Math 323 Homework # 9

5.1. a)

$$p(0,0) = \frac{1}{3^2} = \frac{1}{9}$$

$$p(0,1) = p(1,0) = \frac{2 \times 1}{9} = \frac{2}{9}$$

$$p(1,1) = \frac{2}{9}$$

$$p(0,2) = p(2,0) = \frac{1}{9}$$

b)

$$p_1(0) = \frac{1}{9} + \frac{2}{9} + \frac{1}{9} = \frac{4}{9}$$

$$p_1(1) = \frac{2}{9} + \frac{2}{9} = \frac{4}{9}$$

$$p_1(2) = \frac{1}{9}$$

c)

$$P(X_1 = 1 | X_2 = 1) = \frac{2/9}{4/9} = \frac{1}{2}$$

5.2. b)

$$P(X_1 \leq .2,\ X_2 \leq .4) = .4 \times .2 = .08$$

c)

$$P(.1 \leq X_1 \leq .3,\ X_2 > .4) = (.3 - .1)(1 - .4) = .12$$

5.4. a)

$$k \times \frac{1}{2} \times 2 \times 1 = 1, \qquad k = 1$$

b)

$$P(X_1 \geq 3X_2) = \frac{1}{2} \times 2 \times \frac{2}{3} = \frac{2}{3}$$

5.6. a) For $0 < x_2 < 1$,

$$f_2(x_2) = 2 - 2x_2.$$

b)

$$\begin{aligned} P(X_2 \le .4) &= \int_0^{.4} (2 - 2x_2)dx_2 = 2(.4) - x_2^2|_0^{.4} \\ &= .8 - .4^2 = .64 \end{aligned}$$

c)

$$f_1(x_1) = \frac{1}{2}x_1, \qquad 0 < x_1 < 2.$$

As

$$f_1(x_1)f_2(x_2) = x_1(1 - x_2) \neq f(x_1, x_2),$$

they are not indep.
d) As

$$f(x_2|x_1 = 1) = \frac{1}{1/2} = 2, \qquad 0 < x_2 < \frac{1}{2},$$

we have

$$P\left(X_2 \le \frac{1}{4}|X_1 = 1\right) = \int_0^{1/4} 2dx_2 = \frac{1}{2}$$

5.17. a)

$$\mathbb{E}\ (X_1) = .24$$

$$\mathbb{E}\ (X_1^2) = .24, \qquad V(X_1) = .24 - .24^2 = .1824$$

$$\mathbb{E}\ (X_2) = .16 + 2 \times .29 = .74$$

$$\mathbb{E}\ (X_2^2) = .16 + 2^2 \times .29 = 1.32$$

$$V(X_2) = 1.32 - .74^2 = .7724$$

b)

$$\mathbb{E}\ (X_1X_2) = .02 + 2 \times .05 = .12$$

$$Cov(X_1, X_2) = .12 - .24 \times .74 = -.0576$$

$$\rho = \frac{-.0576}{\sqrt{.1824 \times .7724}} = -.1535$$

c) They are negatively correlated, but the correlation is quite weak.

5.19. a)

$$\mathbb{E}\ (X_1) = \int_0^1 \int_0^{1-x_1} x_1 2 dx_2 dx_1 = \int_0^1 (2x_1 - 2x_1^2) dx_1 = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\mathbb{E}\ (X_1^2) = \int_0^1 \int_0^{1-x_1} x_1^2 2 dx_2 dx_1 = \int_0^1 (2x_1^2 - 2x_1^3) dx_1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

$$V(X_1) = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$$

Similarly,

$$\mathbb{E}\ (X_2) = \frac{1}{3}, \qquad V(X_2) = \frac{1}{18}$$

Note

$$\begin{aligned} \mathbb{E}\ (X_1 X_2) &= \int_0^1 \int_0^{1-x_1} x_1 x_2 2 dx_2 dx_1 = \int_0^1 x_1 (1 - x_1)^2 dx_1 \\ &= \int_0^1 (x_1 - 2x_1^2 + x_1^3) dx_1 = \frac{1}{2} - \frac{2}{3} + \frac{1}{4} = \frac{1}{12} \end{aligned}$$

Thus

$$Cov(X_1, X_2) = \frac{1}{12} - \frac{1}{3}\frac{1}{3} = -\frac{1}{36}.$$

Hence,

$$\mathbb{E}\ (Y) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

and

$$V(Y) = \frac{1}{18} + \frac{1}{18} - 2(-\frac{1}{36}) = \frac{1}{18}.$$

b) Let

$$\frac{1}{k^2} = \frac{1}{2}$$

So, $k = \sqrt{2}$. The interval is

$$\frac{2}{3} \pm \sqrt{2}\frac{1}{\sqrt{18}} = \frac{2}{3} \pm \frac{1}{3} = \left(\frac{1}{3}, 1\right).$$

c)

$$\rho = \frac{-1/36}{1/18} = -\frac{1}{2}$$

negatively correlated.

