Homework #5

5.30. As

$$\bar{X}_1 - \bar{X}_2 \approx N\left(0, \frac{2\sigma^2}{n}\right),$$

we have

$$\begin{aligned} .99 &= P\left(|\bar{X}_1 - \bar{X}_2| < \frac{\sigma}{5}\right) \\ &= P\left(|Z| < \frac{\sigma/5}{\sqrt{2}\sigma/\sqrt{n}}\right) \\ &= P\left(|Z| < \frac{\sqrt{n}}{5\sqrt{2}}\right). \end{aligned}$$

So

$$\frac{\sqrt{n}}{5\sqrt{2}} = 2.326$$

and hence

$$n = 271.$$

5.32. a) Since

$$g(x) = \sqrt{x} \text{ and } g'(x) = \frac{a}{x}$$

are continuous on $(0, \infty)$, the the range of X_i is contained in $(0, \infty)$, and X_i converges in probability, we see that $\sqrt{X_i}$ and $\frac{a}{X_i}$ converges in probability.

b) Since $S_n^2 \to \sigma^2$ in probability, so

$$\frac{\sigma}{S_n} \to 1$$

in probability.

5.33. Since $\lim_{m\to\infty} F_X(-m) = 0$, for any $\epsilon > 0$, there is m_0 such that for any $m \geq m_0$, $F_X(-m) < \epsilon$. We may choose $-m \leq -m_0$ being a continuity point of F_X, since there are only finite many points of discontinuity for F_X. Then

$$\lim_{n\to\infty} F_{X_n}(-m) = F_X(-m) < \epsilon.$$

Since

$$P(Y_n > c + m) \to 0,$$

we have

$$P(Y_n \leq c+m) \to 0.$$

As

$$P(X_n+Y_n \leq c) \leq P(X_n \leq -m) + P(Y_n \leq c+m),$$

we see that

$$\limsup_{n\to\infty} P(X_n+Y_n \leq c) \leq \epsilon.$$

Thus

$$P(X_n+Y_n \leq c) \to 0$$

and hence

$$P(X_n+Y_n > c) \to 1.$$

5.35. a)

$$EX_i = 1 \text{ and } Var(X_i) = 1.$$

So by CLT, we have

$$\frac{\bar{X}_n - 1}{1/\sqrt{n}} \longrightarrow N(0,1).$$

b) As $M_{X_i}(t) = (1-t)^{-1}$, we have

$$M_{\bar{X}}(t) = \left(1 - \frac{t}{n}\right)^{-n}$$

and hence, $\bar{X} \sim gamma\left(n, \frac{1}{n}\right)$. Thus

$$\begin{aligned}\Phi(x) &\approx P\left(\bar{X} \leq 1 + \frac{x}{\sqrt{n}}\right)\\ &= \int_0^{1+\frac{x}{\sqrt{n}}} \frac{n^n}{\Gamma(n)} t^{n-1} e^{-nt} dt.\end{aligned}$$

Hence, the pdf

$$\phi(x) \approx \frac{n^n}{\Gamma(n)}\left(1+\frac{x}{\sqrt{n}}\right)^{n-1} e^{-n+\sqrt{n}x}\frac{1}{\sqrt{n}}$$

Take $x=0$, we get

$$\frac{n^{n+\frac{1}{2}}}{n!}e^{-n} \approx \frac{1}{\sqrt{2\pi}}.$$

Thus

$$n! \approx \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}.$$

5.36. a)

$$E(Y) = E(E(Y|N)) = E(2N) = 2\theta$$

and

$$\begin{aligned} Var(Y) &= E((Y-2\theta)^2) = E\left(E\left((Y-2N)^2 + 4(N-\theta)^2 + 4(Y-2N)(N-\theta)\Big| N\right)\right) \\ &= E\left(4N + 4(N-\theta)^2\right) = 4\theta + 4\theta = 8\theta. \end{aligned}$$

b)

$$\begin{aligned} M_{\frac{Y-2\theta}{\sqrt{8\theta}}}(t) &= M_Y\left(\frac{t}{\sqrt{8\theta}}\right) e^{-\frac{t}{\sqrt{2\theta}}} = \sum_{n=1}^{\infty} \frac{\theta^n}{n!} e^{-\theta} \left(1 - \frac{t}{\sqrt{2\theta}}\right)^{-n} e^{-\frac{t}{\sqrt{2\theta}}} \\ &= e^{-\theta}\left(\exp\left(\theta\left(1-\frac{t}{\sqrt{2\theta}}\right)^{-1}\right) - 1\right) e^{-\frac{t}{\sqrt{2\theta}}} \\ &\to e^{\frac{t^2}{2}}. \end{aligned}$$

5.42. a) Note that

$$f_{(n)}(u) = n\beta(1-u)^{\beta-1}\left(1-(1-u)^\beta\right)^{n-1}.$$

So

$$\begin{aligned} P(n^v(1-X_{(n)}) \le x) &= P(X_{(n)} \ge 1 - n^{-v}x) \\ &= \int_{1-n^{-v}x}^{1} n\beta(1-u)^{\beta-1}\left(1-(1-u)^\beta\right)^{n-1} du. \end{aligned}$$

Let $t = n^v(1-u)$. Then

$$P(n^v(1-X_{(n)}) \le x) = \int_0^x n\beta t^{\beta-1} n^{-v(\beta-1)} \left(1 - n^{-v\beta}t^\beta\right)^{n-1} n^{-v} dt.$$

Take $v = 1/\beta$. Then

$$P(n^v(1-X_{(n)}) \le x) = \int_0^x \beta t^{\beta-1}\left(1-n^{-1}t^\beta\right)^{n-1} dt \to \int_0^x \beta t^{\beta-1} e^{-t^\beta} dt.$$

b) Note that

$$f_{(n)}(u) = ne^{-u}(1-e^{-u})^{n-1}.$$

So

$$\begin{aligned} P(X_{(n)} - a_n \leq x) &= \int_0^{a_n+x} ne^{-u}(1-e^{-u})^{n-1}du \\ &= \int_0^{1-e^{-x-a_n}} nv^{n-1}dv \qquad\qquad v = 1-e^{-u} \\ &= \left(1-e^{-x-a_n}\right)^n. \end{aligned}$$

Take $a_n = \log n$. Then

$$P(X_{(n)} - a_n \leq x) \to e^{-e^{-x}}.$$

5.44. a)

$$EX_i = p, \qquad Var(X_i) = p(1-p).$$

Hence, by CLT,

$$\sqrt{n}(Y_n - p) \longrightarrow N(0, p(1-p)).$$

b) Let $g(p) = p(1-p)$. Then for $p \neq \frac{1}{2}$,

$$g'(p) = 1 - 2p \neq 0.$$

Thus, by δ-method

$$\sqrt{n}(Y_n(1-Y_n) - p(1-p)) \longrightarrow N(0, p(1-p)(1-2p)^2).$$

c) If $p = \frac{1}{2}$, then $g''(p) = -2$ and hence, by second-order δ-method

$$n\left(Y_n(1-Y_n) - \frac{1}{4}\right) \longrightarrow -\frac{1}{4}\chi_1^2.$$