Homework #10

7.37.

$$f(x|\theta) = (2\theta)^{-n} 1_{x_{(1)} \geq -\theta, x_{(n)} \leq \theta}.$$

$(X_{(1)}, X_{(n)})$ is sufficient.

As

$$f_{X_{(n)}}(x) = nf(x)F(x)^{n-1} = n(2\theta)^{-n}(x+\theta)^{n-1},$$

we have

$$\begin{aligned} EX_{(n)} &= \int_{-\theta}^{\theta} xn(2\theta)^{-n}(x+\theta)^{n-1}dx \\ &= n(2\theta)^{-n}\int_0^{2\theta}(y-\theta)y^{n-1}dy \\ &= n(2\theta)^{-n}\left(\frac{(2\theta)^{n+1}}{n+1} - \theta\frac{(2\theta)^n}{n}\right) \\ &= \frac{n-1}{n+1}\theta. \end{aligned}$$

By symmetry, we have

$$EX_{(1)} = -\frac{n-1}{n+1}\theta.$$

Thus

$$E_\theta(X_{(1)} + X_{(n)}) = 0, \forall\theta.$$

So $(X_{(1)}, X_{(n)})$ is not complete. There is not a way to find the best unbiased estimator.

7.38. Note that

$$\log f(x|\theta) = (\theta - 1)\log(x_1 \cdots x_n) + \log\theta.$$

So

$$\partial_\theta \log f(x|\theta) = \log(x_1 \cdots x_n) + \frac{n}{\theta}.$$

We need

$$\frac{W(x) - g(\theta)}{\log(x_1 \cdots x_n) + \frac{n}{\theta}}$$

does not depend on x. We take $W(x) = \frac{1}{n}\log(x_1 \cdots x_n)$ and $g(\theta) = \frac{1}{\theta}$. Note that

$$E\log X_1 = \int_0^1 \theta x^{\theta-1}\log x dx = -\frac{1}{\theta}.$$

We have $EW(X) = g(\theta)$.

b)

$$f(x|\theta) = \left(\frac{\log\theta}{\theta - 1}\right)^n \theta^{n\bar{x}},$$

so

$$\mathcal{L}(\theta|x) = n\log\log\theta - n\log(\theta-1) + n\bar{x}\log\theta.$$

Then

$$\partial_\theta \mathcal{L}(\theta|x) = \frac{n}{\theta\log\theta} - \frac{n}{\theta-1} + \frac{n\bar{x}}{\theta}.$$

So

$$\begin{aligned} W(x) - g(\theta) &= a(\theta)\left(\frac{n}{\theta\log\theta} - \frac{n}{\theta-1} + \frac{n\bar{x}}{\theta}\right) \\ &= \frac{na(\theta)}{\theta}\left(\bar{x} - \theta\left(\frac{1}{\theta-1} - \frac{1}{\theta\log\theta}\right)\right). \end{aligned}$$

Take $W(X) = \bar{X}$ and $g(\theta) = \theta\left(\frac{1}{\theta-1} - \frac{1}{\theta\log\theta}\right)$. Note that

$$E(X_1) = \int_0^1 \frac{\log\theta}{\theta-1}\theta^x x dx = g(\theta).$$

So, $EW(X) = g(\theta)$.

7.40.

$$Var(\bar{X}) = \frac{p(1-p)}{n}.$$

$$\begin{aligned} I_1(p) &= -E_p\partial_p^2\left(X_1\log p + (1-X_1)\log(1-p)\right) \\ &= E_p\left(\frac{X_1}{p^2} + \frac{1-X_1}{(1-p)^2}\right) \\ &= \frac{p}{p^2} + \frac{1-p}{(1-p)^2} = \frac{1}{p(1-p)}. \end{aligned}$$

Thus the CR lower bound is

$$\frac{1}{nI_1(p)} = \frac{p(1-p)}{n} = Var(\bar{X}).$$

7.41. a)

$$E\sum_{i=1}^n a_i X_i = \sum_{i=1}^n a_i \mu = \mu$$

iff $\sum_{i=1}^n a_i = 1$.

b)

$$Var \sum_{i=1}^n a_i X_i = \sum_{i=1}^n a_i^2 \sigma^2.$$

Let

$$f(a_1, \cdots, a_n, \lambda) = \sum_{i=1}^n a_i^2 - 2\lambda \left(\sum_{i=1}^n a_i - 1 \right)$$

Then

$$\partial_i f = 2a_i - 2\lambda = 0, \qquad a_i = \lambda.$$

$$\partial_\lambda f = \sum_{i=1}^n a_i - 1 = 0, \qquad n\lambda = 1.$$

Thus

$$a_i = \frac{1}{n}.$$

Best estimate is $\bar{X}$.

