[ \times \int q^W(0, x_1; s', dv) \int Q^W(s', (v, v); s, d(v_1, v_2)) \]
\[ \times \int Q^W(s, (v_1, v_2); t, d(z_1, z_2)) p_e(u_1 - z_1)p_e(u_2 - z_2). \]
It is easy to show that
\[ \mathbb{E}p^W(0, x; s', dz)q^W(0, x_1; s', dv) = \mathbb{P}^{rc}(s', (x, x_1), d(z, v)) \]
where the notation \( \mathbb{P}^{rc} \) represents the transition probability for the two-dimensional diffusion consists of 1 catalyst point and 1 reactant particle. Similarly, we have
\[ \mathbb{E}p^W(s', z; dy)Q^W(s', (v, v); s, d(v_1, v_2)) = \mathbb{P}^{rc}(s - s', (z, v, v), d(y, v_1, v_2)) \]
and
\[ \mathbb{E}p^W(s, (y_1, y_2); t, d(u_1, u_2))Q^W(s, (v_1, v_2); t, d(z_1, z_2)) = \mathbb{P}^{rrcc}(t - s, (y_1, y_2, v_1, v_2), d(u_1, u_2, z_1, z_2)). \]

Hence
\[ \mathbb{E}I_{312} = \int_0^t ds_0 \int_0^s ds' \int X_0(dx) \int p_0(dx_1) \int \mathbb{P}^{rc}(s', (x, x_1), d(z, v)) \]
\[ \times \int \mathbb{P}^{rc}(s - s', (z, v, v), d(y, v_1, v_2)) \theta^{-1}(y) \]
\[ \times \partial_{y_1 y_2}^2 \mathbb{P}^{rrcc}(t - s, (y_1, y_2, v_1, v_2), d(u_1, u_2, z_1, z_2)) \]
\[ \times p_c(u_1 - z_1)p_c(u_2 - z_2). \] \hfill (20)

Let \( \xi_t = \xi_t(y_1, y_2, v_1, v_2) \) be the 4-dimensional diffusion with transition probability \( \mathbb{P}^{rrcc} \) and initial location \((y_1, y_2, v_1, v_2)\). We now want to estimate
\[ \partial_{y_1 y_2}^2 \mathbb{E}p_c(\xi_t^1 - \xi_t^3)p_c(\xi_t^2 - \xi_t^4) \]
\[ = \mathbb{E} \partial_{y_2} \left( p'_c(\xi_t^1 - \xi_t^3)p_c(\xi_t^2 - \xi_t^4)\partial_{y_1}(\xi_t^3 - \xi_t^4) \right. \]
\[ + p_c(\xi_t^1 - \xi_t^3)p'_c(\xi_t^2 - \xi_t^4)\partial_{y_1}(\xi_t^2 - \xi_t^4) \]
\[ = \mathbb{E} p'_c(\xi_t^1 - \xi_t^3)p_c(\xi_t^2 - \xi_t^4)\partial_{y_1}(\xi_t^3 - \xi_t^4) \]
\[ + 2\mathbb{E} p'_c(\xi_t^1 - \xi_t^3)p'_c(\xi_t^2 - \xi_t^4)\partial_{y_1}(\xi_t^3 - \xi_t^4)\partial_{y_2}(\xi_t^2 - \xi_t^4) \]
\[ + \mathbb{E} p'_c(\xi_t^1 - \xi_t^3)p_c(\xi_t^2 - \xi_t^4)\partial_{y_1 y_2}(\xi_t^3 - \xi_t^4) \]
\[ + \mathbb{E} p_c(\xi_t^1 - \xi_t^3)p'_c(\xi_t^2 - \xi_t^4)\partial_{y_1}(\xi_t^3 - \xi_t^4) \]
\[ + \mathbb{E} p_c(\xi_t^1 - \xi_t^3)p'_c(\xi_t^2 - \xi_t^4)\partial_{y_1 y_2}(\xi_t^3 - \xi_t^4). \]

Now we only consider the first term. Note that
\[ p''_c \leq cc^{-2}. \]
So
\[ 1st \leq cc^{-2}\mathbb{E} |\partial_{y_1}(\xi_t^1 - \xi_t^3)\partial_{y_2}(\xi_t^2 - \xi_t^4)| \leq cc^{-2} \]

11
where the last inequality follows from Theorem 5.4 in ([8], p.122). The other terms can be treated similarly. Now we can continue (20) with
\[ E \int_{0}^{t} ds \int_{0}^{s} ds' \int X_{0}(dx) \int \rho(dx_{1}) = c e^{-2\theta^{-1}}. \]
Take \( e^{-2\theta^{-1}} \to 0 \). We have \( E I_{312} \to 0 \). \( I_{311} \) and \( I_{32} \) can be treated similarly. Therefore, \( E I_{3} \to 0 \). Similar calculations for \( I_{1} \) and \( I_{2} \) yield that
\[ \sup_{\theta > 0, t \leq T} E(I_{1} + I_{2} + I_{3}) < \infty. \]
This proves (19).

Proof of Theorem 1: From (19), we easily get the tightness of \( (X^{\theta}, \langle M^{\theta}(\phi) \rangle, \langle N^{\theta}(\phi) \rangle) \).
Further, it is easy to show that for \( t > 0 \) fixed, \( X_{t}^{\theta}(\phi)^{2} \) and \( \langle N^{\theta}(\phi) \rangle \) are uniformly integrable.
It is clear that \( M^{\theta}(\phi) \to 0 \). Suppose that \( (X^{\theta}, \langle N^{\theta}(\phi) \rangle) \to (X, \Lambda) \). We only need to show that \( \Lambda(t) = \langle L_{X,M}(t), \phi^{2} \rangle \).

