A limit theorem for the trajectory of particle moving in random medium

Vlad Vysotsky

University of Delaware
Motivation: the Lorentz model (1905)

A spherical particle is driven through a random medium by a constant external field \( a \).

The medium is a set of immobile identical spherical obstacles whose centers form a Poisson point process in \( \mathbb{R}^d, d \geq 2 \).

At collisions with obstacles, the particle inelastically reflects with the restitution coefficient \( \alpha \in [0, 1] \):

\[
\nu \mapsto \nu - (1 + \alpha)(\nu, \nu)\nu,
\]

where \( \nu \) is the velocity of the particle and \( \nu \) is the unit normal to the obstacle.
New model: Markov approximation of the LM

Motion in \( \mathbb{R}^d, d \geq 1 \), under a constant external field \( a \). Initial position \( X(0) = 0 \), nonrandom initial velocity \( v_0 \).

Assumptions:
By \( \eta_n \) denote the length of the trajectory between \( n \)th and \((n+1)\)st collisions.
\[ A1: \{\eta_n\}_{n \geq 0} \text{ are i.i.d. exponential r.v.'s with mean } \lambda. \]

By \( v_n \) denote the velocity of the particle at \( n \)th collision.
\[ A2: \text{At collisions, } v_n \mapsto v_n - \frac{1+\alpha}{2}(v_n + |v_n|\sigma_n), \text{ where } \{\sigma_n\}_{n \geq 1} \text{ are i.i.d. random vectors unif. distrd. on the unit sphere } S^{d-1} \subset \mathbb{R}^d. \]

\[ A3: \{\eta_n\}_{n \geq 0}, \{\sigma_n\}_{n \geq 1} \text{ are independent.} \]

V. ('06), Ravishankar and Triolo ('99) for \( \alpha = 1 \).
Velocities at collisions $v_n$ form a Markov chain:

$$v_{n+1} = v_n - \frac{1 + \alpha}{2} (v_n + |v_n|\sigma_n) + aF\left(v_n - \frac{1 + \alpha}{2} (v_n + |v_n|\sigma_n), \eta_n\right),$$

where the nonrandom function $F : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$\int_0^\infty F(v, z) |v + at| dt = z.$$
By $X(T)$ denote the position of the particle at time $T$. Choose a basis of $\mathbb{R}^d$ such that $a = (0, \ldots, 0, |a|)$.

**Theorem (V., ’06)**

Suppose that $0 < \alpha < 1$ and $a \neq 0$. Then there exist constants $c_\nu > 0$ and $c_1, c_2 \geq 0$ (depending on $|a|, \alpha, \lambda$ and $d$) such that for any initial velocity $v_0 \in \mathbb{R}^d$,

$$
\frac{X(sT) - c_\nu sv_0 sT}{\sqrt{T}} \xrightarrow{D} (c_1 W_1(\cdot), \ldots, c_1 W_{d-1}(\cdot), c_2 W_d(\cdot))
$$

in $C([0, 1] \to \mathbb{R}^d)$ as $T \to \infty$, where $W_i(\cdot)$ are independent Wiener processes.
A brief sketch of the proof

Let \( N(T) \) be the number of collisions before \( T \), then
\[
\frac{N(T)}{T} \to c > 0 \text{ a.s.}
\]
By \( \tau_n \) denote the moment of the \( n \)th collision.
Consider a new Markov chain \( \Phi_n := \left( \frac{v_n}{\sigma_n} \right) \). Now

\[
X(sT) = \sum_{i=1}^{N(sT)} X(\tau_k) - X(\tau_{k+1}) + o(\sqrt{T})
\]

\[
= \sum_{i=1}^{N(sT)} g(\Phi_k, \Phi_{k-1}) + o(\sqrt{T})
\]

\[
= \sum_{i=1}^{N(sT)} \tilde{g}(\Phi_k) + o(\sqrt{T}) = \sum_{i=1}^{csT} \tilde{g}(\Phi_k) + o(\sqrt{T})
\]
Relations between the models

Collisions in the Lorentz model:

\[ \nu \mapsto \nu - (1 + \alpha)(\nu, \nu)\nu = \nu - \frac{1 + \alpha}{2} \left( \nu + |\nu|\sigma \right), \]

where \( \sigma \) is a unit vector such that \( \nu \) is directed along the bisectrix of the angle between \( \nu \) and \( \sigma \).

It is easier to describe a collision by \( \sigma \) rather then by \( \nu \).

**Lemma**

We have\(^1\)

1. \( \eta_0^{LM} \overset{D}{=} \eta_0 \) for \( \mathbb{R}^d \) with any \( d \geq 2 \);
2. \( (\eta_0^{LM}, \sigma_1^{LM}) \overset{D}{=} (\eta_0, \sigma_1) \) for \( \mathbb{R}^3 \).

---

\(^1\)under condition that \( |\nu_0| > (r + R)|a| \), where \( r \) and \( R \) are the radii of the particle and an obstacle.
The Markov approximation model is the limiting case of the LM:

The following approach is by Gallavotti and Lanford ('70s).
Recall that $r$ and $R$ are the radii of the particle and an obstacle. By $\rho$ denote the intensity of the Poisson point process which defines positions of obstacles.

Consider the Grad limit: $r + R \to 0, \rho \to \infty, (r + R)^{d-1} \rho = \text{const}$. The last condition implies that the mean free path $\lambda(r + R, \rho)$ also stays constant, denote $\lambda(r + R, \rho) =: \lambda$.

We believe that

$$X^{LM}_{\rho, r, R}(\cdot) \xrightarrow{D} X^{MALM}_{\lambda}(\cdot).$$
Some references


