Least Squares Estimator for Ornstein-Uhlenbeck Processes Driven by Stable Lévy Motions

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1. Formulation of LSE for Generalized O-U Processes

- Generalized Ornstein-Uhlenbeck processes \( \{X_t\} \):

\[
dX_t = -\theta_0 X_t dt + dZ_t, \quad X_0 = x.
\]

(1)
on some probability space \((\Omega, \mathcal{F}, P)\), where \(\theta_0 > 0\) is an unknown parameter.

- \(\{Z_t, t \geq 0\}\) is a standard symmetric \(\alpha\)-stable Lévy motion \((\alpha \in (1, 2))\) so that \(Z_1\) has a symmetric \(\alpha\)-stable distribution \(S_\alpha(1, 0, 0)\) with c.f.

\[
u \mapsto \exp(-|u|^\alpha).
\]

- **Remark:** (1) In general, a random variable \(\eta\) is said to have a stable distribution with index of stability \(\alpha \in (0, 2]\), scale parameter \(\sigma \in (0, \infty)\), skewness parameter \(\beta \in [-1, 1]\), and location parameter \(\mu \in (-\infty, \infty)\) if it has characteristic function of the following form:

\[
\phi_\eta(u) = \mathbb{E}\exp\{iu\eta\} = \begin{cases} 
\exp\left\{-\sigma^\alpha |u|^\alpha \left(1 - i\beta \operatorname{sgn}(u) \tan \frac{\alpha \pi}{2}\right) + i\mu u\right\}, & \text{if } \alpha \neq 1, \\
\exp\left\{-\sigma |u| \left(1 + i\beta^2 \frac{\pi}{2} \operatorname{sgn}(u) \log |u|\right) + i\mu u\right\}, & \text{if } \alpha = 1.
\end{cases}
\]

We denote \(\eta \sim S_\alpha(\sigma, \beta, \mu)\).

(2) Stable distributions are introduced by Paul Lévy (1925). Applications in modeling financial data: Mandelbrot (1963), Fama (1965), Officer (1972), Mittnik and Rachev (2001), Rachev (2003), Nolan (2005).

(3) Stable distributions are heavy-tailed and for \(\eta \sim S_\alpha(\sigma, \beta, \mu)\),

\[
E(|\eta|^q) = \infty \quad (q \geq \alpha).
\]
Brownian motion is almost surely continuous, while stable Lévy motion is a pure jump process (more appropriate to model extreme events).

• Discrete observations: \( (X_{t_i})_{i=0}^{n} \) with \( t_i = ih \).

• Our goal: (i) Construct LSE of \( \theta_0 \) based on discrete observations; (ii) Study the high frequency \( (h \to 0) \) asymptotics of the LSE.


• Our model is simple but can not be covered by those models mentioned above. This is due to the infinite variance property of \( \alpha \)-stable Levy motions.

• Our model has been used in physics, e.g., in geophysical turbulence and climate dynamical changes [see Ditlevsen (1999), Schertzer et al (2001)].
• The SDE (1) can be approximated by Euler scheme [see Jacod (2004), Jacod et al (2005)]

$$X_{t_i} = X_{t_{i-1}} - \theta X_{t_{i-1}} \Delta t_i + \Delta Z_{t_i},$$

where $\Delta t_i = t_i - t_{i-1} = h$ and $\Delta Z_{t_i} = Z_{t_i} - Z_{t_{i-1}}$.

• We may regard $X_{t_{i-1}} - \theta X_{t_{i-1}} \Delta t_i$ as a prediction of $X_{t_i}$.

• Contrast function:

$$\rho_n(\theta) = \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}} + \theta X_{t_{i-1}} \cdot \Delta t_{i-1}|^2. \quad (2)$$

• LSE of $\theta_0$:

$$\hat{\theta}_n = \frac{-\sum_{i=1}^{n}(X_{t_i} - X_{t_{i-1}})X_{t_{i-1}}}{h \sum_{i=1}^{n} X_{t_{i-1}}^2}. \quad (3)$$

• An equivalent expression:

$$\hat{\theta}_n = \frac{1 - e^{-\theta_0 h}}{h} - \frac{\sum_{i=1}^{n} X_{t_{i-1}} \cdot \int_{t_{i-1}}^{t_i} e^{-\theta_0 (t_i - s)} dZ_s}{h \sum_{i=1}^{n} X_{t_{i-1}}^2}. \quad (4)$$

• Strong consistency and asymptotic distribution of $\hat{\theta}_n$. 
2. Strong Consistency of the LSE

2.1. Stable stochastic integrals

- \( L^{\alpha}_{a.s} \) is the class of real-valued predictable processes \( F \) such that for every \( T > 0 \), \( \int_0^T |F(t, \omega)|^\alpha dt < \infty \) a.s.

