Zakai Equation of Nonlinear Filtering with Ornstein-Uhlenbeck Noise: Existence and Uniqueness

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Abstract

We consider a filtering model where the noise is an Ornstein-Uhlenbeck process independent of the signal X. The signal is assumed to be a Markov diffusion process. We derive the (analogue of) Zakai equation for this model. In this case, the (analogue of) the Zakai equation is a system of two measure valued equations satisfied by the unnormalized conditional distribution. We also prove uniqueness of solution to these equations.

1 Introduction

The general nonlinear filtering problem can be described as follows: the process of interest, called the *system* or the *signal* process, X, is unobservable. At time t, we can observe $h(X_t)$, where h is a known function, generally assumed to be non-linear. Moreover, the observations are corrupted by *noise* N. Let the *observation* process be denoted by Y. The aim in filtering theory is to filter out the noise N from the observations Y and get an estimate of the process X. The filtering model, when the observations are accumulated over time, can be written as

$$Y_t = \int_0^t h(X_s)ds + N_t, \qquad 0 \leq t \leq T. \tag{1.1}$$

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The best estimate of X_t, in the squared error sense, is the conditional distribution of X_t given the observations upto time t, $\{Y_s; 0 \leq s \leq t\}$. This is the *optimal filter* and is denoted by π_t, *i.e.* π_t is defined by

$$\pi_t(f) = E[f(X_t)|Y_s; 0 \leq s \leq t], \qquad \text{for all bounded, continuous } f.$$

In the discrete time case, the noise is modelled as an i.i.d. sequence of random variables. It is quite well-known that such a *continuous time* process cannot exist in the Kolmogorov's framework of probability theory. If it did indeed exist, then the time integral of the i.i.d. process would be a Brownian motion with bounded variation paths. However, if we consider the accumulated observations, as against the instantaneous observations in discrete time models, by integrating over t, then we get the observation model (1.1) where now N is a Brownian motion. Such a model has been studied extensively by several authors and is referred to as the *classical model of filtering*. In this case π_t is known to satisfy a measure valued stochastic differential equation called the Fujisaki-Kallianpur-Kunita (FKK) equation, see [3] and [5]. A certain transformation of the optimal filter, which will be denoted by μ_t, has also been studied. This is called the unnormalized conditional distribution of X_t given $\{Y_s; 0 \leq s \leq t\}$. μ also satisfies a stochastic differential equation called the Zakai equation, which is linear in μ and is driven by the observation process Y, see [12]. The filtering problem can be said to be completely solved if the Zakai, or the FKK, equation also characterize μ, or π respectively. Uniqueness of solution to these measure valued equations has been established in [1] under fairly general conditions on the observation function h and on the signal process X, when the noise is a Brownian motion.

However, the classical model of filtering has been objected to by engineers on the ground that while the actual observed paths of the accumalative observation process Y_t are going to be smooth, the classical model gives zero probability to all such smooth paths, see [6] for a discussion on this.

Recently, interest has developed in filtering theory when the noise is a general Gaussian process. This has been initiated by Kunita in [8] and has been followed up in [10] and [2]. These articles derive a Bayes' formula for the optimal filter π. In [10] this is done under the assumption that almost all paths of $h(X_t)$ are differentiable, a very stringent condition, while in [2] the authors assume a lingering effect of the signal over a small time interval. In [4], the results of [10] are used to to derive a Zakai equation for μ when the noise process is connected to a Brownian motion via a certain kernel.

In this article we consider the filtering problem when the noise is a Ornstein-Uhlenbeck (velocity) process independent of the signal. In the next section we introduce the filtering model. We derive the equations of filtering for the unnormalized conditional distribution when the signal is a diffusion Markov process with smooth diffusion and drift coefficients. We need to consider the pair $(\mu_t, \sigma_{s,t})$ (see (2.11)-(2.12)). Here $\sigma_{s,t}$ is the analogous *two* - parameter unnormalized conditional distribution of X given Y. The interesting point is that the (analogue of the) Zakai equation, unlike in the classical case, is a system of two measure valued SDE's. We derive these SDE's via a particle representation proof based on the lines of Kurtz and Xiong [9]. Uniqueness is proved in Section 3. We show that the pair $(\mu_t, \sigma_{s,t})$ is the unique solution of this system of equations under the additional assumption that the law of X_0 has a density. For the sake of notational simplicity we consider the one dimensional signal case. The results are true for a general d-dimensional diffusion as well.