5.26. Multinormial $n = 4$, $p_f = .73$, $p_a = .2$, $p_o = .07$

$$P(Y_f = 2, Y_a = 1, Y_o = 1) = \frac{4!}{2!1!1!}.73^2(.2)(.07) = .0895$$

5.35. $p_L = .4$, $p_R = .25$, $p_S = .35$

a) $n = 5$

$$P(Y_L = 1, Y_R = 1, Y_S = 3) = \frac{5!}{1!1!3!}.4(.25)(.35)^3 = .08575$$

b)

$$\begin{aligned} P(\text{at least one right}) &= 1 - P(\text{no right}) \\ &= 1 - (.4 + .35)^5 = .7627 \end{aligned}$$

c)

$$\mathbb{E}\ (Y_L) = 100 \times .4 = 40, \qquad V(Y_L) = 100(.4)(1 - .4) = 24$$

5.42. a)

$$f_1(x_1) = \int_0^{x_1} 3x_1 dx_2 = 3x_1^2, \qquad 0 < x_1 < 1$$

$$f_2(x_2) = \int_{x_2}^1 3x_1 dx_1 = \frac{3}{2}(1 - x_2^2), \qquad 0 < x_2 < 1$$

b)

$$\begin{aligned} P\left(X_1 \leq \frac{3}{4}, X_2 \leq \frac{1}{2}\right) &= \int_0^{\frac{1}{2}} \int_{x_2}^{\frac{3}{4}} 3x_1 dx_1 dx_2 \\ &= \int_0^{\frac{1}{2}} \left(\frac{27}{32} - \frac{3}{2}x_2^2\right) dx_2 \\ &= \frac{27}{64} - \frac{3}{2}x_2^3|_0^{\frac{1}{2}} = \frac{23}{64} \end{aligned}$$

c) Since $X_2 \leq X_1$, $P(X_1 \leq \frac{1}{2}, X_2 \geq \frac{3}{4}) = 0$. Thus

$$P(X_1 \leq \frac{1}{2}|X_2 \geq \frac{3}{4}) = 0.$$

5.45. a)

$$f(x_1|x_2) = \frac{3x_1}{\frac{2}{3}(1 - x_2^2)} = \frac{2x_1}{1 - x_2^2}, \qquad x_2 < x_1 < 1$$

b)

$$f(x_2|x_1) = \frac{3x_1}{3x_1^2} = \frac{1}{x_1}, \qquad 0 < x_2 < x_1$$

Namely, X_2 is uniformly distributed in $(0, x_1)$.
c) No, they are not independent because

$$f_1(x_1)f_2(x_2) = \frac{9}{2}x_1^2(1-x_2^2) \neq f(x_1, x_2)$$

d)

$$\begin{aligned} P(X_1 \leq \frac{3}{4}|X_2 = \frac{1}{2}) &= \int_{1/2}^{3/4} \frac{2x_1}{1-(1/2)^2} dx_1 \\ &= \frac{4}{3}x_1^2|_{1/2}^{3/4} = \frac{5}{12} \end{aligned}$$

5.60. a)

$$\mathbb{E}\,(X_2|X_1 = x_1) = \frac{1}{2}x_1$$

b)

$$\begin{aligned} \mathbb{E}\,(X_2) &= \mathbb{E}\,(\frac{1}{2}X_1) = \int_0^1 \frac{1}{2}x_1 3x_1^2 dx_1 \\ &= \int_0^1 \frac{3}{2}x_1^3 dx_1 = \frac{3}{2}\frac{1}{4} = \frac{3}{8} \end{aligned}$$

c)

$$\begin{aligned} \mathbb{E}\,(X_2) &= \int_0^1 x_2 \frac{3}{2}(1-x_2^2) dx_2 \\ &= \int_0^1 \left(\frac{3}{2}x_2 - \frac{3}{2}x_2^3\right) dx_2 \\ &= \frac{3}{4} - \frac{3}{8} = \frac{3}{8} \end{aligned}$$