Now we adapt the argument in the proof of Theorem 3.4 in Xiong and Zhou [21]. Note that
\[
\begin{align*}
&\mathbb{E}(\Lambda(t) - \Lambda(s) | \mathcal{F}_{s}) \\
= &\lim_{\theta \to \infty} \mathbb{E}(\langle N^{\theta}(\phi) \rangle_{t} - \langle N^{\theta}(\phi) \rangle_{s} | \mathcal{F}_{s}) \\
= &\lim_{\theta \to \infty} \mathbb{E} \left( \int_{s}^{t} \langle X_{w}^{\theta}, P_{t}(\theta)\rho_{u}\phi^{2} \rangle dw \bigg| \mathcal{F}_{s} \right) \\
= &\lim_{\theta \to \infty} \mathbb{E} \left( \int_{s}^{t} dw \int X_{w}^{\theta}(dx) \int p^{W}(s, x; u, dy) P_{t}(\theta)\rho_{u}(y)\phi^{2}(y) \bigg| \mathcal{F}_{s} \right)
\end{align*}
\]
where the last equality follows from (12). By (18), we can continue with
\[
\begin{align*}
&\mathbb{E}(\Lambda(t) - \Lambda(s) | \mathcal{F}_{s}) \\
= &\lim_{\theta \to \infty} \mathbb{E} \left( \int_{s}^{t} dw \int X_{w}^{\theta}(dx) \int p^{W}(s, x; u, dy) \int dz p_{z}(y - z) \\
&\times \int \rho_{s}(dw) q^{W}(s, w; u, y)\phi^{2}(y) \bigg| \rho_{s}, X_{s}^{\theta} \right) \\
= &\int_{s}^{t} dw \int X_{s}(dx) \int \rho_{s}(dw) \int dy \mathbb{E}^{w}(u - s, (x, w), (y, y))\phi^{2}(y).
\end{align*}
\]
Therefore,

\[
\int_0^t \frac{1}{\delta} \mathbb{E}(\Lambda(s + \delta) - \Lambda(s)|\mathcal{F}_s) \, ds = \int_0^t \frac{1}{\delta} \int_s^{s+\delta} du \int X_s(dx) \int \rho_s(dy) \int dz \mathbb{P}^{rc}(u - s, (x, y), (z, z))\phi^2(z).
\]

Take \(\delta \to 0\), we have

\[
\Lambda(t) = \lim_{\delta \to 0} \int_0^t \frac{1}{\delta} \int_0^\delta du \int X_s(dx) \int \rho_s(dy) \int dz \mathbb{P}^{rc}(u, (x, y), (z, z))\phi^2(z) = \langle L_{[X,\rho]}(t), \phi^2 \rangle.
\]

\[\square\]

3 Moment duality

In this section, we first establish a moment duality for the process \((X_t, \rho_t)\). Then we verify Carleman’s condition to show that the distribution of \((X_t, \rho_t)\) is determined by the moments.

First, we need the following lemma.

**Lemma 10.** If \((\rho, X)\) is a solution to the CMP (1-4), then it solves the following martingale problem (MP):

\[
N^{c,\phi}_t \equiv \langle \rho_t, \phi \rangle - \langle \rho_0, \phi \rangle - \int_0^t \langle \rho_s, a_c\phi'' + b_c\phi' \rangle \, ds
\]

and

\[
N^{r,\psi}_t \equiv \langle X_t, \psi \rangle - \langle X_0, \psi \rangle - \int_0^t \langle X_s, a_r\psi'' + b_r\psi' \rangle \, ds
\]

are continuous martingales with quadratic covariation processes

\[
\langle N^{c,\phi} \rangle_t = \int_0^t \langle \rho_s, \sigma_c\phi' \rangle^2 \, ds + \int_0^t \langle \rho_s, \gamma \phi^2 \rangle \, ds,
\]

\[
\langle N^{r,\psi} \rangle_t = \int_0^t \langle X_s, \sigma_r\psi' \rangle^2 \, ds + \langle L_{[X,\rho]}(t), \psi^2 \rangle,
\]

and

\[
\langle N^{c,\phi}, N^{r,\psi} \rangle_t = \int_0^t \langle \rho_s, \sigma_c\phi' \rangle \langle X_s, \sigma_r\psi' \rangle \, ds.
\]
Proof: Since $M^{c,\phi}$ is a $P^{\rho,W}$-martingale, it is clearly a martingale. For $s < t$, we have
\[
\mathbb{E}(M^{c,\phi}_t M^{c,\psi}_t | \mathcal{F}_s) = \mathbb{E}(\mathbb{E}(M^{c,\psi}_t | \sigma(\rho, W) \vee \mathcal{F}_s) M^{c,\phi}_t | \mathcal{F}_s)
\]
\[
= \mathbb{E}(M^{r,\psi}_s M^{c,\phi}_t | \mathcal{F}_s)
\]
\[
= M^{r,\psi}_s M^{c,\phi}_t.
\]
Hence $\langle M^{c,\phi}, M^{r,\psi} \rangle_t = 0$. Similarly,
\[
\langle M^{r,\psi}, W \rangle_t = \langle M^{c,\phi}, W \rangle_t = 0.
\]
Hence
\[
N^{c,\phi}_t = M^{c,\phi}_t + \int_0^t \langle \rho_s, \sigma_c \phi' \rangle dW_s
\]
is a martingale. Similarly, $N^{r,\psi}_t$ is a martingale. Further, we have
\[
\langle N^{c,\phi}, N^{r,\psi} \rangle_t = \int_0^t \langle X_s, \sigma_r \psi' \rangle \langle \rho_s, \sigma_c \phi' \rangle ds.
\]
The other statement can be proved similarly. \[\square\]

