Chapter 1
A brief survey of superprocesses over a stochastic flow

Jie Xiong

Abstract In this chapter, we survey the recent progress in the study of superprocesses over a stochastic flow. Firstly, the process is defined as the high-density limit of a branching particle system in random environment. Then, the log-Laplace transform is characterized by a nonlinear SPDE. Thirdly, this measure-valued process has a density when $d = 1$ which is governed by an SPDE. Making use of Krylov's L_p theory, the continuity of the density is then established. Fourthly, the longtime behavior of the process is studied. It has persistency for higher spatial dimension and local extinction for lower ones. Finally, the limit behavior of the occupation measure is investigated. An ergodic property is proved for $d \geq 3$ and a fluctuation result is derived when $d = 2$.

1.1 Introduction

We start this section with the introduction of the branching particle system model studied first by Skoulakis and Adler [9]. Let K_n be the number of particles at time 0, spatially distributed in $\mathbb{R}^d$ at points $x_1^n, x_2^n, \cdots, x_{K_n}^n$. Define the deterministic initial atomic measure as

$$\nu_n = \frac{1}{n}\sum_{i=1}^{K_n}\delta_{x_i^n}.$$

Let $\lambda > 0$ be a constant. The lifespan of each particle in the system is $\frac{1}{\lambda n}$. At its death, each particle gives birth to a random number of particles. Let

$$I = \{\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_N),\, N \geq 0,\ \alpha_i \in \mathbb{N},\ 0 \leq i \leq N\}.$$

Department of Mathematics, University of Tennessee, Knoxville, TN 37996-1300, USA and Department of Mathematics, Hebei Normal University, Shijiazhuang 050016, PRC e-mail: jxiong@math.utk.edu

We shall use $\alpha \in I$ to denote a particle in this system.

Let N_n be a $\mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$ valued random variable such that

$$\mathbb{E}N_n = 1 + \frac{\gamma_n}{n}, \qquad 0 \le \gamma_n \to \gamma_0 \text{ as } n \to \infty$$

and

$$\mathrm{Var}(N_n) = \sigma_n^2 \to \sigma^2 \text{ as } n \to \infty.$$

We further assume that

$$\mathbb{E}N_n^p \le M, \qquad \forall\, n \in \mathbb{N}$$

for constants $p > 2$ and $M > 0$. Let $\{N^{\alpha,n} : \alpha \in I\}$ be i.i.d. copies of N_n. $N^{\alpha,n}$ stands for the number of offsprings of the particle α at its death. The notation $\alpha \sim_n t$ means that the particle $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_N)$ is alive at time t, i.e.,

$$\frac{N}{\lambda n} \le t < \frac{N+1}{\lambda n} \text{ and } \alpha_{i+1} \le N^{\alpha - N + i, n},$$

where $\alpha - N + i = (\alpha_0, \alpha_1, \cdots, \alpha_i)$, $i = 0, 1, \cdots, N-1$.

Let $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma_1 : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ and $\sigma_2 : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be measurable functions. Let $W(t)$ be an m-dimensional Brownian motion applied to the whole system and $\{B_\alpha(t) : \alpha \in I\}$ be a family of independent d-dimensional Brownian motions. During its lifetime, the motion of the particle α is governed by the following stochastic differential equation (SDE):

$$d\xi_\alpha(t) = b(\xi_\alpha(t))dt + \sigma_1(\xi_\alpha(t))dW(t) + \sigma_2(\xi_\alpha(t))dB_\alpha(t). \tag{1.1}$$

To obtain a unique strong solution to (1.1), we impose the following Lipschitz condition on the coefficients. Namely, there exists a constant K such that

$$|b(x) - b(y)| + \|\sigma_1(x) - \sigma_1(y)\| + \|\sigma_2(x) - \sigma_2(y)\| \le K|x - y|, \qquad \forall\, x,\ y \in \mathbb{R}^d. \tag{1.2}$$

Finally, we define the empirical measure process of the system as

$$X_t^n = \frac{1}{n} \sum_{\alpha \sim_n t} \delta_{\xi_\alpha(t)},$$

where δ_x is the Dirac measure at x.

The rest of this chapter is organized as follows: In next section, we consider the limit of the sequence $\{X^n : n \ge 1\}$ of measure-valued process and characterize it by a (conditional) martingale problem. Then, in Section 1.3, we establish a stochastic partial differential equation (SPDE) for the conditional log-Laplace transform of the limit process X. In Section 1.4, for the case of $d = 1$, we derive an SPDE for the density of X and study its continuity in the spatial variable. In Section 1.5, we consider the persistency and local extinction property of the superprocess X. Finally, in Section 1.6, we study the limiting behavior of the occupation time process of X.

1.2 Limit of the branching particle system

In this section, we consider the limit of X^n. To this end, we need to make the following

Assumption U: The mappings b, σ_1, σ_2 have bounded and continuous first and second partial derivatives. Furthermore, the matrix $a = \sigma_1\sigma_1^* + \sigma_2\sigma_2^*$ is uniformly elliptic, where σ^* stands for the transpose of the matrix σ.

