Homework #6

5.56. a)

$$F_Y(y) = \int_{-\infty}^{y} f_Y(x)dx = \int_{-\infty}^{y} \frac{1}{\pi}\frac{1}{1+x^2}dx = \frac{1}{\pi}\tan^{-1} y.$$

b) As $F_Y^{-1}(y) = \tan \frac{2}{\pi} y$, if $U_1,\ U_2,\ \cdots,\ U_n$ i.i.d. unif(0,1), then $\tan \frac{2}{\pi} U_1,\ \tan \frac{2}{\pi} U_2,\ \cdots,\ \tan \frac{2}{\pi} U_n$ i.i.d. Cauchy

5.58. a) If $n = 1$, then

$$P(S_1 \le t) = t.$$

Assume it holds for $n = k$, i.e.,

$$P(S_k \le t) = \frac{t^k}{(k)!}, \qquad 0 < t < 1.$$

Then for $n = k + 1$, we have

$$\begin{aligned} P(S_n \le t) &= \int_0^t P(S_k \le t - u_{k+1}) du_{k+1} \\ &= \int_0^t \frac{(t - u_{k+1})^k}{(k)!} du_{k+1} \\ &= \frac{t^{k+1}}{(k+1)!}. \end{aligned}$$

b) As

$$\begin{aligned} P(N = n) &= P(S_{n-1} \le 1,\ S_n > 1) \\ &= P(S_{n-1} \le 1) - P(S_n \le 1) \\ &= \frac{1}{(n-1)!} - \frac{1}{n!} = \frac{n-1}{n!}, \end{aligned}$$

we have

$$E(N) = \sum_{n=2}^{\infty} n\frac{n-1}{n!} = \sum_{n=2}^{\infty} \frac{1}{(n-2)!} = e.$$