Apply Itô’s formula to (3) and (1), we have
\[
d\Pi^n_t (X_t, \phi_i)
\]
\[
= \sum_{j=1}^{n} (\Pi_{i \neq j} \langle X_t, \phi_i \rangle) \left( \langle X_t, a_r \phi''_j + b_r \phi'_j \rangle dt + \langle X_t, \sigma_r \phi'_j \rangle dW_t + dM^{r,\phi}_t \right)
\]
\[
+ \frac{1}{2} \sum_{1 \leq j \neq k \leq m} (\Pi_{i \neq j,k} \langle X_t, \phi_i \rangle)
\]
\[
\left( \langle X_t, \sigma_r \phi'_j \rangle \langle X_t, \sigma_r \phi'_k \rangle dt + \langle L[X, \rho](dt), \phi_j \phi_k \rangle \right)
\]
and
\[
d\Pi^n_t (\rho_t, \psi_i)
\]
\[
= \sum_{j=1}^{n} (\Pi_{i \neq j} \langle \rho_t, \psi_i \rangle) \left( \langle \rho_t, a_c \psi''_j + b_c \psi'_j \rangle dt + \langle \rho_t, \sigma_c \psi'_j \rangle dW_t + dM^{c,\psi}_t \right)
\]
\[
+ \frac{1}{2} \sum_{1 \leq j \neq k \leq m} (\Pi_{i \neq j,k} \langle \rho_t, \psi_i \rangle)
\]
\[
\left( \langle \rho_t, \sigma_c \psi'_j \rangle \langle \rho_t, \sigma_c \psi'_k \rangle + \langle \rho_t, \gamma \psi_j \psi_k \rangle \right) dt.
\]
Let
\[
f(x_1, \ldots, x_m; y_1, \ldots, y_n) = \Pi^n_{i=1} \phi_i(x_i) \Pi^n_{j=1} \psi_j(y_j)
\]
and
\[
F_{m,n,f}(X, \rho) = \langle X^\otimes m \otimes \rho^\otimes n, f \rangle.
\]
Define the generator for \( m + n \)-points motion as

\[
A_{m,n} f(x_1, \cdots, x_m; y_1, \cdots, y_n) = \sum_{j=1}^m \left( a_r(x_j) \frac{\partial^2 f}{\partial x_j^2} + b_r(x_j) \frac{\partial f}{\partial x_j} \right) + \frac{1}{2} \sum_{1 \leq j \neq k \leq m} \sigma_r(x_j) \sigma_r(x_k) \frac{\partial^2 f}{\partial x_j \partial x_k}
\]

\[
+ \sum_{j=1}^n \left( a_c(y_j) \frac{\partial^2 f}{\partial y_j^2} + b_c(y_j) \frac{\partial f}{\partial y_j} \right) + \frac{1}{2} \sum_{1 \leq j \neq k \leq n} \sigma_c(y_j) \sigma_c(y_k) \frac{\partial^2 f}{\partial y_j \partial y_k}
\]

\[
+ \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^n \sigma_r(x_j) \sigma_c(y_k) \frac{\partial^2 f}{\partial x_j \partial y_k}.
\]

Apply Itô’s formula to (21) and (22), we see that

\[
F_{m,n,f}(X_t, \rho_t) - \int_0^t \frac{1}{2} \sum_{1 \leq j \neq k \leq m} \left< X_s^\otimes (m-2) \otimes L_{[X,\rho]}(ds) \otimes \rho_s^\otimes n, G_{jk}^r f \right>
\]

\[
- \int_0^t \left( F_{m,n,A.m.n,f}(X_s, \rho_s) + \frac{1}{2} \sum_{1 \leq j \neq k \leq n} F_{m,n-1,G_{jk}^c f}(X_s, \rho_s) \right) ds \tag{23}
\]

is a martingale, where

\[
G_{jk}^r f(x_1, \cdots, x_{m-2}; y; y_n) = f(x_1, \cdots, x, \cdots, x, \cdots, x_{m-2}; y_1, \cdots, y_n)
\]

and

\[
G_{jk}^c f(x_1, \cdots, x_m; y_1, \cdots, y_{n-2}, y) = \gamma f(x_1, \cdots, x; y_1, \cdots, y_1, \cdots, y_{n-2}),
\]

where the \( x \) and \( y \) are at their respective \( j \)th and \( k \)th places. By approximation, we can show that (23) holds for other bounded smooth function \( f \) (instead of a product). Note that

\[
\int_0^t \frac{1}{2} \sum_{1 \leq j \neq k \leq m} \left< G_{jk}^r f, X_s^\otimes (m-2) \otimes L_{[X,\rho]}(ds) \otimes \rho_s^\otimes n \right>
\]

\[
= \lim_{\delta \to 0} \int_0^t \frac{1}{2} \sum_{1 \leq j \neq k \leq m} F_{m-1,n+1,G_{jk}^r \delta f}(X_s, \rho_s) ds,
\]

where

\[
G_{jk}^r \delta f(x_1, \cdots, x_{m-2}, x; y_1, \cdots, y_n)
\]

\[
= \frac{1}{\delta} \int_0^\delta \! dc \int_{\mathbb{R}} \! d\zeta P_{r c}(\epsilon, (x, y), (z, z)) \times f(x_1, \cdots, z, \cdots, z, \cdots, x_{m-2}; y_1, \cdots, y_n).
\]
Now we construct an approximate dual process \((m_t, n_t, f^\delta_t)\) as follows: \((m_t, n_t)\) is a birth-death process with a rate \(\frac{1}{2}m(m - 1)\) to jump from \((m, n)\) to \((m - 1, n + 1)\) and a rate \(\frac{1}{2}n(n - 1)\) from \((m, n)\) to \((m - 1, n + 1)\). Namely, the second component is Kingman’s coalescent process but gives a birth to the second type during its coalescent event.