Theorem 1 (Skoulakis-Adler). *Assume that $\nu^n \to \nu$ in $\mathscr{M}_F(\mathbb{R}^d)$. Then, under the assumption U, the sequence $\{X^n\}$ converges weakly to X, where $X \in C(\mathbb{R}_+, \mathscr{M}_F(\mathbb{R}^d))$ is the unique solution of the following martingale problem (MP): For all $f \in C_b^2(\mathbb{R}^d)$,*

$$Z_t(f) \equiv X_t(f) - \nu(f) - \int_0^t X_s(Lf)ds - \xi \int_0^t X_s(f)ds$$

is a continuous square integrable $\{\mathscr{F}_t^X\}$-martingale such that $Z_0(f) = 0$ and

$$\langle Z(f)\rangle_t = \int_0^t \left(\gamma X_s(f^2) + |X_s(\sigma_1^*\nabla f)|^2\right) ds,$$

where $\xi = \lambda\gamma_0$ and $\gamma = \lambda\sigma^2$ and

$$Lf = \sum_{i=1}^d b^i \partial_i f + \frac{1}{2}\sum_{i,j=1}^d a^{ij}\partial_{ij}^2 f,$$

∂_i means the partial derivative with respect to the ith component of $x \in \mathbb{R}^d$, $\partial_{ij} = \partial_i\partial_j$, $\nabla = (\partial_1, \cdots, \partial_d)^*$ is the gradient operator and $\mu(f)$ represents the integral of the function f with respect to the measure μ.

We shall also use $\langle \mu, f\rangle$ to denote the integral of the function f with respect to the measure μ, especially when the expression for f is very long.

For simplicity of notations, we shall consider the case of $\gamma_0 = 0$ in the rest of this chapter.

To extend the results to the case of μ being σ-finite, we use the following lemma about conditional martingale problem (CMP), which is proved in [13].

Lemma 1. (i) *If X_t is the solution to MP, then there exists a Brownian motion W such that X_t is the solution to CMP with this W. That is, for any $\phi \in C_0^2(\mathbb{R})$,*

$$N_t(\phi) \equiv \langle X_t, \phi\rangle - \langle \mu, \phi\rangle - \int_0^t \langle X_s, L\phi\rangle ds - \int_0^t \langle X_s, \nabla^*\phi\sigma_1\rangle dW_s \tag{1.3}$$

is a continuous $(\mathbb{P}, \mathscr{G}_t)$-martingale with quadratic variation process

$$\langle N(\phi)\rangle_t = \gamma \int_0^t \left\langle X_s, \phi^2\right\rangle ds \tag{1.4}$$

where $\mathscr{G}_t = \mathscr{F}_t \vee \mathscr{F}_\infty^W$.

(ii) If X_t is a solution to CMP with a Brownian motion W, then it is a solution to MP.

If μ is σ-finite, we can take $\mu = \sum_{n=1}^{\infty} \mu^n$ with μ^n finite. Given initial values $X_0^n = \mu^n$, it is not hard to show that the solutions X_t^n to CMP and the corresponding noises W^n take values in a Polish space. Furthermore the W^n are equal in distribution.

To get the solution with a common W, we need the following lemma proved in [5].

Lemma 2. *Let W be a random variable, and let $(X_i, W_i) : i = 1, 2, \ldots$ be a sequence of random vectors with components X_i, W_i taking values in Polish spaces $\mathscr{X}, \mathscr{W}$ respectively. Suppose that for each i, W_i and W are equal in law. Then we can realize W and the vectors (X_i, W_i) on a common probability space such that the following holds. For all i, $W_i = W$. Furthermore, given W the random variables $\{X_i\}$ are conditionally independent.*

By the lemma above, we may consider the X^n as driven by a single noise W, and we may assume that the X^n are conditionally independent given W. We can also check that

$$X_t = \sum_{n=1}^{\infty} X_t^n$$

is the solution to CMP with initial μ.

1.3 Stochastic log-Laplace equation

When $\sigma_1 = 0$, X_t is the well-known super-diffusion process which has been studied by many authors. Here we mention the works of Dawson [1], Watanabe [11], Dynkin [2], Perkins [8], and Le Gall [6]. The branching property of the superprocess plays a central role in these studies. More specifically, for any test function $f \in C_b(\mathbb{R}^d)$, the Laplace transform has the form

$$\mathbb{E}_\mu e^{-\langle X_t, f\rangle} = e^{-\left\langle \mu, \bar{y}_{0,t}\right\rangle}$$

where $\{\bar{y}_{s,t} :\ 0 \le s \le t\}$ is the unique solution to the following PDE

$$y_{s,t}(x) = f(x) + \int_s^t \left(\bar{L} y_{r,t}(x) - \gamma y_{r,t}(x)^2\right) dr$$

and $\bar{L}$ is defined as L with $\sigma_1 = 0$.

Writing the CMP formally, we see that, given W, X_t is the solution to the following MP: $\forall\ f \in C_b^2(\mathbb{R}^d)$,

$$N_t(f) \equiv \langle X_t, f\rangle - \langle \mu, f\rangle - \int_0^t \langle X_s, \bar{L} f\rangle\, ds - \int_0^t \langle X_s, \nabla^* f \sigma_1\rangle\, \dot{W}_s ds$$

is a continuous square-integrable $\mathbb{P}^W$-martingale with quadratic variation process

$$\langle N(f)\rangle_t = \int_0^t \left\langle X_s, f^2\right\rangle ds,$$

where $\mathbb{P}^W(\cdot) \equiv \mathbb{P}(\cdot|\mathscr{F}_\infty^W)$. Thus,

$$\mathbb{E}_\mu^W e^{-\langle X_t, f\rangle} = e^{-\left\langle \mu, y_{0,t}\right\rangle}$$

where

$$y_{s,t}(x) = f(x) + \int_s^t \left(Ly_{r,t}(x) - \gamma y_{r,t}(x)^2\right) dr + \int_s^t \nabla^* y_{r,t}(x)\sigma_1(x)\dot{W}_r dr.$$

In this section, we shall make the above procedure rigorous. To this end, we need the following
Boundedness condition (BC): $f \geq 0$, b, σ_1, σ_2 are bounded functions with bounded first and second derivatives. Denote a bound by K. Further, σ_2 is bounded away from 0, σ_1 has third continuous and bounded derivative, and f is of compact support.

