Homework #3

5.10. a) $\theta_1 = \mu$, $\theta_2 = \sigma^2$, $\theta_3 = 0$ and

$$\begin{aligned}
\theta_4 &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} (x-\mu)^4 dx \\
&= \sigma^4 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} z^4 dz \qquad \text{define } z = \tfrac{x-\mu}{\sigma} \\
&= \sigma^4 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} z^3 d\left(-e^{-\frac{z^2}{2}}\right) \\
&= \sigma^4 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} 3z^2 e^{-\frac{z^2}{2}} dz \\
&= 3\sigma^4.
\end{aligned}$$

b)

$$\begin{aligned}
Var(S^2) &= \frac{1}{n}\left(3\sigma^4 - \frac{n-3}{n-1}\sigma^4\right) \\
&= \frac{2\sigma^4}{n-1}.
\end{aligned}$$

c) As

$$\xi \equiv \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1},$$

we have

$$M_\xi(t) = (1-2t)^{-(n-1)/2} \text{ and } Var(\xi) = \frac{(n-1)^2}{\sigma^4} Var(S^2).$$

As

$$M'(t) = (n-1)(1-2t)^{-(n+1)/2} \text{ and } M''(t) = (n^2-1)(1-2t)^{-(n+3)/2},$$

we see that

$$E(\xi) = n-1 \text{ and } E(\xi^2) = n^2 - 1.$$

Thus

$$Var(S^2) = \frac{\sigma^4}{(n-1)^2} Var(\xi) = \frac{2\sigma^4}{n-1}.$$

11. Proof: For any t, we have

$$0 \le E(S+t)^2 = ES^2 + 2ESt + t^2.$$

Thus

$$(2ES)^2 - 4E(S^2) \leq 0.$$

Thus

$$ES \leq \sigma.$$

In case of $E(S) = \sigma$, we then have

$$ES^2 + 2ESt + t^2 = 0 \text{ when } t = -\sigma.$$

Therefore

$$E((S-\sigma)^2) = 0.$$

Thus $S = \sigma$ a.s. and $S^2 = \sigma^2$ a.s.

As

$$(n-1)S^2 = \left(1 - n^{-1}\right) X_n^2 - 2n^{-1} \sum^{n-1} X_{i-1} X_n + n^{-1} \left(\sum_{i=1}^{n-1} X_i \right)^2$$

and X_n is independent of $(X_1, \cdots, X_{n-1})$, X_n can take only two possible values $x_\pm$. Then

$$2n^{-1} \sum^{n-1} X_{i-1} = x_+ + x_- \qquad a.s.$$

As X_1 is independent of other r.v., we then get X_1 can take only one possible value. Thus, $\sigma = 0$.

5.12. As $\bar{X} \sim N(0, n^{-1})$, we have

$$\begin{aligned} EY_1 &= \int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi/n}} e^{-\frac{(x-\mu)^2}{2/n}} dx \\ &= \frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} |z| \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\ &= \frac{2}{\sqrt{n}} \int_0^{\infty} \frac{1}{\sqrt{2\pi}} d\left(-e^{-\frac{z^2}{2}}\right) \\ &= \sqrt{2/(n\pi)}. \end{aligned}$$

Take $n = 1$ above, we have $E(|X_1|) = \sqrt{2/\pi}$. Thus

$$EY_2 = \sqrt{2/\pi} > EY_1.$$

5.13.

$$\begin{aligned}
E\sqrt{S} &= \frac{\sigma}{\sqrt{n-1}} E\sqrt{\chi^2_{n-1}} \\
&= \frac{\sigma}{\sqrt{n-1}} \int_0^\infty \sqrt{x} \frac{1}{\Gamma((n-1)/2)2^{(n-1)/2}} x^{\frac{n-1}{2}-1} e^{-x/2} dx \\
&= \frac{\sigma}{\sqrt{n-1}} \frac{1}{\Gamma((n-1)/2)2^{(n-1)/2}} \int_0^\infty x^{\frac{n}{2}-1} e^{-x/2} dx \\
&= \frac{\sigma}{\sqrt{n-1}} \frac{\Gamma(n/2)2^{n/2}}{\Gamma((n-1)/2)2^{(n-1)/2}} = \frac{2}{\sqrt{n-1}} \frac{\Gamma(n/2)}{\Gamma((n-1)/2)} \sigma.
\end{aligned}$$

Thus

$$c = \frac{\sqrt{n-1}}{2} \frac{\Gamma((n-1)/2)}{\Gamma(n/2)}.$$

5.16. Let

$$Z_i = \frac{X_i - i}{i}, \qquad i = 1, 2, 3.$$

a)

$$Z_1^2 + Z_2^2 + Z_3^2 \sim \chi_3^2.$$

b)

$$\frac{Z_1}{\sqrt{(Z_1^2 + Z_2^2)/2}} \sim t_2.$$

c)

$$\frac{2Z_1^2}{Z_1^2 + Z_2^2} \sim F_{1,2}.$$

5.19. a)

$$\begin{aligned}
& P(\chi_p^2 > a) - P(\chi_q^2 > a) \\
= \;& P(Z_1^2 + \cdots + Z_p^2 > a) - P(Z_1^2 + \cdots + Z_q^2 > a) \\
= \;& P(Z_1^2 + \cdots + Z_p^2 > a, \; Z_1^2 + \cdots + Z_q^2 \le a) > 0.
\end{aligned}$$

b) Let $k > j$ and $\xi_1, \; \xi_2, \; \xi_3$ be indep. r.v. such that $k\xi_1 \sim \chi_k^2$, $j\xi_2 \sim \chi_j^2$ and $\nu\xi_3 \sim \chi_\nu^2$. Then

$$\begin{aligned}
P(kF_{k,\nu} > a) &= P(k\xi_1 > a\xi_3) \\
&= \int_R P(k\xi_1 > au) dF_{\xi_3}(u) \\
&> \int_R P(j\xi_2 > au) dF_{\xi_3}(u) \\
&= P(jF_{j,\nu} > a).
\end{aligned}$$

c) As

$$\begin{aligned} P((k-1)F_{k-1,\nu} > (k-1)F_{\alpha,k-1,\nu}) &= \alpha \\ &= P(kF_{k,v} > kF_{\alpha,k,\nu}) \\ &> P((k-1)F_{k-1,\nu} > kF_{\alpha,k,\nu}), \end{aligned}$$

we have

$$(k-1)F_{\alpha,k-1,\nu} < kF_{\alpha,k,\nu}.$$