- First define stochastic integral w.r.t \( Z_t \) for simple predictable process and then for general predictable process in \( L^{\alpha}_{a.s.} \):
  \[
  \int_0^t F(s) dZ_s. \quad \text{[see Rosinski and Woyczynski (1986), Kallenberg (1992)]}
  \]

- **Random time change:**
  Let \( F \in L^{\alpha}_{a.s} \) such that \( \tau(u) = \int_0^u |F|^\alpha dt \to \infty \) as \( u \to \infty \). Then, there exists an independent process \( Z' \) (with same distribution as \( Z \)) such that
  \[
  \int_0^t F dZ = Z'(\tau(t)).
  \]

- **Limit Theorem:** Let \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) be increasing. If \( \tau(u) \to \infty \) a.s. as \( u \to \infty \), then
  \[
  \limsup_{t \to \infty} \left| \int_0^t F dZ \right| / \varphi(\tau(t)) = 0 \quad \text{or} \quad = \infty \quad \text{a.s}
  \]
  according to
  \[
  \int_1^\infty \varphi(t)^{-\alpha} dt < \infty \quad \text{or} \quad = \infty.
  \]
2.2. Strong Consistency

- $X_t$ is ergodic ($\theta_0 > 0$) and $X_t \Rightarrow X_\infty$ as $t \to \infty$, where $X_\infty = \int_0^\infty e^{-\theta_0 t} dZ_t$ [see Sato (1999)].

- **Theorem 1**: Assume that $h \to 0$ and $t_n = nh \to \infty$. Then,
  \[ \hat{\theta}_n \to \theta_0 \text{ almost surely as } n \to \infty. \]  
  \[ (5) \]

- **Proof ideas**:
  - Let $\phi_n(t) = \sum_{i=1}^n X_{t_{i-1}} e^{-\theta(t_{i-1} - t)} 1_{(t_{i-1}, t_i]}(t)$ and $\tau_n(t_n) = \int_0^{t_n} |\phi_n(t)|^\alpha dt$.
  - Rewrite:
    \[ \hat{\theta}_n = \frac{1 - e^{-\theta_0 h}}{h} - \frac{\int_0^{t_n} \phi_n(t) dZ_t}{\tau_n(t_n)} \cdot \frac{\tau_n(t_n)}{h \sum_{i=1}^n X_{t_{i-1}}^2}. \]
  \[ (6) \]
  - Ergodic theorem:
    \[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n |X_{t_{i-1}}|^2 = E[X_\infty^2] = \infty \text{ a.s.} \]
  - Ergodic theorem plus limit theorem imply the almost sure convergence.
3. Asymptotic Distribution of LSE

3.1. Main Result

• **(A1):** As \( n \to \infty, h \to 0, nh^{1+\alpha}/\log n \to 0, nh^{2\alpha-1}\log n \to \infty, \) and \( nh^{-\alpha/2+\rho} \to \infty \) for some \( \rho > 0. \)

• Let \( C_\alpha = \left( \int_0^\infty x^{-\alpha} \sin(x) \, dx \right)^{-1} = \left[ \Gamma(1 - \alpha) \cos(\pi \alpha/2) \right]^{-1}, \) \( \sigma_1 = C_{\alpha/2}^{-2/\alpha}, \) and \( \sigma_2 = C_\alpha^{-1/\alpha}. \)

• **Theorem 2:** Under the condition (A1), we have
  \[
  \left( \frac{n}{\log n} \right)^{1/\alpha} h^{1/\alpha} (\hat{\theta}_n - \theta_0) \Rightarrow \frac{2\theta_0(\alpha \theta_0)^{-1/\alpha} \tilde{Y}}{Y_0},
  \]
  where \( Y_0 \) and \( \tilde{Y} \) are independent stable random variables, \( Y_0 \) is positive \( \alpha/2 \)-stable with distribution \( S_{\alpha/2}(\sigma_1, 1, 0) \), \( \tilde{Y} \) is symmetric \( \alpha \)-stable with distribution \( S_{\alpha}(\sigma_2, 0, 0) \).