To make precise, we replace $\dot{W}_r dr$ by backward Itô integral. Namely, we consider the following backward stochastic log-Laplace equation (LLE):

$$y_{s,t}(x) = f(x) + \int_s^t \left(Ly_{r,t}(x) - \gamma y_{r,t}(x)^2\right) dr + \int_s^t \nabla^* y_{r,t}(x)\sigma_1(x)\hat{d}W_r, \tag{1.5}$$

where the stochastic integral is the backward Itô's integral, namely, for any partition $s = t_0 < t_1 < \cdots < t_n = t$,

$$\int_s^t \nabla^* y_{r,t}(x)\sigma_1(x)\hat{d}W_r = \lim_{n\to\infty} \sum_{i=1}^n \nabla^* y_{t_i,t}(x)\sigma_1(x)(W_{t_i} - W_{t_{i-1}}).$$

For t fixed, we define

$$\tilde{W}_s = W_t - W_{t-s} \text{ and } y_s = y_{t-s,t}.$$

Then, y_s satisfies the following forward version of the LLE:

$$y_s(x) = f(x) + \int_0^s \left(b(x)\partial_x y_r(x) + a(x)\partial_x^2 y_r(x) - y_r(x)^2\right) dr + \int_0^s c(x)\partial_x y_r(x) d\tilde{W}_r. \tag{1.6}$$

For simplicity of notation, we replace $\tilde{W}$ on the right hand side of (1.6) by W. This will not affect the results we shall present below because W and $\tilde{W}$ have the same law.

Theorem 2. *Suppose that the condition (BC) holds. Then, the LLE (1.6) has a unique solution $y_t(x)$.*

Next, we consider the Wong-Zakai type approximation to LLE (1.6). For simplicity of notations, we take $d = m = 1$.

$$\begin{aligned} y_t^\varepsilon(x) &= f(x) + \int_0^t \left(\bar{b}(x)\partial_x y_r^\varepsilon(x) + \bar{a}(x)\partial_x^2 y_r^\varepsilon(x) - \gamma y_r^\varepsilon(x)^2\right) dr \\ &\quad + \int_0^t \sigma_1(x)\partial_x y_r^\varepsilon(x)\dot{W}_r^\varepsilon dr \end{aligned} \tag{1.7}$$

where $\bar{b}(x) = b(x) - \frac{1}{2}\sigma_2(x)\sigma_2'(x)$, $\bar{a}(x) = \frac{1}{2}\sigma_1(x)^2$ and for $k\varepsilon \le r < (k+1)\varepsilon$, $\dot{W}_r^\varepsilon = \varepsilon^{-1}(W_{(k+1)\varepsilon} - W_{k\varepsilon})$.

Theorem 3. *Suppose that the condition (BC) holds. Then for any $t \ge 0$,*

$$\mathbb{E}\int |y_t^\varepsilon(x) - y_t(x)|^2 dx \to 0$$

as $\varepsilon \to 0$.

Now, we consider the Wong-Zakai approximation to the measure-valued process X. Let X^ε be the solution to the following *conditional martingale problem (CMP)*: X^ε is a continuous $\mathscr{M}_F(\mathbb{R})$-valued process such that for any $f \in C_b^2(\mathbb{R})$,

$$M_t^\varepsilon(f) \equiv \langle X_t^\varepsilon, f\rangle - \langle X_0^\varepsilon, f\rangle - \int_0^t \left\langle X_s^\varepsilon, (\bar{b} + c\dot{W}_s^\varepsilon)f' + \bar{a}f''\right\rangle ds$$

is a continuous $\mathbb{P}^W$-martingale with quadratic variation process

$$\langle M^\varepsilon(f)\rangle_t = \int_0^t \left\langle X_s^\varepsilon, f^2\right\rangle ds.$$

Let $\bar{\mathbb{R}} \equiv \mathbb{R} \cup \{\partial\}$ be the one-point compactification of $\mathbb{R}$. Denote by $\mathscr{M}_F(\bar{\mathbb{R}})$ the space of all finite measures on $\bar{\mathbb{R}}$ with the weak convergence topology. Note that $\mathscr{M}_F(\mathbb{R})$ can be regarded as a subset of $\mathscr{M}_F(\bar{\mathbb{R}})$ by extending the measure at ∂ as 0.