Let \(\tau_0 = 0\) and \(\tau_{m_0 + n_0 - 1} = \infty\), and let \(\{\tau_k : 1 \leq k \leq m_0 + n_0 - 2\}\) be the sequence of jump times of \(((m_t, n_t) : t \geq 0)\). Let \(\{\Gamma^\delta_k \in [0, \infty) : 1 \leq k \leq m_0 + n_0 - 2\}\) be a sequence of random operators which are conditionally independent given \(((m_t, n_t) : t \geq 0)\) and satisfy

\[
P(\Gamma^\delta_k = G^\delta_{ij} | \tau_k = \ell, n_{\tau_k} = \ell - 1) = \frac{1}{\ell(\ell - 1)}, \quad 1 \leq i \neq j \leq \ell
\]

and

\[
P(\Gamma^\delta_k = G^\delta_{ij} | m_{\tau_k} = \ell, m_{\tau_k} = \ell - 1) = \frac{1}{\ell(\ell - 1)}, \quad 1 \leq i \neq j \leq \ell.
\]

For \(\tau_k \leq t < \tau_{k+1}\), we define

\[
f^\delta_t = P^{(m_{\tau_k}, n_{\tau_k})}_t \Gamma^\delta_k P^{(m_{\tau_{k-1}}, n_{\tau_{k-1}})}_{\tau_k - \tau_{k-1}} \Gamma^\delta_{k-1} \ldots P^{(m_{i}, n_{i})}_{\tau_{i-1} - \tau_i} \Gamma^\delta_{i} P^{(m_{0}, n_{0})}_0 f_0
\]  
(24)

where \(P^{(m, n)}_t\) is the semigroup generated by \(\mathcal{A}^{m,n}\) of an \(m + n\)-dimensional diffusion process. Then \((m_t, n_t, f^\delta_t)\) is a Markov process taking values on \(\mathbb{N}^2 \times \mathbb{C}\) where \(\mathbb{C} = \cup_{m \geq 1} C(\mathbb{R}^m)\).

**Theorem 11.**

\[
E(X^{\otimes m} \otimes \rho^{\otimes n}, f) = \lim_{\delta \to 0} E(\mathbb{E}(X^{\otimes m} \otimes \rho^{\otimes n}, f^\delta_t) \exp \left( \frac{1}{2} \int_0^t (m_s(m_s - 1) + n_s(n_s - 1)) ds \right))
\]  
(25)

Proof: When \(m = 0\), this theorem has been proved by [6] for all \(n\). In that case, the process \((n_t, f_t)\) does not depend on \(\delta\) and is the dual process of \(\rho_t\). For \(m = 1\), there is no coalescence for the first component and hence, the theorem follows from the case of \(m = 0\) directly. Now we prove our theorem by induction in \(m\).

We assume that (25) holds for \(m\) and prove it for \(m + 1\). To this end, we first verify that

\[
E\left(\langle X_t, 1 \rangle^{m+1} \langle \rho_t, 1 \rangle^n \right) < \infty.
\]  
(26)

By (23),

\[
\langle X_t, 1 \rangle^{m+1} \langle \rho_t, 1 \rangle^n - \int_0^t \frac{m(m + 1)}{2} \langle X_s, 1 \rangle^{m-1} \langle \rho_s, 1 \rangle^n \langle L_{X, \rho}(ds), 1 \rangle
\]

\[
- \int_0^t \frac{n(n - 1)}{2} \langle X_s, 1 \rangle^{m+1} \langle \rho_s, 1 \rangle^{n-1} ds
\]  
(27)
is a martingale. We estimate
\[
\mathbb{E} \int_0^t (X_s, 1)^{m-1} (\rho_s, 1)^n \langle L_{[X_s, \rho]}(ds), 1 \rangle \\
\leq c \liminf_{\delta \to 0} \mathbb{E} \int_0^t (X_s, 1)^{m-1} (\rho_s, 1)^n \left( X_s \otimes \rho_s, \frac{1}{\delta} \int_0^\delta d \varphi_p(x - y) \right) ds.
\]

Take
\[
f_0(x_1, \ldots, x_m; y_1, \ldots, y_{n+1}) = \frac{1}{\delta} \int_0^\delta d \varphi_{2\delta}(x_m - y_{n+1}).
\]

Now we estimate the process defined by (24). By Friedman [8] (Theorem 6.4.5, P141), we know that $P_t^{(m,n)}$ has a density dominated by $c \varphi_t^{(m+n)}$. Hence, for $\tau_1 \leq t$,
\[
P_{\tau_1}^{(m_0,n_0)} f_0 \leq c \frac{1}{\delta} \int_0^\delta d \varphi_{2\delta}(x - y) \leq c \frac{1}{\sqrt{\tau_1}}
\]
(note that the constant $c$ depends on $t$ but not on $\tau_1$). Hence
\[
f_t^\delta \leq c \Pi_{j=1}^k (\tau_j - \tau_{j-1})^{-1/2}.
\]

Therefore
\[
\mathbb{E} \left( X_s, 1 \right)^{m-1} (\rho_s, 1)^n \left( X_s \otimes \rho_s, \frac{1}{\delta} \int_0^\delta d \varphi_p(x - y) \right) \leq c \mathbb{E} \left( \Pi_{j=1}^k (\tau_j - \tau_{j-1})^{-1/2} \right) < \infty,
\]
where the finiteness above follows from the fact that $\tau_j - \tau_{j-1} \geq 1$ are independent exponential random variables. Applying induction to (27) with respect to $n$, we can show that (26) holds. Now we proceed to proving (25) for $m + 1$.

