Homework #11

8.5. a) As

$$L(\theta, v|x) = \frac{\theta^n v^{n\theta}}{w^{\theta+1}} 1_{x_{(1)} \geq v}$$

is increasing in $v \leq x_{(1)}$, $\hat{v} = x_{(1)}$, where $w = \Pi_{i=1}^n x_i$. Note that

$$\mathcal{L}(\theta, v|x) = n \log \theta + n\theta \log v - (\theta + 1) \log w,$$

we have

$$\partial_\theta \mathcal{L}(\theta, v|x) = \frac{n}{\theta} + n \log v - \log w = \frac{n}{\theta} - T^{-1}.$$

Thus, $\hat{\theta} = \frac{1}{n}T$.

b) As

$$\begin{aligned}
\lambda(x) &= \frac{L(\hat{\theta}_0, \hat{v}_0|x)}{L(\hat{\theta}, \hat{v}|x)} \\
&= \frac{\hat{v}^n / w^2}{(T/n)^n \hat{v}^T / w^{T/n+1}} \\
&= n^n T^{-n} e^{-(1-\frac{1}{n})T},
\end{aligned}$$

we have $\lambda(x) \leq c$ iff $T \leq c_1$ or $T \geq c_2$.

c) As

$$f_{X_{(1)},\cdots,X_{(n)}}(u_1, \cdots, u_n) = n! \frac{\theta^n v^{n\theta}}{(u_1 \cdots u_n)^{\theta+1}} 1_{v \leq u_1 \leq \cdots \leq u_n}$$

and

$$f_{X_{(1)}}(u_1) = \frac{n\theta v^{n\theta}}{u_1^{n\theta+1}} 1_{u_1 \geq v},$$

the conditional density is

$$f_{(X_{(2)},\cdots,X_{(n)})|X_{(1)}=u_1}(u_2, \cdots, u_n) = (n-1)! \frac{\theta^{n-1} u_1^{(n-1)\theta}}{(u_2 \cdots u_n)^{\theta+1}} 1_{u_1 \leq u_2 \leq \cdots \leq u_n}.$$

Thus, $X_{(2)}, \cdots, X_{(n)}$ are the order statistic of conditionally i.i.d. random variables $\tilde{X}_2, \cdots, \tilde{X}_n$ with common pdf $f(x|\theta, X_{(1)})$. Then, given $X_{(1)}$, the random variables $\tilde{X}_2/X_{(1)}, \cdots, \tilde{X}_n/X_{(1)}$ are conditionally i.i.d. with common pdf

$$g(x|\theta) = \frac{\theta}{x^{\theta+1}} 1_{x>1}.$$

Since the density of $2\log \tilde{X}_2/X_{(1)}$ is the same as χ_2^2, we see that $2T \sim \chi^2_{2(n-1)}$.
8.6. a), b) As

$$L(\theta, \mu|x, y) = \frac{1}{\theta^n \mu^m} \exp\left(-\frac{1}{\theta}\sum x - \frac{1}{\mu}\sum y\right),$$

we have

$$\mathcal{L}(\theta, \mu|x, y) = -n\log\theta - m\log\mu - \frac{1}{\theta}\sum x - \frac{1}{\mu}\sum y.$$

Thus

$$\partial_\theta \mathcal{L}(\theta, \mu|x, y) = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum x$$

and

$$\partial_\mu \mathcal{L}(\theta, \mu|x, y) = -\frac{m}{\mu} + \frac{1}{\mu^2}\sum y.$$

So

$$\hat{\theta} = \bar{x} \text{ and } \hat{\mu} = \bar{y}.$$

If $\theta = \mu$, then

$$\mathcal{L}(\theta, \theta|x, y) = -(n+m)\log\theta - \frac{1}{\theta}\left(\sum x + \sum y\right).$$

So

$$\partial_\theta \mathcal{L}(\theta, \theta|x, y) = -\frac{n+m}{\theta} + \frac{1}{\theta^2}\left(\sum x + \sum y\right).$$

Hence

$$\hat{\theta}_0 = \frac{\sum x + \sum y}{n+m}.$$

Therefore

$$\begin{aligned} \lambda(x) &= \frac{\frac{1}{\hat{\theta}_0^{n+m}}\exp(-n-m)}{\frac{1}{\hat{\theta}^n \hat{\mu}^m}\exp(-n-m)} \\ &= T^n(1-T)^m. \end{aligned}$$

c) As $\sum x \sim gamma(n, \theta)$ and $\sum y \sim gamma(m, \theta)$, we have $T \sim beta(n, m)$.
8.7. As

$$L(\theta, \lambda|x) = \frac{1}{\lambda^n} e^{-\frac{n}{\lambda}(\bar{x}-\theta)} 1_{x_{(1)} \geq \theta}$$

is increasing in $\theta \leq x_{(1)}$, we have $\hat{\theta} = X_{(1)}$.