Theorem 4. *As $\varepsilon \to 0$, if $X_0^\varepsilon \to \mu$ in $\mathscr{M}_F(\mathbb{R})$, then $X^\varepsilon \to X$ in $C([0,\infty), \mathscr{M}_F(\bar{\mathbb{R}}))$ in conditional law $\mathbb{P}^W$ for almost all W. As a consequence, we have*

$$\mathbb{E}^W \exp\left(-\langle X_t, f\rangle\right) = \exp\left(-\langle \mu, y_{0,t}\rangle\right) \qquad a.s. \tag{1.8}$$

Idea of the proof: By the classical superprocess theory, for $f_n \in C_0^2(\mathbb{R}^d)$, we have

$$\mathbb{E}^W \exp\left(-\langle X_t^\varepsilon, f_n\rangle\right) = \exp\left(-\left\langle \mu, y_{0,t}^\varepsilon\right\rangle\right).$$

Taking $\varepsilon \to 0$, then taking $f_n \to f$, we see that (1.8) holds. □

1.4 SPDE and Continuity

Let $p_0(t,x,y)$ and $q_0(t,(x_1,x_2),(y_1,y_2))$ be the transition density functions of the Markov processes $\xi_1(t)$ and $(\xi_1(t), \xi_2(t))$ respectively. The following identities about the first and second moments, essentially due to [9], are proved by Theorem 1.5 of [12]:

$$\mathbb{E}\Big[\langle X_t, f\rangle\Big] = \int_{\mathbb{R}^2} f(y) p_0(t,x,y) dy \mu(dx) \tag{1.9}$$

and

$$\begin{aligned}
&\mathbb{E}\Big[\langle X_t, f\rangle \langle X_t, g\rangle\Big] \qquad (1.10)\\
&= \int_{\mathbb{R}^4} f(y_1) g(y_2) q_0(t,(x_1,x_2),(y_1,y_2)) dy_1 dy_2 \mu(dx_1)\mu(dx_2)\\
&\quad +2\int_0^t ds \int_{\mathbb{R}^4} p_0(t-s,z,y) f(z_1) g(z_2) q_0(s,(y,y),(z_1,z_2)) dz_1 dz_2 dy \mu(dz).
\end{aligned}$$

Based on the identities above, we can prove that

Theorem 5. *If $\mu(\mathbb{R}) < \infty$, then X_t has a density $X(t,\cdot) \in H_2^0 = L^2(\mathbb{R})$ for almost every t a.s.*

Idea of the proof: Applying (1.10) to $f = p_0(\varepsilon,x,\cdot)$ and $g = p_0(\varepsilon',x,\cdot)$, it follows from direct calculation that

$$\mathbb{E}\left\langle X_t, p_0(\varepsilon,x,\cdot) - p_0(\varepsilon',x,\cdot)\right\rangle^2 \to 0, \qquad \text{as } \varepsilon,\ \varepsilon' \to 0.$$

Thus, $\langle X_t, p_0(\varepsilon,x,\cdot)\rangle$ converges. This implies the existence of the density. □

Applying the same arguments as those at the end of Section 1.2, we can extend the last theorem to the case of $\mu \in M_{tem}(\mathbb{R}^d)$, i.e., $\int_{\mathbb{R}^d} e^{-\lambda|x|}\mu(dx) < \infty$ for some $\lambda > 0$.

Theorem 6. *If $\mu \in \mathscr{M}_{tem}(\mathbb{R})$, then X_t has a density $X(t,x)$.*

Using the CMP and the martingale representation theorem, we can prove that

Theorem 7. *X_t is the unique (in law) solution to the SPDE*

$$\partial_t X = L^* X - \partial_x(\sigma_1 X)\dot{W}_t + \sqrt{X}\dot{B}_{tx} \tag{1.11}$$

in the sense that, for any f satisfying conditions in (BC) and $t > 0$,

$$\begin{aligned}
\langle X(t,\cdot), f\rangle = \langle \mu, f\rangle &+ \int_0^t \langle X(s,\cdot), Lf\rangle ds + \int_0^t \left\langle X(s,\cdot), \sigma_1 f'\right\rangle dW_s\\
&+ \int_0^t \int_{\mathbb{R}} \sqrt{X(s,x)} f(x) B(dsdx) \qquad (1.12)
\end{aligned}$$

holds a.s., where B is a Brownian sheet and L^* is the adjoint operator of L.

Finally, we consider the continuity of the random field.

Remark 1. Suppose that we apply the usual integral equation as in [10], Chapter 3, for (1.11) in order to prove the Hölder continuity. Then formally we have

$$X(t,x) = \int p_0(t,x,y)X(0,y)dy + \int_0^t \int \sigma_1(y)X(s,y)\partial_y p_0(t-s,x,y)dydW_s$$
$$+ \int_0^t \int \sqrt{X(s,y)}p_0(t-s,x,y)B(dsdy).$$

However, the second term on the right hand side of the above equation is about

$$\int_0^t (t-s)^{-1/2}dW_s$$

which is *not* convergent. Therefore, the convolution argument used by Konno and Shiga [3] does not apply to our model. This also shows that (1.12) does not have a mild solution.

We shall use Krylov's L_p theory for SPDE to derive the continuity. To this end, we need to introduce some notations. For any real number n and $p \in [2,\infty)$, we let H_p^n denote the space of Bessel potentials defined on $\mathbb{R}$ with norm

$$\|u\|_{n,p} = \|(I-\Delta)^{n/2}u\|_p.$$

(see, for instance, p.186 - p.187 in [4] for an explanation of this space.)

The following estimate plays an important role in the prove of continuity using Krylov's L_p theory for SPDE.