Let $0 = t_0 < t_1 < \cdots < t_k = t$ be a partition of $[0, t]$ such that
\[
\max\{t_i - t_{i-1} : 1 \leq i \leq k\} \to 0.
\]
Then
\[
\begin{align*}
&\mathbb{E} \left( X_t^{\otimes m} \otimes \rho_t^{\otimes n}, f \right) \\
&- \mathbb{E} \left[ \left( X_0^{\otimes m} \otimes \rho_0^{\otimes n}, f^\delta_0 \right) \exp \left\{ \frac{1}{2} \int_0^t (m_s(m_s - 1) + n_s(n_s - 1)) \, ds \right\} \right] \\
&= \sum_{i=1}^k \left\{ \mathbb{E} \left[ \left( X_{t_i}^{\otimes m_{t-i}} \otimes \rho_{t_i}^{\otimes n_{t-i}}, f^\delta_{t_i} \right) \exp \left\{ \frac{1}{2} \int_0^{t_{t-i}} (m_s(m_s - 1) + n_s(n_s - 1)) \, ds \right\} \right] \\
&- \mathbb{E} \left[ \left( X_{t_i-1}^{\otimes m_{t-i-1}} \otimes \rho_{t_i-1}^{\otimes n_{t-i-1}}, f^\delta_{t_i-1} \right) \exp \left\{ \frac{1}{2} \int_0^{t_{t-i-1}} (m_s(m_s - 1) + n_s(n_s - 1)) \, ds \right\} \right] \right\}.
\end{align*}
\]

Note that
\[
\lim_{k \to \infty} \sum_{i=1}^k \left\{ \mathbb{E} \left[ \left( X_{t_i}^{\otimes m_{t-i}} \otimes \rho_{t_i}^{\otimes n_{t-i}}, f^\delta_{t_i} \right) \exp \left\{ \frac{1}{2} \int_0^{t_{t-i}} (m_s(m_s - 1) + n_s(n_s - 1)) \, ds \right\} \right] \\
- \mathbb{E} \left[ \left( X_{t_i-1}^{\otimes m_{t-i-1}} \otimes \rho_{t_i-1}^{\otimes n_{t-i-1}}, f^\delta_{t_i-1} \right) \exp \left\{ \frac{1}{2} \int_0^{t_{t-i-1}} (m_s(m_s - 1) + n_s(n_s - 1)) \, ds \right\} \right] \right\} = - \int_0^t \mathbb{E} \left( \exp \left\{ \frac{1}{2} \int_u^t (m_s(m_s - 1) + n_s(n_s - 1)) \, ds \right\} \right)
\]

\[
\begin{align*}
&\left( F_{m_{t-u}, n_{t-u}, A_{m_{t-u}, n_{t-u}}^{f_{t-u}}}(X_u, \rho_u) \\
&+ \frac{1}{2} \sum_{j,k=1}^{n_{t-u}} F_{m_{t-u}, n_{t-u}-1, G_{j,k}^{f_{t-u}}}(X_u, \rho_u) \\
&- \frac{m_{t-u}(m_{t-u} - 1) + n_{t-u}(n_{t-u} - 1)}{2} F_{m_{t-u}, n_{t-u}, f_{t-u}^\delta}(X_u, \rho_u) \right) \, du \\
&- \int_0^t \mathbb{E} \left( \exp \left\{ \frac{1}{2} \int_0^u (m_s(m_s - 1) + n_s(n_s - 1)) \, ds \right\} \\
&\frac{1}{2} \sum_{j,k=1}^{m_{t-u}} \left( X_u^{\otimes (m_{t-u} - 2)} \otimes L_{X,\rho}^t(du) \otimes \rho_u^{n_{t-u}}, G_{j,k}^{f_{t-u}} \right) \right) \, du.
\end{align*}
\]
\[
\lim_{k \to \infty} \sum_{i=1}^{k} \{ \mathbb{E} \left[ \left. X_{t_i}^{\otimes m_{t_{i-1}}} \otimes \rho_{t_{i-1}}, f_{t_{i-1}}^{\delta} \right| X_{t_{i-1}}^{\otimes m_{t_{i-1}}} \otimes \rho_{t_{i-1}}, f_{t_{i-1}}^{\delta} \right] \\
- \mathbb{E} \left[ \left. X_{t_{i-1}}^{\otimes m_{t_{i-1}}} \otimes \rho_{t_{i-1}}, f_{t_{i-1}}^{\delta} \right| X_{t_{i-1}}^{\otimes m_{t_{i-1}}} \otimes \rho_{t_{i-1}}, f_{t_{i-1}}^{\delta} \right] \exp \left\{ \frac{1}{2} \int_{0}^{t-t_{i-1}} (m_{s}(m_{s} - 1) + n_{s}(n_{s} - 1)) \, ds \right\} \} \\
= \int_{0}^{t} \mathbb{E} \left( \exp \left\{ \frac{1}{2} \int_{0}^{t-u} (m_{s}(m_{s} - 1) + n_{s}(n_{s} - 1)) \, ds \right\} \\
+ \frac{1}{2} \sum_{j,k=1}^{n_{t-u}} \left\langle X_{u}^{\otimes (m_{t-u} - 2)} \otimes L_{X_{u}^{\otimes n_{t-u}}, G_{j,k}^{f_{t-u}}}, \rho_{u}^{\otimes n_{t-u}}, G_{j,k}^{f_{t-u}} \right\rangle \right) \, du \\
+ \int_{0}^{t} \left( \exp \left\{ \frac{1}{2} \int_{0}^{t-u} (m_{s}(m_{s} - 1) + n_{s}(n_{s} - 1)) \, ds \right\} \\
+ \frac{1}{2} \sum_{j,k=1}^{m_{t-u}} \left\langle X_{u}^{\otimes (m_{t-u} - 2)} \otimes L_{X_{u}^{\otimes n_{t-u}}, G_{j,k}^{f_{t-u}}}, \rho_{u}^{\otimes n_{t-u}}, G_{j,k}^{f_{t-u}} \right\rangle \right) \, du \\
\right)
\]

