Homework #2

5.5.

$$\begin{aligned} P(\bar{X} \le x) &= P\left(\sum_{i=1}^{n} X_i \le nx\right) \\ &= \int_{-\infty}^{nx} f_{X_1+\cdots+X_n}(t)dt \\ &= \int_{-\infty}^{x} f_{X_1+\cdots+X_n}(ns)nds. \end{aligned}$$

Thus

$$f_{\bar{X}}(x) = nf_{X_1+\cdots+X_n}(nx).$$

5.6. a) Let $u = x - y, \;\; v = y$. Then $J = 1$ and hence,

$$\begin{aligned} P(X - Y \le z) &= \int\int 1_{x-y\le z} f_X(x) f_Y(y) dxdy \\ &= \int_{-\infty}^{z}\int_{-\infty}^{\infty} f_X(u+v) f_Y(v) dvdu. \end{aligned}$$

Thus

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x+v) f_Y(v) dv.$$

b) Let $u = xy, \;\; v = y$. Then $J = v^{-1}$ and hence,

$$\begin{aligned} P(XY \le z) &= \int\int 1_{xy\le z} f_X(x) f_Y(y) dxdy \\ &= \int_{-\infty}^{z}\int_{-\infty}^{\infty} f_X(u/v) f_Y(v) |v^{-1}| dvdu. \end{aligned}$$

Thus

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z/v) f_Y(v) |v^{-1}| dv.$$

c) Let $u = x/y, \;\; v = y$. Then $J = v$ and hence,

$$\begin{aligned} P(X/Y \le z) &= \int\int 1_{x/y\le z} f_X(x) f_Y(y) dxdy \\ &= \int_{-\infty}^{z}\int_{-\infty}^{\infty} f_X(uv) f_Y(v) |v| dvdu. \end{aligned}$$

Thus

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(zv) f_Y(v) |v| dv.$$

5.8. a)

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}(X_i - X_j)^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}(X_i^2 - 2X_iX_j + X_j^2) \\ &= 2n\sum_{i=1}^{n} X_i^2 - 2n^2\bar{X}^2 \\ &= 2n(n-1)S^2.\end{aligned}$$

Thus

$$S^2 = \frac{1}{2n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}(X_i - X_j)^2.$$

b) Note that

$$S^2 = \frac{1}{n(n-1)}\sum_{1\le i<j\le n}(Y_i - Y_j)^2.$$

For $i \neq j$, we have

$$E(Y_i - Y_j)^2 = E(Y_i^2 - 2Y_iY_j + Y_j^2) = 2\theta_2.$$

We write

$$S^2 - \theta_2 = \frac{1}{n(n-1)}\sum_{1\le i<j\le n}\left((Y_i - Y_j)^2 - 2\theta_2\right).$$

For $i < j$, we have

$$\begin{aligned}E\left((Y_i - Y_j)^2 - 2\theta_2\right)^2 &= E\left((Y_i^2 - 2Y_iY_j + Y_j^2)^2 - 4\theta_2^2\right) \\ &= \theta_4 + 4\theta_2^2 + \theta_4 + 2\theta_2^2 - 4\theta_2^2 = 2(\theta_4 + \theta_2^2)\end{aligned}$$

and for distinct $i,\ j,\ k$, we have

$$\begin{aligned}& E\left((Y_i - Y_j)^2 - 2\theta_2\right)\left((Y_i - Y_k)^2 - 2\theta_2\right) \\ =\ & E\left((Y_i - Y_j)^2(Y_i - Y_k)^2 - 4\theta_2^2\right) \\ =\ & \theta_4 - \theta_2^2.\end{aligned}$$

Therefore

$$\begin{aligned}Var(S^2) &= \frac{1}{n^2(n-1)^2}\sum_{1\le i<j\le n}\sum_{1\le k<\ell\le n} E\left(\left((Y_i - Y_j)^2 - 2\theta_2\right)\left((Y_k - Y_\ell)^2 - 2\theta_2\right)\right) \\ &= \frac{1}{n^2(n-1)^2}\left(2(\theta_4 + \theta_2^2)\frac{n(n-1)}{2} + (\theta_4 - \theta_2^2)n(n-1)(n-2)\right) \\ &= \frac{1}{n}\left(\theta_4 - \frac{n-3}{n-1}\theta_2^2\right).\end{aligned}$$

c)

$$\begin{aligned}
Cov(\bar{X}, S^2) &= E(\Big((\bar{X}-\theta_1)S^2\Big) \\
&= \frac{1}{2n^2(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} E\left((X_i - X_j)^2(X_k-\theta_1)\right) \\
&= \frac{1}{n^2(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n} E\left((Y_i - Y_j)^2 Y_i\right) \\
&= \frac{1}{n^2(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n} E\left(Y_i^3 - 2Y_i^2 Y_j + Y_j^2 Y_i\right) \\
&= \frac{1}{n^2(n-1)}\theta_3(n^2-n) = \frac{\theta_3}{n}.
\end{aligned}$$

So $Cov(\bar{X}, S^2) = 0$ iff $\theta_3 = 0$.