2 Zakai equation

Fix a probability space $(\Omega, \mathcal{F}, P)$. We will assume that the signal process, X, is a $I\!R$ valued diffusion process governed by the SDE

$$dX_t = b(X_t)dt + c(X_t)dB_t \tag{2.1}$$

where b and c are bounded Lipschitz continuous real valued functions and B is a standard Brownian motion independent of X_0. It is well-known that SDE (2.1) admits a unique solution. Furthermore, the paths of this solution are continuous. Let W be a Standard Brownian motion independent of X. We will investigate the nonlinear filtering model where the observation process Y is given by

$$Y_t = \int_0^t h(X_s)ds + O_t. \tag{2.2}$$

The observation function h is assumed to be a bounded continuous function and the noise process O_t is an Ornstein-Uhlenbeck process satisfying the SDE

$$dO_t = -\beta O_t dt + dW_t \tag{2.3}$$

with $\beta > 0$. The optimal filter π_t is given by

$$\pi_t f = \mathbb{E}(f(X_t)|\mathcal{F}_t^Y), \quad \forall f \in C_b(I\!R)$$

where $\mathcal{F}_t^Y = \sigma\{Y_s : 0 \le s \le t\}$ is the σ-field generated by all observations upto time t.

We will recast (2.2) in a form which helps us in deriving an equation for the filter. For this purpose we proceed as follows: let $C([0,T], I\!R)$ denote the space of all continuous functions from $[0,T]$ to $I\!R$ and $E = [0,T] \times C([0,T], I\!R)$, then define $H : E \to I\!R$ by

$$\begin{aligned} H(t, \mathbf{x}) &= \frac{d}{dt}\left[e^{\beta t}\int_0^t h(\mathbf{x}(r))dr\right] \\ &= \beta e^{\beta t}\int_0^t h(\mathbf{x}(r))dr + e^{\beta t}h(\mathbf{x}(t)). \end{aligned} \tag{2.4}$$

For any $\mathbf{x} \in C([0,T], I\!R)$, let $\mathbf{x}^t \in C([0,T], I\!R)$ denote the path $\mathbf{x}$ stopped at t, *i.e.* $\mathbf{x}^t(r) = \mathbf{x}(t \wedge r), r \in [0,1]$.

Define an operator A on $C_b(I\!R)$ as follows: let $D(A) = C_b^2(I\!R)$, the space of twice continuously differentiable functions on $I\!R$ with bounded derivatives and define A by

$$Af(x) = \frac{1}{2}c^2(x)f''(x) + b(x)f'(x). \tag{2.5}$$

Then A uniquely determines the Markov processes X as a solution of its martingale problem, see Stroock and Varadhan [11].

Define $S_t \equiv (t, X^t)$. It is well-known that S is an E-valued Markov process. Also S is a unique solution of the martingale problem for an operator $\bar{A}$ on $C_b(E)$ which can be defined as follows: let $D(\bar{A})$ be the algebra generated by functions of the form $\{F : E \to I\!R : F(t, \mathbf{x}) = g(t)f(\mathbf{x}_t), g \in C[0,T] \cap C^1(0,T), f \in D(A)\}$. Define

$$\bar{A}F(t,\mathbf{x}) = g'(t)f(\mathbf{x}_t) + g(t)Af(\mathbf{x}(t)). \tag{2.6}$$

Now let $M_t = e^{\beta t}O_t$. Then (2.3) implies that

$$M_t = \int_0^t e^{\beta s} dW_s.$$

Clearly M is a $\mathcal{F}_t^W$ martingale. Correspondingly, let $Z_t = e^{\beta t}Y_t$. The filtering model (2.2) can now be rewritten as

$$\begin{aligned} Z_t &= e^{\beta t} \int_0^t h(X_s)ds + M_t \\ &= \int_0^t H(S_u)du + M_t \end{aligned} \tag{2.7}$$

where H is as in (2.4). Let

$$\Lambda_t^{-1} \equiv \exp\left\{ -\int_0^t e^{-2\beta u} H(S_u) dM_u - \frac{1}{2}\int_0^t e^{-2\beta u} |H(S_u)|^2 du \right\}.$$

Note that independence of X and W implies that S is independent of M, hence Λ_t^{-1} is a P-martingale. Morevoer, P_0 defined by

$$\frac{dP_0}{dP} = \Lambda_T^{-1}$$

is a probability measure on $(\Omega, \mathcal{F})$. Also, Girsanov's theorem implies that under P_0, Z is a martingale independent of S and that the law of S under the two measures P and P_0 remains unchanged. In particular, under P_0, S is a Markov process and is the unique solution of the martingale problem for $\bar{A}$.