and

\[
\lim_{k \to \infty} \sum_{i=1}^{k} \{ \mathbb{E} \left[ \left. X_{t_{i-1}}^{\otimes m_{t_{i-1}}} \otimes \rho_{t_{i-1}}, f_{t_{i-1}}^{\delta} \right| X_{t_{i-1}}^{\otimes m_{t_{i-1}}} \otimes \rho_{t_{i-1}}, f_{t_{i-1}}^{\delta} \right] \\
- \mathbb{E} \left[ \left. X_{t_{i-1}}^{\otimes m_{t_{i-1}}} \otimes \rho_{t_{i-1}}, f_{t_{i-1}}^{\delta} \right| X_{t_{i-1}}^{\otimes m_{t_{i-1}}} \otimes \rho_{t_{i-1}}, f_{t_{i-1}}^{\delta} \right] \exp \left\{ \frac{1}{2} \int_{0}^{t-t_{i-1}} (m_{s}(m_{s} - 1) + n_{s}(n_{s} - 1)) \, ds \right\} \} \\
= - \int_{0}^{t} \left( \exp \left\{ \frac{1}{2} \int_{0}^{t-u} (m_{s}(m_{s} - 1) + n_{s}(n_{s} - 1)) \, ds \right\} \\
+ \frac{1}{2} \sum_{j,k=1}^{m_{t-u}} \left\langle X_{u}^{\otimes (m_{t-u} - 2)} \otimes L_{X_{u}^{\otimes n_{t-u}}, G_{j,k}^{f_{t-u}}}, \rho_{u}^{\otimes n_{t-u}}, G_{j,k}^{f_{t-u}} \right\rangle \right) \, du.
\]
Put together, we have

\[
\mathbb{E} \left( X_t^{m} \otimes \rho_t^{n}, f \right) - \mathbb{E} \left[ X_0^{m} \otimes \rho_0^{n}, f_1 \right] \exp \left\{ \frac{1}{2} \int_0^t (m_s(m_s - 1) + n_s(n_s - 1)) \, ds \right\} = - \frac{1}{2} \int_0^t \mathbb{E} \left( \exp \left\{ \frac{1}{2} \int_0^{t-u} (m_s(m_s - 1) + n_s(n_s - 1)) \, ds \right\} \right.

\]

\[
\sum_{j,k=1}^{m_t-u} \left( \left< X_u^{(m_t-u-2)} \otimes L_{[X,t]}(du) \otimes \rho_u^{n_t-u}, G_{jk}^{t-u} \right> - \left< X_u^{(m_t-u-2)} \otimes L_{[X,t]}(du) \otimes \rho_u^{n_t-u}, G_{jk}^{t-u} \right> \right) \, du. \tag{29}
\]

Take a subsequence if necessary, we may and will assume that \( L_{[X,t]}^\delta \rightarrow L_{[X,t]} \) in \( C([0,t], \mathcal{M}_F(\mathbb{R})) \) a.s. as \( \delta \rightarrow 0 \). Note that \( m_s(m_s - 1) + n_s(n_s - 1) \) is bounded. The other factor on the right hand side of (29) is dominated by the right hand side of (28). By the dominated convergence theorem, we have

\[
\lim_{\delta \rightarrow 0} \mathbb{E} \int_0^t \left( \exp \left\{ \frac{1}{2} \int_0^{t-u} (m_s(m_s - 1) + n_s(n_s - 1)) \, ds \right\} \right.

\]

\[
\left( \left< X_u^{(m_t-u-2)} \otimes L_{[X,t]}^\delta(du) \otimes \rho_u^{n_t-u}, G_{jk}^{t-u} \right> - \left< X_u^{(m_t-u-2)} \otimes L_{[X,t]}(du) \otimes \rho_u^{n_t-u}, G_{jk}^{t-u} \right> \right) \, du = \mathbb{E} \lim_{\delta \rightarrow 0} \int_0^t \left( \exp \left\{ \frac{1}{2} \int_0^{t-u} (m_s(m_s - 1) + n_s(n_s - 1)) \, ds \right\} \right.

\]

\[
\left( \left< X_u^{(m_t-u-2)} \otimes L_{[X,t]}^\delta(du) \otimes \rho_u^{n_t-u}, G_{jk}^{t-u} \right> - \left< X_u^{(m_t-u-2)} \otimes L_{[X,t]}(du) \otimes \rho_u^{n_t-u}, G_{jk}^{t-u} \right> \right) \, du. \]
Therefore, we only need to prove that

\[
\lim_{\delta \to 0} \int_0^t \left( \exp \left\{ \frac{1}{2} \int_0^{t-u} (m_s(m_s-1) + n_s(n_s-1)) \, ds \right\} \\
\left( \left\langle X_u^{(m_s-2)} \otimes L_{[X,\rho]}(du) \otimes \rho_u^{\otimes m_s-2}, G_{jk}^\delta f_t^{\delta} \right\rangle \\
- \left\langle X_u^{(m_s-2)} \otimes L_{[X,\rho]}(du) \otimes \rho_u^{\otimes m_s-2}, G_{jk}^\delta f_t^{\delta} \right\rangle \right) \right)
\]

= 0 \quad a.s. \quad (30)

Now we fix \( \omega \). Since there are only finite many jumps for the process \((m_{t-u}, n_{t-u}, f_{t-u}^\delta)\), we estimate the limit for the integral over subinterval \([a,b] \) such that no jump occurs in it. In this case, \( f_{t-u}^\delta \) is bounded and equi-continuous in \( u \in [a,b] \). Therefore

\[
\lim_{\delta \to 0} \int_a^b \left( \exp \left\{ \frac{1}{2} \int_0^{t-u} (m_s(m_s-1) + n_s(n_s-1)) \, ds \right\} \\
\left( \left\langle X_u^{(m_s-2)} \otimes L_{[X,\rho]}(du) \otimes \rho_u^{\otimes m_s-2}, G_{jk}^\delta f_t^{\delta} \right\rangle \\
- \left\langle X_u^{(m_s-2)} \otimes L_{[X,\rho]}(du) \otimes \rho_u^{\otimes m_s-2}, G_{jk}^\delta f_t^{\delta} \right\rangle \right) \right)
\]

= 0.