Lemma 3. *If μ is finite and satisfies*

$$\sup_{t,x} \langle \mu, \varphi_t(x-\cdot)\rangle < \infty, \tag{1.13}$$

then

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} X(t,x)^n dxdt\right] < \infty \tag{1.14}$$

for all $n \in \mathbb{N}$.

Let us explain our idea on Hölder Continuity of X. By freezing the nonlinear term of SPDE (1.11), we consider the following auxiliary linear SPDE :

$$\begin{cases} \partial_t Y = L^*Y + \sqrt{X}\dot{B}_{tx} \\ Y_0 = \mu \end{cases} \tag{1.15}$$

with $\mu \in H_p^{\frac{1}{2}-\varepsilon-\frac{2}{p}}$. Then $Z := X - Y$ satisfies another linear SPDE :

$$\begin{cases} \partial_t Z = L^*Z - (\partial_x(\sigma_1 Z) + \partial_x(\sigma_1 Y))\dot{W}_t \\ Z_0 = 0. \end{cases} \tag{1.16}$$

Hence, we can estimate X via Y and Z by using linear SPDE theory if the coefficients of (1.15) and (1.16) are good for doing so. It turns out that (BC) serves this purpose very well.

Theorem 8. *We take a finite time T and let $\varphi_t(x)$ be the normal density with mean 0 and variance t. If, in addition to the conditions of Theorem 7, μ is finite satisfying (1.13) and μ is in $H_p^{\frac{1}{2}-\varepsilon-\frac{2}{p}}$ for some $\varepsilon \in (0, \frac{1}{2})$ and p satisfying $\frac{1}{2}-\varepsilon-\frac{1}{p} > 0$, then the density $X(t,x)$ is Hölder continuous in x with index $\frac{1}{2}-\varepsilon-\frac{1}{p}$ for $(a.e.)$ $t \in [0,T]$ $(a.s.)$.*

Idea of the proof: We use the notations of Krylov [4]. Applying Theorem 8.5 of [4] to (1.15), we have a unique solution Y in $\mathscr{H}_p^{\frac{1}{2}-\varepsilon}(T)$ with estimate

$$\|Y\|_{\mathscr{H}_p^{\frac{1}{2}-\varepsilon}(T)} \leq N(\|\sqrt{X}\|_{\mathbb{L}_p(T)} + \|\mu\|_{\frac{1}{2}-\varepsilon-2/p,p}) \tag{1.17}$$

where N depends only on $\varepsilon, p, \delta, K, T$.

Next, we apply Theorem 5.1 of [4] to (1.16) to get a unique solution Z in $\mathscr{H}_p^{\frac{1}{2}-\varepsilon}(T)$ with

$$\|Z\|_{\mathscr{H}_p^{\frac{1}{2}-\varepsilon}(T)} \leq N\|\partial_x(\sigma_1 Y)\|_{\mathbb{H}_p^{-\frac{1}{2}-\varepsilon}(T)} \leq N\|\sqrt{X}\|_{\mathbb{L}_p(T)} + N\|\mu\|_{\frac{1}{2}-\varepsilon-2/p,p} \tag{1.18}$$

where $N = N(\varepsilon, p, \delta, K, T)$.

Thus, combining (1.17) and (1.18), we have $X = Y + Z \in \mathscr{H}_p^{\frac{1}{2}-\varepsilon}(T)$ with estimate

$$\|X\|_{\mathscr{H}_p^{\frac{1}{2}-\varepsilon}(T)} \leq N\|\sqrt{X}\|_{\mathbb{L}_p(T)} + N\|\mu\|_{\frac{1}{2}-\varepsilon-2/p,p}. \tag{1.19}$$

By the embedding Theorem 7.1 in [4], this implies

$$\left(E\int_0^T \|X_t\|^p_{C^{\frac{1}{2}-\varepsilon-\frac{1}{p}}} dt\right)^{1/p} \leq N\|X\|_{\mathscr{H}_p^{\frac{1}{2}-\varepsilon}(T)} \leq N\|\sqrt{X}\|_{\mathbb{L}_p(T)} + N\|\mu\|_{\frac{1}{2}-\varepsilon-2/p,p}.$$

So, for large $p > \frac{1}{\varepsilon}$, we have

$$\|X_t\|_{C^{\frac{1}{2}-2\varepsilon}} < \infty$$

for (a.e.) $t \in [0,T]$ a.s.. since 2ε takes any value in $(0, \frac{1}{2})$, we are done with the proof. □

1.5 Longtime limit

In this section, we consider the limit of the process X_t when $t \to \infty$. We will start the process from a measure μ which is invariant in a sense we shall define below. First, μ will not be a finite measure. Otherwise, the total mass $X_t(1)$ is Feller's diffusion which will converge to 0 as $t \to \infty$. Second, this measure need to be invariant with respect to the particle's motion with the environment W given.

To be more specific, we consider the motion of a typical point in the system

$$d\xi_t = b(\xi_t)dt + \sigma_1(\xi_t)dW(t) + \sigma_2(\xi_t)dB(t). \tag{1.20}$$

The next lemma establish the Markovian property of ξ_t.