Now, the optimal filter $\bar{\pi}_t$ for the model (2.7) satisfies $\forall F \in C_b(E)$,

$$\begin{aligned}
\bar{\pi}_t F &\equiv \mathbf{E}(F(S_t)|\mathcal{F}_t^Z) \\
&= \mathbf{E}(F(S_t)|\mathcal{F}_t^Y) \\
&= \mathbf{E}_{P_0}(F(S_t)\Lambda_t|\mathcal{F}_t^Y)/\mathbf{E}_{P_0}(\Lambda_t|\mathcal{F}_t^Y) \\
&\equiv \bar{\mu}_t F/\bar{\mu}_t \mathbf{1},
\end{aligned}$$

where $\bar{\mu}_t$ is the unnormalized conditional distribution of S_t given $\mathcal{F}_t^Z$. We now derive the Zakai equation for $\bar{\mu}_t$.

Proposition 2.1. *$\bar{\mu}_t$ satisfies the equation*

$$\bar{\mu}_t F = \bar{\mu}_0 F + \int_0^t \bar{\mu}_s(\bar{A}F)ds + \int_0^t e^{-2\beta s}\bar{\mu}_s(HF)dZ_s, \qquad \forall F \in D(\bar{A})$$

Proof: Fix $F \in D(\bar{A})$. Consider independent copies S^i of S. Let

$$N_t^i = F(S_t^i) - F(S_0^i) - \int_0^t \bar{A}F(S_u^i)du.$$

Then $\{N^i, i \geq 1\}$ are independent P_0-martingales that are also independent of the P_0-martingale Z_t. Let

$$d\Lambda_t^i = e^{-2\beta t}\Lambda_t^i H(S_t^i)dZ_t.$$

Then it is easy to see that

$$\Lambda_t^i = \exp\left\{\int_0^t e^{-2\beta u}H(S_u^i)dZ_u - \frac{1}{2}\int_0^t e^{-2\beta u}|H(S_u^i)|^2 du\right\}.$$

By Itô's formula, we have

$$d(F(S_t^i)\Lambda_t^i) = e^{-2\beta t}F(S_t^i)H(S_t^i)\Lambda_t^i dZ_t + \Lambda_t^i[\bar{A}F(S_t^i)]dt + \Lambda_t^i dN_t^i. \tag{2.8}$$

It is clear that the sequence of processes $\{(\Lambda^i, F(S^i)) : i \geq 1\}$ is exchangable. Thus $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \Lambda_t^i F(S_t^i)$ exists under P_0 and the ergodic theorem implies that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \Lambda_t^i F(S_t^i) = \mathbf{E}_{P_0}(\Lambda_t F(S_t)|\mathcal{I}) \tag{2.9}$$

where $\mathcal{I}$ is the invariant σ-field of the stationary sequence $\{(S^i, N^i, Z) : i \geq 1\}$. As in Kurtz and Xiong ([9, Theorem 2.3]) we use the independence of (S^i, N^i) to note that $\mathcal{I}$ is contained in the completion of the σ-field generated by Z. Now (2.9) implies

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \Lambda_t^i F(S_t^i) &= \mathbf{E}_{P_0}(\Lambda_t F(S_t)|\mathcal{F}_T^Z) \\ &= \mathbf{E}_{P_0}(\Lambda_t F(S_t)|\mathcal{F}_t^Z) \\ &= \bar{\mu}_t F. \end{aligned} \tag{2.10}$$

Arguing similarly as above, using (2.8), (2.9), (2.10) and the fact that N^1 and Z are independent under P_0 we get

$$\bar{\mu}_t F = \bar{\mu}_0 F + \int_0^t e^{-2\beta s}\bar{\mu}_s(HF)dZ_s + \int_0^t \bar{\mu}_s(\bar{A}F)ds.$$

This completes the proof. □

Remark 2.2. *The above Proposition can also be proved along the lines of the proof of the classical Zakai equation. Here we have given a different* **particle-representation** *proof, see Kurtz and Xiong [9].*

Let μ_t denote the unnormalized conditional distribution of X_t given $\mathcal{F}_t^Y$, *i.e.*

$$\mu_t f = \mathbf{E}_{P_0}\left[f(X_t)\Lambda_t|\mathcal{F}_t^Y\right]. \tag{2.11}$$

Also, for $0 \le s,t \le T$, let $\sigma_{s,t}$ be defined by

$$\sigma_{s,t} f = \mathbf{E}_{P_0}\left[h(X_s)f(X_t)\Lambda_t | \mathcal{F}_t^Y\right]. \tag{2.12}$$

Note that $\sigma_{t,t} f = \mu_t(hf)$. We will now use Proposition 2.1 to derive the analogue of the Zakai equation for (μ_t) which involves $(\sigma_{s,t})$.