7.42. a)

$$E \sum_{i=1}^n a_i W_i = \sum_{i=1}^n a_i \theta = \theta$$

iff $\sum_{i=1}^n a_i = 1$.

$$Var \sum_{i=1}^n a_i W_i = \sum_{i=1}^n a_i^2 \sigma_i^2.$$

Let

$$f(a_1, \cdots, a_n, \lambda) = \sum_{i=1}^n a_i^2 \sigma_i^2 - 2\lambda \left(\sum_{i=1}^n a_i - 1 \right)$$

Then

$$\partial_i f = 2a_i \sigma_i^2 - 2\lambda = 0, \qquad a_i = \lambda / \sigma_i^2.$$

$$\partial_\lambda f = \sum_{i=1}^n a_i - 1 = 0, \qquad \lambda \sum_{j=1}^n \sigma_j^{-2} = 1.$$

Thus

$$a_i = \frac{\sigma_i^{-2}}{\sum_{j=1}^n \sigma_j^{-2}}.$$

Best estimate is

$$W^* = \frac{\sum_{i=1}^n W_i \sigma_i^{-2}}{\sum_{j=1}^n \sigma_j^{-2}}.$$

b)

$$Var\sum_{i=1}^{n} a_i W_i = \sum_{i=1}^{n} a_i^2 \sigma_i^2 = \frac{1}{\sum_{j=1}^{n} \sigma_j^{-2}}.$$

7.44. As

$$\begin{aligned} f(x|\theta) &= \Pi_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x_i-\mu)^2\right) \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2}\sum_{i=1}^{n} x_i^2\right) \exp\left(-\frac{n}{2}\mu^2\right) \exp\left(n\mu\bar{x}\right), \end{aligned}$$

$\bar{X}$ is complete and sufficient.

Note that

$$\begin{aligned} E\left(\bar{X}^2 - \frac{1}{n}\right) &= Var\bar{X} + \left(E\bar{X}\right)^2 - \frac{1}{n} \\ &= \frac{1}{n} + \mu^2 - \frac{1}{n} = \mu^2. \end{aligned}$$

So, $\bar{X}^2 - \frac{1}{n}$ is UMVUE for μ^2.

$$\begin{aligned} Var\left(\bar{X}^2 - \frac{1}{n}\right) &= Var(\bar{X}^2) = Var\left(\left(\frac{1}{n}Z + \mu\right)^2\right) \\ &= E\left(\left(\frac{1}{n}Z + \mu\right)^4\right) - \left(E\left(\left(\frac{1}{n}Z + \mu\right)^2\right)\right)^2 \\ &= 3n^{-4} + 6n^{-2}\mu^2 + \mu^4 - \left(n^{-2} + \mu^2\right)^2 \\ &= 2n^{-4} + 4n^{-2}\mu^2. \end{aligned}$$

$$I_1(\mu) = -E_\mu \partial_\mu^2 \left(-\frac{(X_1-\mu)^2}{2}\right) = 1.$$

Thus

$$LB = \frac{4\mu^2}{n} > 2n^{-4} + 4n^{-2}\mu^2.$$

7.47. $A_i = \pi r_i^2$, $r_i \in N(\mu, \sigma^2)$. If σ^2 is known, it is similar to problem 7.44. Suppose σ^2 is unknown. Then $(\bar{r}, S^2)$ is sufficient and complete for (μ, σ^2). As

$$E\bar{r}^2 = \frac{\sigma^2}{n} + \mu^2 \text{ and } ES^2 = \sigma^2,$$

we have

$$E\left(\bar{r}^2 - \frac{1}{n}S^2\right) = \mu^2$$

and hence, $\pi\left(\bar{r}^2 - \frac{1}{n}S^2\right)$ is unbiased for A. It is UMVUE.

