Homework #4

5.21.

$$\begin{aligned} P\left(X_{(2)} > \frac{1}{2}(X_1 + X_2)\right) &= P(X_2 > X_1) + P(X_1 > X_2) \\ &= 1 - P(X_1 = X_2) = 1. \end{aligned}$$

For general n, we still have

$$P(X_{(n)} > median) = 1.$$

In fact, for n odd,

$$P(X_{(n)} > X_{((n+1)/2)}) = 1 - P(X_{(n)} < X_{((n+1)/2)}) = 1;$$

and for n even

$$P\left(X_{(n)} > \frac{1}{2}(X_{(n/2)} + X_{(n/2+1)})\right) = 1 - P(X_{(n)} = X_{(n/2)} = X_{(n/2+1)}) = 1.$$

5.22. Let $\phi(x)$, $\Phi(x)$ be the pdf and cdf of the standard normal. Then

$$f_Z(x) = 2\phi(x)(1 - \Phi(x)).$$

So

$$\begin{aligned} P(Z^2 \le z) &= \int_{x^2 \le z} 2\phi(x)(1 - \Phi(x))dx \\ &= \int_{-\sqrt{z}}^{\sqrt{z}} 2\phi(x)(1 - \Phi(x))dx. \end{aligned}$$

Then the pdf of Z^2 is

$$\begin{aligned} f_{Z^2}(z) &= 2\phi(\sqrt{z})(1 - \Phi(\sqrt{z}))\frac{1}{2\sqrt{z}} + 2\phi(-\sqrt{z})(1 - \Phi(-\sqrt{z}))\frac{1}{2\sqrt{z}} \\ &= \frac{\phi(\sqrt{z})}{\sqrt{z}}\left(2 - \Phi(\sqrt{z}) - \Phi(-\sqrt{z})\right) \\ &= \frac{1}{\sqrt{2\pi}} z^{-1/2} e^{-z/2}. \end{aligned}$$

Hence, $Z^2 \sim \chi^2_1$.

5.23. For $0 < z < 1$, we have

$$\begin{aligned} P(Z \leq z) &= 1 - P(Z > z) \\ &= 1 - P(U_1 \geq z, \cdots, U_X \geq z) \\ &= 1 - \sum_{n=1}^{\infty} \frac{c}{n!} P(U_1 > z)^n \\ &= 1 - \frac{e^{1-z} - 1}{e - 1}. \end{aligned}$$

5.24.

$$\begin{aligned} f_{X_{(1)},X_{(n)}}(u,v) &= \frac{n!}{(n-2)!} \frac{1}{\theta^2} 1_{0<u<v<\theta} \left(\frac{v-u}{\theta}\right)^{n-2} \\ &= \frac{n(n-1)}{\theta^n} (v-u)^{n-2} 1_{0<u<v<\theta}. \end{aligned}$$

Define

$$s = u/v, \qquad t = v.$$

Then

$$u = st \text{ and } = t.$$

The Jacobian is $J = t$. Thus the joint density of $X_{(1)}/X_{(n)}$ and $X_{(n)}$ is

$$\begin{aligned} f(s,t) &= \frac{n(n-1)}{\theta^n} (t(1-s))^{n-2} t 1_{0<s<1, 0<t<\theta} \\ &= \frac{n}{\theta^n} t^{n-1} 1_{0<t<\theta} (n-1)(1-s)^{n-2} 1_{0<s<1}. \end{aligned}$$

$X_{(1)}/X_{(n)}$ and $X_{(n)}$ are independent.

5.27. a) For $u \leq v$, we have

$$\begin{aligned} f_{X_{(i)}|X_{(j)}}(u|v) &= \frac{f_{X_{(i)},X_{(j)}}(u,v)}{f_{X_{(j)}}(v)} \\ &= \frac{\frac{n!}{(i-1)!(j-1-i)!(n-j)!} f(u)f(v)F(u)^{i-1}(F(v)-F(u))^{j-i-1}(1-F(v))^{n-j}}{\frac{n!}{(j-1)!(n-j)!} f(v)F(v)^{j-1}(1-F(v))^{n-j}} \\ &= \frac{(j-1)!}{(i-1)!(j-i-1)!} f(u)F(u)^{i-1}(F(v)-F(u))^{j-i-1}F(v)^{1-j}. \end{aligned}$$

b) As

$$f_{R,V}(r,v) = n(n-1)r^{n-2}, \qquad 0 < r < 1, \;\; \frac{r}{2} < v < 1 - \frac{r}{2}$$

and

$$f_R(r) = n(n-1)r^{n-2}(1-r),$$

for $\frac{r}{2} < v < 1 - \frac{r}{2}$, we have

$$f_{V|R=r}(v) = \frac{n(n-1)r^{n-2}}{n(n-1)r^{n-2}(1-r)} = \frac{1}{1-r}.$$