Proposition 2.3. *μ_t and $\sigma_{s,t}$ satisfy the system of equations*

$$\mu_t f = \mu_0 f + \int_0^t \mu_s(Af)ds + \int_0^t e^{-\beta s}\left(\beta \int_0^s \sigma_{u,s} f du + \mu_s(hf)\right) dZ_s, \tag{2.13}$$

$$\sigma_{s,t} f = \mu_s(hf) + \int_s^t \sigma_{s,u}(Af)du + \int_s^t e^{-2\beta(u-s)}\sigma_{s,u}(hf)dZ_u, \qquad \forall f \in D(A). \tag{2.14}$$

Proof: Fix $f \in D(A)$. Let $F \in D(\bar{A})$ be defined by $F(t,\mathbf{x}) = f(\mathbf{x}_t)$. Note that $Af(\mathbf{x}_t) = \bar{A}F(t,\mathbf{x}^t)$. It follows from Proposition 2.1 that

$$\mu_t f = \mu_0 f + \int_0^t \mu_s(Af)ds + \int_0^t e^{-2\beta s}\bar{\mu}_s(HF)dZ_s.$$

Equation (2.13) now follows, by noting that

$$\begin{aligned}(HF)(s,\mathbf{x}^s) &= \left(\beta e^{\beta s}\int_0^s h(\mathbf{x}_u)du + e^{\beta s}h(\mathbf{x}_s)\right) f(\mathbf{x}_s) \\ &= \beta e^{\beta s}\int_0^s h(\mathbf{x}_u)f(\mathbf{x}_u)du + e^{\beta s}h(\mathbf{x}_s)f(\mathbf{x}_s).\end{aligned}$$

The same arguments as in the proof of Proposition 2.1 together with the fact that

$$N_{s,t} = h(X_s)\left[f(X_t) - f(X_s) - \int_s^t Af(X_u)du\right]$$

is a martingale for $t \ge s$, imply (2.14). $\square$

3 Uniqueness

In this section we will show uniqueness of solution to the system of equations (2.13)-(2.14). For $f \in C_b(I\!\!R)$, let $F(t,x^t) = f(x_t)$, then $\mu_t f = \bar{\mu}_t F$. For $\delta > 0$, let p_δ denote the density kernel of a normal random variable with variance δ. For a measure ν on $I\!R$, let $T_\delta \nu$ be the function defined by

$$T_\delta \nu(x) = \int p_\delta(x-y)\nu(dy).$$

Also, let

$$\sigma^\delta_{s,t} = T_\delta \sigma_{s,t} \text{ and } \mu^\delta_t = T_\delta \mu_t,$$

where $\sigma^\delta_{s,t}$ and μ^δ_t are $H_0 \equiv L^2(I\!R)$-valued processes. With an abuse of notation, for $f \in C_b(I\!R)$, $T_\delta f$ will denote the function $\int p_\delta(x-y)f(y)dy$. Note that $T_\delta f \in D(A)$. Thus, recalling the definition of the operator A (see (2.5)) and using (2.14), we get

$$\begin{aligned}
\left\langle \sigma^\delta_{s,t}, f\right\rangle_0 &= \sigma_{s,t}(T_\delta f) \\
&= \mu_s(hT_\delta f) + \int_s^t \sigma_{s,u}(\frac{c^2}{2}(T_\delta f)'' + b(T_\delta f)')du + \int_s^t e^{-2\beta(u-s)}\sigma_{s,u}(hT_\delta f)dZ_u \\
&= \langle T_\delta(h\mu_s), f\rangle_0 + \int_s^t \left\langle (T_\delta(\frac{c^2}{2}\sigma_{s,u}))'' - (T_\delta(b\sigma_{s,u})', f \right\rangle_0 du \\
&\qquad + \int_s^t e^{-2\beta(u-s)} \langle (T_\delta(h\sigma_{s,u}), f\rangle_0 \, dZ_u.
\end{aligned}$$