7.50. a)

$$E(a\bar{X} + (1-a)cS) = a\theta + (1-a)\theta = \theta.$$

b)

$$\begin{aligned}
Var\left(a\bar{X} + (1-a)cS\right) &= a^2Var(\bar{X}) + (1-a)^2c^2Var(S) \\
&= a^2\frac{\theta^2}{n} + (1-a)^2c^2\left(\theta^2 - (\theta/c)^2\right) \\
&= \left(\frac{a^2}{n} + (1-a)^2(c^2-1)\right)\theta^2.
\end{aligned}$$

Let

$$f(a) = \frac{a^2}{n} + (1-a)^2(c^2-1).$$

It attains its minimum at

$$a = \frac{n(c^2-1)}{1+n(c^2-1)}.$$

c) As

$$E_\theta(\bar{X} - cS) = 0, \qquad \forall\theta,$$

and $\bar{X} - cS \neq 0$, $(\bar{X}, S^2)$ is not complete.

7.62. a)

$$\begin{aligned}
R(\theta,\delta) &= E\left(|a\bar{X} + b - \theta|^2\right) \\
&= E|a(\bar{X}-\theta) + b + (a-1)\theta|^2 \\
&= a^2\frac{\sigma^2}{n} + (b + (a-1)\theta)^2.
\end{aligned}$$

b) Since the posterior is normal with mean $\frac{n\tau^2\bar{x}+\sigma^2\mu}{n\tau^2+\sigma^2}$ and variance $\frac{\tau^2+\sigma^2}{n\tau^2+\sigma^2}$. The Bayes estimate is

$$\delta^\pi(x) = \frac{n\tau^2\bar{x} + \sigma^2\mu}{n\tau^2+\sigma^2}.$$

Namely, $a = 1-\eta$ and $b = \eta\mu$ Thus

$$\begin{aligned} R(\theta, \delta^\pi) &= (1-\eta)^2\frac{\sigma^2}{n} + (\eta\mu - \eta\theta)^2 \\ &= (1-\eta)^2\frac{\sigma^2}{n} + \eta^2(\mu-\theta)^2. \end{aligned}$$

c)

$$\begin{aligned} B(\pi, \delta^\pi) &= E^\pi\left((1-\eta)^2\frac{\sigma^2}{n} + \eta^2(\mu-\theta)^2\right) \\ &= (1-\eta)^2\frac{\sigma^2}{n} + \eta^2\tau^2 \\ &= \left(\frac{n\tau^2}{n\tau^2+\sigma^2}\right)^2\frac{\sigma^2}{n} + \eta^2\tau^2 \\ &= \eta\frac{n\tau^4}{n\tau^2+\sigma^2} + \eta^2\tau^2 = \eta\tau^2. \end{aligned}$$

7.63.

$$R(\mu, \delta_1) = \left(1 - \frac{1}{n+1}\right)^2\frac{1}{n} + \frac{1}{(n+1)^2}\mu^2$$

and

$$R(\mu, \delta_2) = \left(1 - \frac{1}{10n+1}\right)^2\frac{1}{n} + \frac{1}{(10n+1)^2}\mu^2.$$

Except for μ near 0, δ_2 has a smaller Bayes risk. Namely, The more spread out the prior, the smaller the Bayes risk.

7.65. b) The posterior expected loss is

$$E\left(e^{c(\delta(x)-\theta)} - c(\delta(x)-\theta) - 1|x\right) = f(\delta(x))$$

where

$$f(\delta) = e^{c\delta}E(e^{-c\theta}|x) - c\delta + cE(\theta|x) - 1.$$

Taking derivatives, we have

$$f'(\delta) = ce^{c\delta}E(e^{-c\theta}|x) - c$$

and

$$f''(\delta) = c^2e^{c\delta}E(e^{-c\theta}|x) > 0.$$

Thus

$$\delta^\pi(x) = -\frac{1}{c}\log E(e^{-c\theta}|x).$$

c) Since the posterior is normal with mean $\bar{x}$ and variance $\frac{\sigma^2}{n}$. We have

$$\begin{aligned}\delta^\pi(x) &= -\frac{1}{c}\log e^{-c\bar{x}+\frac{\sigma^2 c^2}{2n}} \\ &= \bar{x} - \frac{c\sigma^2}{2n}.\end{aligned}$$

d) Posterior expected loss is

$$e^{c\delta}e^{-c\bar{x}+\frac{\sigma^2 c^2}{2n}} - c\delta + c\bar{x} - 1.$$

e)

$$E^\pi(\bar{x} - \frac{c\sigma^2}{2n} - \theta) = \frac{\sigma^2}{n} + \left(\frac{c\sigma^2}{2n}\right)^2.$$