Sum up over all such subintervals, we see that (30) holds.

Now we show that the moments grow not too fast so that the moment problem determines the distribution.

**Proof of Theorem 3**: First we estimate \( \mathbb{E} \left( (X_1, \phi)^m (\rho_1, \psi)^n \right) \). To this end, let \( f_0 = \phi^\otimes m \otimes \psi^\otimes n \). Suppose that \( \phi \) and \( \psi \) are bounded by \( c \). Then

\[
P_{(m,n)}^{(m,n)} f_0 \leq c^{m+n}. \quad (31)
\]

Now we seek the bound for \( P_{\tau_2-\tau_1} \Gamma_1^\delta P_{(m,n)}^{(m,n)} f_0 \). At \( \tau_1 \), if two catalyst particle coalesce, then the bound in (31) remains valid for \( P_{\tau_2-\tau_1} \Gamma_1^\delta P_{(m,n)}^{(m,n)} f_0 \). Suppose that two reactant particles coalesce at time \( \tau_1 \), then

\[
\Gamma_1^\delta P_{(m,n)}^{(m,n)} f_0(x_1, \ldots, x_{m-1}; y_1, \ldots, y_{n+1})
\]

\[
\leq \frac{1}{\delta} \int_0^\delta de \int_{\mathbb{R}} dz p_e(z-x_{m-1}) p_e(z-y_1) c^{m+n}
\]

\[
= \frac{1}{\delta} \int_0^\delta de p_{2e}(x_{m-1} - y_1) c^{m+n}
\]

21
Note that the value of $c$ may have been changed. In fact, we always use $c$ to denote a constant when its specific value is of no concern to us. Hence

$$P_{\tau_2 - \tau_1} \Gamma_{\tau_1}^\delta P_{(m,n)}^{(m,n)} f_0 \leq \frac{1}{\delta} \int_0^\delta \int_0^\delta dp_2(x_m - y_n + 1) c^{m+n}. \quad (32)$$

At $\tau_2$, there are three cases: two catalyst particles coalesce, two reactant particles (not include $x_{m-1}$), or $x_{m-1}$ coalesces with another reactant particle. For the first case, there is no change for the bound. For the second case, it is easy to see that the bound is

$$c^{m+n} \frac{1}{\delta} \int_0^\delta d\epsilon_2 \frac{1}{\delta} \int_0^\delta d\epsilon_1 p_{2\epsilon_2} (x_m - y_n + 1) p_{2\epsilon_1 + 2\epsilon_2} (x_{m-2} - y_{n+1}).$$

For the third case,

$$\Gamma_2^\delta P_{\tau_2 - \tau_1} \Gamma_1^\delta P_{(m,n)}^{(m,n)} f_0 \leq c^{m+n} \frac{1}{\delta} \int_0^\delta d\epsilon_2 \int_0^\delta d\epsilon_1 p_{2\epsilon_1 + 2\epsilon_2} (x_{m-2} - y_{n+1}) \rho_{\epsilon_2} (z - x_{m-2}) \rho_{\epsilon_2} (z - y_{n+1}).$$

Hence

$$P_{\tau_3 - \tau_2} \Gamma_2^\delta P_{\tau_2 - \tau_1} \Gamma_1^\delta P_{(m,n)}^{(m,n)} f_0 \leq c^{m+n} \frac{1}{\delta} \int_0^\delta d\epsilon_2 \int_0^\delta d\epsilon_1 p_{2\epsilon_1 + 2\epsilon_2} (z - y_{n+1}) \rho_{\epsilon_2 + \tau_2 - \tau_1} (z - x_{m-2}) \rho_{\epsilon_2 + \tau_2 - \tau_1} (z - y_{n+2}) \leq c^{m+n} \frac{1}{\sqrt{\tau_3 - \tau_2}} p_{2\epsilon_2 + \tau_2 - \tau_1} (x_{m-2} - y_{n+2}).$$

Note that $\tau_3 - \tau_2$ is exponential with parameter no greater than $(m - 1)$. Note that, for the next step, the worst case will involve the second case of the present step. Therefore, the $\tau_4 - \tau_3$ there will be exponential with parameter no greater than $2(m - 2)$. The pattern will be $k(m - k)$. In summary, the worst case estimate will be

$$f_t \leq \Pi_j (\tau_j - \tau_{j-1})^{-1/2} \Pi_k \frac{1}{\delta} \int_0^\delta d\epsilon_k p_{2\epsilon_k + \alpha_k} (x_{i_k} - y_{j_k}).$$

Note that

$$E \left( (\tau_j - \tau_{j-1})^{-1/2} \right) \leq c^m m!.$$ 

Hence

$$E((X_t, \phi)^m (\rho_t, \psi)^n) \leq c^{m+n} m!.$$
Then
\[ \mathbb{E} ( (\langle X_t, \phi \rangle + \langle \rho_t, \psi \rangle)^m ) \leq \sum_{k=0}^{m} \binom{m}{k} c^m k! \leq c^m m!. \]

Note that
\[
\sum_{m \geq 1} \left( \mathbb{E} \left( (\langle X_t, \phi \rangle + \langle \rho_t, \psi \rangle)^{2m} \right) \right)^{-1/2m} \geq \sum_{m \geq 1} \left( c^{2m} (2m)! \right)^{-1/2m} \\
\geq c \sum_{m \geq 1} (2m)^{-1} = \infty.
\]

Namely, Carleman’s condition is satisfied and the moment problem determine the distribution (cf. Chung [2]).

\section{The existence of density}

In this section, we study the absolute continuity of the \( \mathcal{M}_F(\mathbb{R}) \)-valued random element \( X_t \). Namely, we prove the existence of the density by a standard second moment calculation. We continue to assume \( X_0 \) and \( \rho_0 \) to be finite.