Lemma 4. *ξ_t is a conditional Markov process in the sense that for any $t > s$ and $f \in C_b(\mathbb{R}^d)$,*

$$\mathbb{E}^W(f(\xi_t)|\mathscr{F}_t^\xi) = \mathbb{E}^W(f(\xi_t)|\xi_s).$$

Denote the conditional transition probability of ξ by

$$p^W(s,x;t,\cdot) = \mathbb{P}^W(\xi_t \in \cdot|\xi_s = x).$$

Definition 1. μ is an invariant measure of ξ_t for given environment W if

$$\int p^W(s,x;t,\cdot)\mu(dx) = \mu, \qquad a.s. \tag{1.21}$$

Next, we seek sufficient conditions for (1.21) to be satisfied. To this end, we write (1.20) in Stratonovich form:

$$d\xi_t = \left(\bar{b}(\xi_t)dt + \sigma_2(\xi_t)dB(t)\right) + \sigma_1(\xi_t) \circ dW(t)$$

where

$$\bar{b}^i = b^i - \frac{1}{2}\sum_{j,k=1}^d \partial_k \sigma_1^{ij}\sigma_1^{kj}.$$

Then, μ is invariant for each W if and only if μ is invariant for η_t and ζ_t given by:

$$d\eta_t = \bar{b}(\eta_t)dt + \sigma_2(\eta_t)dB(t)$$

and

$$\dot{\zeta}_t = \sigma_1(\zeta_t)\dot{W}_t.$$

So

$$\bar{L}^*\mu = 0 \text{ and } \nabla(\sigma_1^*\mu) = 0$$

where

$$\bar{L}f = \sum_{i=1}^d \bar{b}^i\partial_i f + \frac{1}{2}\sum_{i,j=1}^d \bar{a}^{ij}\partial_{ij}^2 f,$$

and

$$\bar{a}^{ij} = \sum_{k=1}^d \sigma_2^{ik}\sigma_2^{kj}.$$

The following theorem is proved in [13].

Theorem 9. *Suppose $\mu \in C^2(R^d)^+$ and*

$$|\nabla \log \mu(x)| \le K(1+|x|).$$

If

$$\bar{L}^*\mu = 0 \textit{ and } \nabla(\sigma_1^*\mu) = 0, \tag{1.22}$$

then (1.21) holds.

Idea of the proof: Let $\xi^\varepsilon(t)$ be the Wong-Zakai approximation for the process $\xi(t)$:

$$d\xi^\varepsilon(t) = \left(\bar{b}(\xi^\varepsilon(t)) + \sigma_1(\xi^\varepsilon(t))\dot{W}_t^\varepsilon\right)dt + \sigma_2(\xi^\varepsilon(t))dB_1(t)$$

where $\dot{W}_t^\varepsilon = \varepsilon^{-1}(W_{(k+1)\varepsilon} - W_{k\varepsilon})$ if $k\varepsilon \le t \le (k+1)\varepsilon$, $k = 0, 1, \cdots$. Then, given W^ε, ξ^ε is a Markov process with generator

$$L_t^\varepsilon\phi = \bar{L}\phi + (\dot{W}_t^\varepsilon)^T\sigma_1\nabla\phi.$$

It follows from the same arguments as in [14] that μ is invariant for ξ^ε, namely

$$\int_{\mathbb{R}^d}\mathbb{E}_x^W f(\xi^\varepsilon(t))\mu(x)dx = \int_{\mathbb{R}^d} f(x)\mu(x)dx.$$

Let $F(W)$ be a bounded continuous function of W. Then

$$\int_{\mathbb{R}^d}\mathbb{E}_x(f(\xi^\varepsilon(t))F(W))\mu(x)dx = \int_{\mathbb{R}^d} f(x)\mu(x)dx\mathbb{E}(F(W)). \tag{1.23}$$

Taking $\varepsilon \to 0$, we get

$$\int_{\mathbb{R}^d}\mathbb{E}_x(f(\xi(t))F(W))\mu(x)dx = \int_{\mathbb{R}^d} f(x)\mu(x)dx\mathbb{E}(F(W)).$$

This implies the conclusion of the proposition. □

Remark 2. If b, σ_1, σ_2 const., then $\mu(x) = 1$ is a solution. If $\sigma_2 = I$, then $\exists\sigma_1$ s.t. $\mu(x) = e^{2b^*x}$ and $\mu(x) = 1$ are solutions. Thus, σ-finite invariant measure is not unique.

Before we state the main limit results in this and the next section, we would like to mention a couple of *open problems*. First, we think that (1.22) is also necessary for (1.21). This conjecture remains to be settled. Second, it is clear that (1.22) implies $L^*\mu = 0$. The question is whether the results we shall present in the remaining of this chapter are still true when μ satisfies $L^*\mu = 0$? Namely, μ will only be assumed to be invariant for the annealed motion.

Now we proceed to studying the limiting behavior of X_t. Note that

$$\begin{aligned}\mathbb{E}_\mu e^{-\langle X_t, f\rangle} &= \mathbb{E}e^{-\langle\mu, y_{0,t}\rangle}\\ &= \mathbb{E}e^{-\langle\mu, y_t\rangle}\end{aligned}$$

where

$$\begin{aligned}y_s(x) = f(x) &+ \int_0^s\left(Ly_r(x) - y_r(x)^2\right)dr\\ &+ \int_0^s\nabla^T y_r(x)\sigma_1(x)dW(r)\end{aligned}$$

Write into the convolution form

$$y_t(x) = \int p^W(0,x;t,du)f(u) - \int_0^t dr \int p^W(r,x;t,du)y_r(u)^2.$$

Taking integral with respect to the measure μ, we then get

$$\begin{aligned} \langle \mu, y_t \rangle &= \langle \mu, f \rangle - \int_0^t \left\langle \mu, y_r^2 \right\rangle dr \\ &\to \langle \mu, f \rangle - \int_0^t \left\langle \mu, y_r^2 \right\rangle dr. \end{aligned}$$

As a consequence, $X_t \Rightarrow X_\infty$

Definition 2. The process X_t is persistent if $\mathbb{E}(X_\infty) = \mu$.