By Itô's formula, we have

$$\begin{aligned}
\left\langle \sigma^\delta_{s,t}, f\right\rangle_0^2 \;=\; & \langle T_\delta(h\mu_s), f\rangle_0^2 + \int_s^t 2\left\langle \sigma^\delta_{s,u}, f\right\rangle_0 \left\langle (T_\delta(\frac{c^2}{2}\sigma_{s,u}))'' - (T_\delta(b\sigma_{s,u})', f \right\rangle_0 du \\
& + \int_s^t 2\left\langle \sigma^\delta_{s,u}, f\right\rangle_0 \langle (T_\delta(h\sigma_{s,u}), f\rangle_0 \, e^{-2\beta(u-s)}dZ_u \\
& + \int_s^t e^{-2\beta(u-s)} \langle (T_\delta(h\sigma_{s,u}), f\rangle_0^2 \, du. \qquad (3.1)
\end{aligned}$$

We use Lemmas 3.2 and 3.3 from [9] which give estimates for expressions involving the convolution operator T_δ. By Lemma 3.2 in the above reference, we get that there exists a constant K_1, which does not depend on $\sigma_{s,u}$, satisfying

$$|\left\langle \sigma^\delta_{s,u}, (T_\delta(b\sigma_{s,u})'\right\rangle_0| \le K_1 \|T_\delta(|\sigma_{s,u}|)\|_0^2.$$

Similarly, using Lemma 3.3 in [9], we get the existence of a constant K_2 independent of $\sigma_{s,u}$ and satisfying

$$2\left\langle \sigma^\delta_{s,u}, (T_\delta(\frac{c^2}{2}\sigma_{s,u}))''\right\rangle_0 \le K_2\|T_\delta(|\sigma_{s,u}|)\|_0^2.$$

Let $\{f_i : i \ge 1\}$ be a CONS in H_0. Equation (3.1) holds for each f_i. Adding over i and using the above two inequalitites, we get that there exists a constant K which does not depend on $\sigma_{s,u}$ such that

$$\begin{aligned}
\mathbf{E}\|\sigma^\delta_{s,t}\|_0^2 &= \mathbf{E}\|T_\delta(h\mu_s)\|_0^2 + \mathbf{E}\int_s^t 2\left\langle \sigma^\delta_{s,u}, (T_\delta(\frac{c^2}{2}\sigma_{s,u}))'' - (T_\delta(b\sigma_{s,u})'\right\rangle_0 du \\
&\quad + \mathbf{E}\int_s^t e^{-2\beta(u-s)}\|T_\delta(h\sigma_{s,u})\|_0^2 du \\
&\le K\mathbf{E}\|\mu^\delta_s\|_0^2 + K\mathbf{E}\int_s^t \|T_\delta(|\sigma_{s,u}|)\|_0^2 du. \qquad (3.2)
\end{aligned}$$

We cannot directly use Gronwall's inequality here, hence we must proceed by a different route by setting $\xi^\pm_t$ to be independent Markov processes which are solutions of the martingale problem for $(A, \mu_s(h^\pm\cdot)/\mu_s(h^\pm))$, respectively, then

$$\begin{aligned}
\sigma_{s,t}f &= \mathbf{E}\left(f(\xi^+_t)\exp\left(\int_s^t e^{-2\beta(u-s)}h(\xi^+_u)\circ dZ_u\right)\middle|\mathcal{F}^Z_u\right)\mu_s(h^+) \\
&\quad - \mathbf{E}\left(f(\xi^-_t)\exp\left(\int_s^t e^{-2\beta(u-s)}h(\xi^-_u)\circ dZ_u\right)\middle|\mathcal{F}^Z_u\right)\mu_s(h^-) \\
&\equiv \sigma^+_{s,t}f - \sigma^-_{s,t}f.
\end{aligned}$$

As in (2.13), we have

$$\sigma^\pm_{s,t}f = \mu_s(h^\pm f) + \int_s^t \sigma^\pm_{s,u}(Af)du + \int_s^t e^{-2\beta(u-s)}\sigma^\pm_{s,u}(hf)dZ_u.$$