Proof of Theorem 4: By (25), we have

\[
\mathbb{E} \langle X_t, g \rangle \langle X_t, h \rangle \\
= \int \int X_0(dx_1)X_0(dx_2) \int \mathbb{P}^{rr}(t, (x_1, x_2), d(u_1, u_2))g(u_1)h(u_2) \\
+ \lim_{\delta \to 0} \int_0^t ds \int X_0(dx_1) \int \rho_0(dx_2) \int \mathbb{P}^{\tau}(t-s, (x_1, x_2), d(y_1, y_2)) \\
\times \frac{1}{\delta} \int_0^\delta dz \mathbb{P}(z-y_1)p_\tau(z-y_2) \int \mathbb{P}^{rr}(s, (z, z), d(u_1, u_2))g(u_1)h(u_2) \\
= \int \int X_0(dx_1)X_0(dx_2) \int \mathbb{P}^{rr}(t, (x_1, x_2), d(u_1, u_2))g(u_1)h(u_2) \\
+ \int_0^t ds \int X_0(dx_1) \int \rho_0(dx_2) \int dz \mathbb{P}^{rr}(t-s, (x_1, x_2), (z, z)) \\
\times \int \mathbb{P}^{rr}(s, (z, z), d(u_1, u_2))g(u_1)h(u_2). \quad (33)
\]

Take 
\[ g = p(\epsilon, x, \cdot) \text{ and } h = p(\epsilon', x, \cdot) \]

Note that
\[
\int \int \mathbb{P}^{rr}(t, (x_1, x_2), d(u_1, u_2))p(\epsilon, x, u_1)p(\epsilon', x, u_2) \to \mathbb{P}^{rr}(t, (x_1, x_2), (x, x)) \quad (34)
\]
and
\[
\int \int \mathbb{P}_{\epsilon}(t, (x_1, x_2), d(u_1, u_2))p(\epsilon, x, u_1)p(\epsilon', x, u_2)
\leq cp(t + \epsilon, x_1 - x)p(t + \epsilon', x_2 - x)
\to cp(t, x_1 - x)p(t, x_2 - x).
\] (35)

As
\[
\int_0^T dt \int dx \int \int X_0(dx_1)X_0(dx_2)p(t + \epsilon, x_1 - x)p(t + \epsilon', x_2 - x)
= \int_0^T dt \int \int X_0(dx_1)X_0(dx_2)p(2t + \epsilon + \epsilon', x_1 - x_2),
\]
and
\[
p(2t + \epsilon + \epsilon', x_1 - x_2) \leq \frac{c}{\sqrt{t}},
\]
by the dominated convergence theorem, we have
\[
\int_0^T dt \int dx \int \int X_0(dx_1)X_0(dx_2)p(t + \epsilon, x_1 - x)p(t + \epsilon', x_2 - x)
\to \int_0^T dt \int dx \int \int X_0(dx_1)X_0(dx_2)p(t, x_1 - x)p(t, x_2 - x).
\] (36)

Making use of the extended dominated convergence theorem (cf. Kallenberg [9], p12), by (34), (35) and (36), we have
\[
\int_0^T dt \int dx \int \int X_0(dx_1)X_0(dx_2)
\times \int \int \mathbb{P}_{\epsilon}(t, (x_1, x_2), d(u_1, u_2))p(\epsilon, x, u_1)p(\epsilon', x, u_2)
\to \int_0^T dt \int dx \int \int X_0(dx_1)X_0(dx_2)\mathbb{P}_{\epsilon}(t, (x_1, x_2), (x, x)).
\] (37)

Note that
\[
\int \int \mathbb{P}_{\epsilon}(s, (z, z), d(u_1, u_2))p(\epsilon, x, u_1)p(\epsilon', x, u_2)
\leq cp(s + \epsilon, z - x)p(s + \epsilon', z - x)
\leq \frac{c}{\sqrt{s}}p(s + \epsilon, z - x).
\]
Similar to the arguments leading to (37), we have that
\[
\int_0^T dt \int dx \int X_0(dx_1)\rho_0(dx_2) \int dz \mathbb{P}^{rc}(t-s,(x_1,x_2),(z,z)) \\
\times \int \mathbb{P}^{rr}(s,(z,z),d(u_1,u_2))p(\epsilon,x,u_1)p(\epsilon',x,u_2) \\
\rightarrow \int_0^T dt \int dx \int \int X_0(dx_1)\rho_0(dx_2) \\
\times \int dz \mathbb{P}^{rc}(t-s,(x_1,x_2),(z,z))\mathbb{P}^{rr}(s,(z,z),(x,x)).
\]
(38)

(33), (37) and (38) show that
\[
\int_0^T \int \mathbb{E}(\langle X_t, p(\epsilon,x,\cdot) \rangle \langle X_t, p(\epsilon',x,\cdot) \rangle) dx dt \\
\rightarrow \int_0^T dt \int dx \int \int X_0(dx_1)X_0(dx_2)\mathbb{P}^{rr}(t,(x_1,x_2),(x,x)) \\
+ \int_0^T dt \int dx \int_0^t ds \int X_0(dx_1) \int \rho_0(dx_2) \\
\times \int dz \mathbb{P}^{rc}(t-s,(x_1,x_2),(z,z))\mathbb{P}^{rr}(s,(z,z),(x,x)).
\]
Therefore, as \(\epsilon \to 0\), \(\langle X_t, p(\epsilon,x,\cdot) \rangle\) converges weakly as elements in \(L^2(\Omega \times [0,T] \times L^2(\mathbb{R}))\). This implies the existence of the density \(X_t(x)\) of \(X_t\). Further, \(\int_0^T dt \int dx \mathbb{E}X_t(x)^2 < \infty\).

5 Acknowledgement

Xiong’s research is supported partially by Alexander von Humboldt Foundation and by NSA. Zhou’s research is supported by NSERC grant N00761.

References


