MATH 323 Second Midterm Exam

March 11, 2009

Name ____________________

ID number ____________________

1. (18 points) Let X be a continuous random variable with probability density function

$$f(x) = \frac{3}{8}x^2, \qquad 0 \leq x \leq 2.$$

1) Find $P(0 \leq X \leq 1)$.
2) Find the expected value and the standard deviation of X.
3) Calculate $E(X^3)$.

Solution. 1)

$$P(0 \leq X \leq 1) = \int_0^1 \frac{3}{8}x^2 dx = \frac{1}{8}x^3|_0^1 = \frac{1}{8}$$

2)

$$\mathbb{E}\ (X) = \int_0^2 x\frac{3}{8}x^2 dx = \frac{3}{32}x^4|_0^2 = \frac{3}{2}$$

$$\mathbb{E}\ (X^2) = \int_0^2 x^2\frac{3}{8}x^2 dx = \frac{3}{40}x^5|_0^2 = \frac{12}{5}$$

$$V(X) = \frac{12}{5} - \left(\frac{3}{2}\right)^2 = \frac{3}{20}$$

$$SD(X) = \sqrt{\frac{3}{20}} = .387$$

3)

$$\mathbb{E}\ (X^3) = \int_0^2 x^3\frac{3}{8}x^2 dx = \frac{3}{48}x^6|_0^2 = 4$$

2. (18 points) Suppose X has a uniform distribution on $(1, 3)$.
1) What is the probability density function of X?
2) What are $E(X)$ and $V(X)$ equal to?
3) Calculate $E(X^4)$.
Solution. 1)

$$f(x) = \frac{1}{2}, \qquad 1 < x < 3$$

2)

$$\mathbb{E}\,(X) = \frac{1+3}{2} = 2, \qquad V(X) = \frac{(3-1)^2}{12} = \frac{1}{3}$$

3)

$$\mathbb{E}\,(X^4) = \int_1^3 x^4 \frac{1}{2} dx = \frac{1}{10} x^5|_1^3 = 24.1$$

3. (18 points) Suppose X has an exponential distribution with mean 25.
1) What is the distribution function of X?
2) What is the variance of X?
3) Find $P(10 < X < 30)$.
Solution. 1)

$$F(x) = \begin{cases} 1 - e^{-\frac{x}{25}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

2)

$$V(X) = \theta^2 = 25^2 = 625$$

3)

$$\begin{aligned} P(10 < X < 30) &= 1 - e^{-\frac{30}{25}} - \left(1 - e^{-\frac{10}{25}}\right) \\ &= e^{-.4} - e^{-1.2} = .369 \end{aligned}$$

4. (18 points) Let X be a normal random variable with mean 20 and variance 25.
1) Find $P(X \leq 25)$.
2) What is the distribution of $Y = 2X + 3$? Specify its mean and variance.
3) Find a level x_0 such that $P(X \leq x_0) = .05$.
Solution. 1)

$$\begin{aligned} P(X \leq 25) &= P\left(Z \leq \frac{25-20}{5}\right) = P(Z \leq 1) \\ &= .5 + .3413 = .8413 \end{aligned}$$

2) Normal with $\mu = 2 \times 20 + 3 = 43$ and $\sigma^2 = 4 \times 25 = 100$
3)

$$P\left(Z \leq \frac{x_0 - 20}{5}\right) = .05$$

$$\frac{x_0 - 20}{5} = -1.645$$

$$x_0 = 11.775$$

5. (18 points) Let X_1 and X_2 be two random variables with joint probability function given by

(Table Draw by hand)

1) Find the marginal probability function of X_1.
2) Calculate $E(X_1)$ and $V(X_1)$.
3) Find $E(X_2|X_1 = 1)$.

Solution. 1)

$$p_1(-2) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$
$$p_1(1) = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$$

2)

$$\mathbb{E}\ (X_1) = (-2)\frac{1}{3} + \frac{2}{3} = 0$$
$$V(X) = (-2)^2\frac{1}{3} + \frac{2}{3} = 2$$

3)

$$p(-2|x_1 = 1) = \frac{\frac{1}{6}}{\frac{2}{3}} = \frac{1}{4}$$
$$p(1|x_1 = 1) = \frac{\frac{1}{2}}{\frac{2}{3}} = \frac{3}{4}$$

Thus

$$\mathbb{E}\ (X_2|X_1 = 1) = (-2)\frac{1}{4} + \frac{3}{4} = \frac{1}{4}$$

6. (10 points) Suppose X_1 and X_2 have joint probability density function

$$f(x_1, x_2) = 8x_1x_2, \qquad 0 \le x_1 \le x_2 \le 1.$$

Calculate $E(X_1X_2)$.

Solution.

$$\begin{aligned}
\mathbb{E}\ (X_1X_2) &= \int_0^1 \int_0^{x_2} x_1x_2 8x_1x_2 dx_1 dx_2 \\
&= \int_0^1 8x_2^2 \int_0^{x_2} x_1^2 dx_1 dx_2 \\
&= \int_0^1 8x_2^2 \frac{1}{3} x_2^3 dx_2 \\
&= \int_0^1 \frac{8}{3} x_2^5 dx_2 = \frac{4}{9}
\end{aligned}$$