As

$$\mathcal{L}(\theta, \lambda|x) = -n\log\lambda - \frac{n}{\lambda}(\bar{x} - \theta),$$

we have

$$\partial_\lambda \mathcal{L}(\theta, \lambda|x) = -\frac{n}{\lambda} + \frac{n}{\lambda^2}(\bar{x} - \theta),$$

and hence,

$$\hat{\lambda} = \bar{X} - X_{(1)}.$$

Similarly, we have

$$\hat{\theta}_0 = 0 \wedge X_{(1)} \text{ and } \hat{\lambda}_0 = \bar{X} - \hat{\theta}_0.$$

If $x_{(1)} \leq 0$, then $\lambda(x) = 1$. If $x_{(1)} > 0$, then $\hat{\theta}_0 = 0$ and $\hat{\lambda}_0 = \bar{x}$. Hence

$$\lambda(x) = \frac{\frac{1}{\bar{x}^n} e^{-n}}{\frac{1}{(\bar{x} - x_{(1)})^n} e^{-n}} = \left(\frac{\bar{x} - x_{(1)}}{\bar{x}} \right)^n.$$

Reject H_0 if

$$\frac{\bar{x} - x_{(1)}}{\bar{x}} \leq c'.$$

b) Similar.

8.8. a) Note that

$$\begin{aligned} L(\theta, a|x) &= \Pi_{i=1}^n \frac{1}{\sqrt{2\pi a\theta}} \exp\left(-\frac{1}{2a\theta}(x_i - \theta)^2 \right) \\ &= (2\pi a\theta)^{-n/2} \exp\left(-\frac{n}{2a} \left(\frac{m_2}{\theta} - 2\bar{x} + \theta \right) \right). \end{aligned}$$

So

$$\mathcal{L}(\theta, a|x) = -\frac{n}{2} \log(2\pi a\theta) - \frac{n}{2a} \left(\frac{m_2}{\theta} - 2\bar{x} + \theta \right).$$

Taking derivative, we have

$$\partial_\theta \mathcal{L}(\theta, a|x) = -\frac{n}{2}\frac{1}{\theta} - \frac{n}{2a} \left(-\frac{m_2}{\theta^2} + 1 \right) = 0$$

and

$$\partial_a \mathcal{L}(\theta, a|x) = -\frac{n}{2}\frac{1}{a} + \frac{n}{2a^2} \left(\frac{m_2}{\theta} - 2\bar{x} + \theta \right) = 0.$$

Thus

$$\hat{\theta} = \bar{x} \text{ and } \hat{a} = \frac{m_2 - \bar{x}^2}{\bar{x}}.$$

Suppose $a = 1$, then

$$\mathcal{L}(\theta|x) = -\frac{n}{2} \log(2\pi\theta) - \frac{n}{2} \left(\frac{m_2}{\theta} - 2\bar{x} + \theta \right).$$

So

$$\partial_\theta \mathcal{L}(\theta|x) = -\frac{n}{2}\frac{1}{\theta} - \frac{n}{2}\left(-\frac{m_2}{\theta^2} + 1\right) = 0$$

and

$$\partial_\theta^2 \mathcal{L}(\theta|x) = \frac{n}{2}\frac{1}{\theta^2}\left(1 - \frac{2m_2}{\theta}\right).$$

Thus

$$\hat{\theta}_0 = \frac{-1 + \sqrt{1 + 4m_2}}{2}.$$

Therefore

$$\begin{aligned} \lambda(x) &= \frac{(2\pi\hat{\theta}_0)^{-n/2} \exp\left(-\frac{n}{2}\left(\sqrt{1+4m_2} - 2\bar{x}\right)\right)}{(2\pi(m_2 - \bar{x}^2))^{-n/2} \exp\left(-\frac{n}{2}\right)} \\ &= \left(\frac{m_2 - \bar{x}^2}{\hat{\theta}_0}\right)^{-n/2} \exp\left(-\frac{n}{2}\left(\sqrt{1+4m_2} - 2\bar{x} - 1\right)\right). \end{aligned}$$

Reject H_0 if

$$\frac{m_2 - \bar{x}^2}{\sqrt{1+4m_2} - 1} \exp\left(\sqrt{1+4m_2} - 2\bar{x}\right) < c'.$$