The following theorem, proved in [13], establishes the persistency when for high spatial dimension.

Theorem 10. *Suppose $d \geq 3$, (1.21) holds and*

$$\mu(x) \leq c_1 e^{c_2|x|}.$$

Then $X_t \Rightarrow X_\infty$ as $t \to \infty$. Furthermore, the process X_t is persistent.

Idea of the proof: Note that, $\forall\, f \in C_b^2(\mathbb{R}^d)$,

$$\begin{aligned} \mathbb{E}_\mu \langle X_t, f \rangle &= \mathbb{E}\left(\mathbb{E}_\mu^W \langle X_t, f \rangle\right) \\ &= \mathbb{E} \langle \mu, y_{0,t} \rangle \\ &\leq \mathbb{E} \int \mu(dx) \int p^W(0,x;t,du) f(u) \\ &= \int \mu(du) f(u) < \infty. \end{aligned} \tag{1.24}$$

By Fatou's lemma, we have

$$\mathbb{E} \langle X_\infty, f \rangle \leq \liminf_{t\to\infty} \mathbb{E}_\mu \langle X_t, f \rangle \leq \langle \mu, f \rangle,$$

where the second inequality follows from (1.24).

On the other hand, by Jensen's inequality

$$\begin{aligned} e^{-\mathbb{E}\langle X_\infty, f \rangle} &\leq \mathbb{E} e^{-\langle X_\infty, f \rangle} \\ &= \mathbb{E} \exp\left(-\langle \mu, f \rangle + \int_0^\infty \left\langle \mu, y_r^2 \right\rangle dr \right) \end{aligned}$$

and hence

$$\mathbb{E} \langle X_\infty, f \rangle \geq -\log \mathbb{E} \exp\left(-\langle \mu, f \rangle + \int_0^\infty \left\langle \mu, y_r^2 \right\rangle dr \right).$$

Replace f by εf, we have

$$\begin{aligned}
\langle \mu, f\rangle &\geq \mathbb{E}\langle X_\infty, f\rangle \\
&\geq -\varepsilon^{-1}\log\mathbb{E}\exp\left(-\varepsilon\langle \mu, f\rangle + \int_0^\infty \left\langle \mu, y_r^2(\varepsilon f)\right\rangle dr\right) \\
&= \langle \mu, f\rangle - \varepsilon^{-1}\log\mathbb{E}\exp\left(\int_0^\infty \left\langle \mu, y_r^2(\varepsilon f)\right\rangle dr\right) \qquad (1.25)
\end{aligned}$$

here $y_r(\varepsilon f)$ is defined as in (1.6) with f replaced by εf. A large deviation type argument then implies that

$$\varepsilon^{-1}\log\mathbb{E}\exp\left(\int_0^\infty \left\langle \mu, y_r^2(\varepsilon f)\right\rangle dr\right) \to 0 \qquad \text{as } \varepsilon \to 0.$$

The persistency then follows from (1.25). □

For low spatial dimension, the process behavior differently. The following theorem, proved in [13], establishes its local extinction property.

Theorem 11. *Suppose $d \leq 2$ and (1.21) holds. Assume*

$$\mu << \lambda \text{ and } 0 < c_1 \leq \frac{d\mu}{d\lambda} \leq c_2 < \infty.$$

Then for any bounded set B,

$$\lim_{t\to\infty} X_t(B) = 0.$$

We *conjecture* that for $d = 1$, the finite time local extinction holds. Namely, for each bounded set B, there exists a stopping time τ_B such that $X_t(B) = 0$ when $t \geq \tau_B$. This conjecture remains to be proved.

1.6 Occupation time limit

In this section, we consider the limiting behavior of the occupation measure

$$Y_t = \int_0^t X_s ds.$$

This section is based on the paper of Li *et al* [7]. For simplicity of notation, it assumes that $b = 0$.

As a tool for studying this problem, we need to establish the log-Laplace equation for the occupation process. The following theorem is derived in [7].

Theorem 12.

$$\mathbb{E}^W_{r,\nu}\exp\left(-\int_r^t \langle X_s, f_s\rangle ds\right) = \exp\left(-\langle \nu, u_{r,t}\rangle\right),$$

where $u_{r,t}$ solves

$$\begin{aligned} u_{r,t}(x) = &\int_r^t \left(Lu_{s,t}(x) - u_{s,t}^2(x) + f_s(x)\right) ds \\ &+ \int_r^t \nabla^* u_{s,t}(x) \sigma_1(x) \hat{d}W_s, \quad 0 \le r \le t. \end{aligned}$$

Idea of the proof: Let $s_2 \ge s_1 \ge 0$. For $s_1 \le r \le s_2$ let $\psi_{r,s_2}(x)$ be given by

$$\psi_{r,s_2}(x) = f_2(x) + \int_r^{s_2} [L\psi_{s,s_2}(x) - \psi_{s,s_2}^2(x)] ds + \int_r^{s_2} \sigma_1^*(x) \nabla \psi_{s,s_2}(x) \hat{d}W_s.$$