Since $\sigma^\pm_{s,t}$ are positive measures, arguing as in (3.2) we get

$$\begin{aligned}
\mathbf{E}\|\sigma^{\pm,\delta}_{s,t}\|_0^2 &\le K\mathbf{E}\|\mu^\delta_s\|_0^2 + K\mathbf{E}\int_s^t \|T_\delta(|\sigma^\pm_{s,u}|)\|_0^2 du \\
&= K\mathbf{E}\|\mu^\delta_s\|_0^2 + K\mathbf{E}\int_s^t \|\sigma^{\pm,\delta}_{s,u}\|_0^2 du.
\end{aligned}$$

Now Gronwall's inequality implies that

$$\mathbf{E}\|\sigma_{s,t}^{\pm,\delta}\|_0^2 \leq K_1 \mathbf{E}\|\mu_s^\delta\|_0^2,$$

therefore

$$\mathbf{E}\|T_\delta(|\sigma_{s,t}|)\|_0^2 \leq K_2 \mathbf{E}\|\mu_s^\delta\|_0^2. \tag{3.3}$$

Now working with $\langle \mu_t^\delta, f \rangle_0 = \mu_t(T_\delta f)$, using (2.13) and arguing as above, as an analogue of (3.2) for μ_t^δ, we get

$$\mathbf{E}\|\mu_t^\delta\|_0^2 \leq \|\mu_0^\delta\|_0^2 + K \int_0^t \mathbf{E}\|\mu_s^\delta\|_0^2 ds. \tag{3.4}$$

As a result we have the following proposition.

Proposition 3.1. *If $\mu_0 \in H_0$, then $\mu_t \in H_0$ and $\sigma_{s,t} \in H_0$ a.s.*

Proof: Applying Gronwall's inequality to (3.4) we get

$$\mathbf{E}\|\mu_t^\delta\|_0^2 \leq \|\mu_0^\delta\|_0^2 e^{Kt},$$

letting $\delta \to 0$, we have

$$\mathbf{E}\|\mu_t\|_0^2 \leq \|\mu_0\|_0^2 e^{Kt} < \infty,$$

and hence $\mu_t \in H_0$ a.s.. The assertion that $\sigma_{s,t} \in H_0$ a.s. also follows similarly from (3.3). □

Now we are ready to state and prove the main theorem.

Theorem 3.2. *Suppose that μ_0 has a square integrable density. Then the measure valued processes $(\mu_t, \sigma_{s,t} : \ 0 \leq s \leq t \leq T)$ are such that $\pi_t = \mu_t \langle \mu_t, 1 \rangle^{-1}$ and $(\mu_t, \sigma_{s,t})$ are the unique solutions to the system of equations* (2.13)–(2.14).

Proof: Let μ_t and $\sigma_{s,t}$ be defined by (2.11) and (2.12). Suppose that μ_t^1 and $\sigma_{s,t}^1$, $0 \le s \le t \le T$, are also solutions of the equations (2.13) and (2.14). Let $\tilde{\mu}_t$ and $\tilde{\sigma}_{s,t}$ denote the differences $\mu_t - \mu_t^1$ and $\sigma_{s,t} - \sigma_{s,t}^1$ respectively. Then, $\tilde{\sigma}_{s,t}$, $\tilde{\mu}_t \in H_0$ a.s., and with an argument similar to (3.2), we have

$$\mathbf{E}\|\tilde{\sigma}_{s,t}^\delta\|_0^2 \le K\mathbf{E}\|T_\delta(|\tilde{\mu}_s|)\|_0^2 + K\mathbf{E}\int_s^t \|T_\delta(|\tilde{\sigma}_{s,u}|)\|_0^2 du.$$

Taking $\delta \to 0$, we have

$$\begin{aligned}\mathbf{E}\|\tilde{\sigma}_{s,t}\|_0^2 &\le K\mathbf{E}\||\tilde{\mu}_s|\|_0^2 + K\mathbf{E}\int_s^t \||\tilde{\sigma}_{s,u}|\|_0^2 du \\ &= K\mathbf{E}\|\tilde{\mu}_s\|_0^2 + K\mathbf{E}\int_s^t \|\tilde{\sigma}_{s,u}\|_0^2 du.\end{aligned}$$

Similarly, applying (3.4), we have

$$\mathbf{E}\|\tilde{\mu}_t\|_0^2 \le K\int_0^t \mathbf{E}\|\tilde{\mu}_s\|_0^2 ds$$

hence, $\mathbf{E}\|\tilde{\mu}_t\|_0^2 = 0$. This in turn implies that $\mathbf{E}\|\tilde{\sigma}_{s,u}\|_0^2 = 0$. □

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