For $r \le s_1$ let $\phi_{r,s_1}(x)$ be the solution to

$$\begin{aligned} \phi_{r,s_1}(x) = &f_1(x) + \psi_{s_1,s_2}(x) + \int_r^{s_1} [L\phi_{s,s_1}(x) - \phi_{s,s_1}^2(x)] ds \\ &+ \int_r^{s_1} \sigma_1^*(x) \nabla \phi_{r,s_1}(x) \hat{d}W_s. \end{aligned}$$

By Theorem 4 for $r \le s_1 \le s_2$ we have

$$\begin{aligned} \mathbb{E}_{r,\mu}^W \exp\left\{ -\langle X_{s_1}, f_1\rangle - \langle X_{s_2}, f_2\rangle \right\} &= \mathbb{E}_{r,\mu}^W \exp\left\{ -\langle X_{s_1}, f_1 + \psi_{s_1,s_2}\rangle \right\} \\ &= \exp\{-\langle \mu, \phi_{r,s_1}\rangle\}. \end{aligned}$$

Now we define

$$u(r,x) = \begin{cases} \psi_{r,s_2}(x), \ s_1 \le r \le s_2, \\ \phi_{r,s_1}(x), \ \ r < s_1. \end{cases}$$

It is easy to see that

$$\begin{aligned} u(r,x) = &f_1(x) 1_{\{r<s_1\}} + f_2(x) 1_{\{r<s_2\}} + \int_r^{s_2} [Lu(s,x) - u^2(s,x)] ds \\ &+ \int_r^{s_2} \sigma_1^*(x) \nabla u(s,x) \hat{d}W_s. \end{aligned}$$

By similar arguments as the above we get

$$\mathbb{E}_{r,\nu}^W \exp\left\{ -\sum_{i=1}^n \left\langle X_{s_i}, \frac{1}{n} f_{s_i} \right\rangle \right\} = \exp\{-\langle \nu, u_t^n(r,\cdot)\rangle\}, \tag{1.26}$$

where $s_i = it/n$ and $u_t^n(\cdot,\cdot)$ is the solution to

$$u_t^n(r,x) = f_r^n(x) + \int_r^t [Lu_t^n(s,x) - u_t^n(s,x)^2] ds + \int_r^t \sigma_1^*(x) \nabla u_t^n(s,x) \hat{d}W_s, \tag{1.27}$$

where

$$f_r^n(x) := \frac{1}{n} \sum_{i=1}^n f_{s_i}(x) 1_{\{r<s_i\}} \to \int_r^t f_s(x) ds.$$

We can prove that $u_t^n \to u_t$, and hence, the conclusion of the theorem follows by taking $n \to \infty$ in (1.26) and (1.27). $\square$

Apply the previous theorem directly, we will get the following law of large number type theorem for high spatial dimension case. This result dues to Li et al [7].

Theorem 13. *Suppose that $d \geq 3$, μ is invariant in the sense of (1.21) and the Condition (BC) hold. Then, as $t \to \infty$, we have*

$$\frac{1}{t} Y_t \to \mu$$

in probability.

Finally, we consider the limit when the spatial dimension is $d = 2$. In this case, a central limit type result is obtained in [7]. To this end, we need some additional conditions (C):

- $$\lim_{|x| \to \infty} \mu(x) = \mu(\infty),$$

  and $\exists\, c_1, c_2 > 0$ so that $c_1 \leq \mu(x) \leq c_2$ for all $x \in \mathbb{R}^2$;
- There are matrices $(\tilde{\sigma}_1^{ij})$ and $(\tilde{\sigma}_2^{ij})$ so that

$$\sigma_1^{ij}(x) \to \tilde{\sigma}_1^{ij}, \quad \sigma_2^{ij}(x) \to \tilde{\sigma}_2^{ij}, \qquad |x| \to \infty,\ i, j = 1, 2.$$

Let $\tilde{p}^W$ be conditional transition with σ_i replaced by $\tilde{\sigma}_i$.

Theorem 14. *Suppose that $d = 2$, μ is invariant in the sense of (1.21), and the Conditions (BC) and (C) hold. If $X_0 = \mu$, then for any $f \in C_0^\infty(\mathbb{R}^2)^+$, we have*

$$\frac{1}{t} \langle Y_t, f \rangle \Rightarrow \xi$$

where ξ is a random variable such that for $\theta > 0$

$$\mathbb{E}\big[\exp\{-\theta\xi\}\big] = \mathbb{E}\exp\left\{ -\theta \langle \mu, f \rangle + \mu(\infty) \int_0^1 \big\langle \lambda, v^2(s, \cdot; \theta) \big\rangle \, ds \right\},$$

and $v(r, x; \theta)$ is the unique positive solution to

$$\begin{aligned} v(r,x) &+ \int_r^1 ds \int_{\mathbb{R}^2} v^2(s,y) \tilde{p}^W(r,x,s,y) dy \\ = \langle \lambda, \theta f \rangle &\int_r^1 \tilde{p}^W(r,x,s,0) ds \end{aligned}$$

with $0 \leq r \leq 1$ and $x \in \mathbb{R}^2$.

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