• **Remark:** The rate of convergence \( \left( \frac{\log n}{nh} \right)^{1/\alpha} \) is considerably faster than the rate \( (nh)^{-1/2} \) in the classical Brownian motion case.
3.2. Some Preliminaries

- We have (denoting $X_{t_i}$ by $X_i$)

  \[
  \left( \frac{n}{\log n} \right)^{1/\alpha} h^{1/\alpha} (\hat{\theta}_n - \theta_0)
  \]
  \[
  = \left( \frac{n}{\log n} \right)^{1/\alpha} h^{1/\alpha} [h^{-1} (1 - e^{-\theta_0 h}) - \theta_0] \]
  \[
  = (n \log n)^{-1/\alpha} h^{-1/\alpha} \sum_{i=1}^n X_{i-1} \int_{t_{i-1}}^{t_i} e^{-\theta_0 (t_i - s)} dZ_s \]
  \[
  n^{-2/\alpha} h^{1-\frac{2}{\alpha}} \sum_{i=1}^n X_{i-1}^2 \]
  \[
  := \Lambda_n - \frac{\Phi_1(n)}{\Phi_2(n)}. \tag{8}
  \]

- Note that

  \[
  X_i = e^{-\theta_0 h} X_0 + \sum_{k=1}^i e^{-\theta_0 i h} \int_{t_{k-1}}^{t_k} e^{\theta_0 s} dZ_s. \tag{9}
  \]

- Let $V_{k-1} = \int_{t_{k-1}}^{t_k} e^{\theta_0 s} dZ_s$ and $\tau_{k-1} = \int_{t_{k-1}}^{t_k} |e^{\theta_0 s}|^\alpha ds$

  \[
  U_{k-1} = V_{k-1}/\tau_{k-1}^{\alpha}. \]

  By scaling property of stable distribution, we know that $U_0, U_1, U_2, \ldots$ are i.i.d. with the same distribution $S_\alpha(1, 0, 0)$. 


• \( X_i \) can be represented as

\[
X_i = e^{-\theta_0ih} X_0 + \left( \frac{e^{\alpha \theta_0 h} - 1}{\alpha \theta_0} \right)^{1/\alpha} \sum_{k=1}^{i} e^{-\theta_0(i-k+1)h} U_{k-1} \\
= c_{i,h} X_0 + \gamma_h \sum_{j=1}^{i} c_{j,h} U_{i-j},
\]

where \( c_{i,h} = e^{-\theta_0ih} \) and \( \gamma_h = \left( \frac{e^{\alpha \theta_0 h} - 1}{\alpha \theta_0} \right)^{1/\alpha} \).

• **Remark:** For symmetric \( \alpha \)-stable random variable \( U_1 \sim S_\alpha(1, 0, 0) \), we have

\[
\lim_{x \to \infty} x^\alpha P(U_1 > x) = C_\alpha/2 \quad \text{and} \quad \lim_{x \to \infty} x^\alpha P(U_1 < -x) = C_\alpha/2.
\]

• We introduce some scaling quantities:

\[
a_n = \inf \{ x : P(|U_1| > x) \leq n^{-1} \} \quad \text{and} \quad \tilde{a}_n = \inf \{ x : P(|U_0 U_1| > x) \leq n^{-1} \}.
\]

• Thanks to the tail behavior, we may take

\[
a_n = (C_\alpha n)^{\frac{1}{\alpha}} \quad \text{and} \quad \tilde{a}_n = C_\alpha^2 (n \log n)^{\frac{1}{\alpha}}.
\]
Lemma 1 [Davis and Resnick (1986)]: Let \( \{U_i\}_{i=0}^{\infty} \) be i.i.d. with the same stable distribution \( S_{\alpha}(1, 0, 0) \). Then, for \( a_n \) and \( \tilde{a}_n \) defined as above, we have for \( m \in \mathbb{N} \)

\[
\left( a_n^{-2} \sum_{i=1}^{n} U_i^2, \tilde{a}_n^{-1} \sum_{i=1}^{n} U_i U_{i+1}, \ldots, \tilde{a}_n^{-1} \sum_{i=1}^{n} U_i U_{i+m} \right) \\
\Rightarrow (Y_0, Y_1, \ldots, Y_m),
\]

where \( Y_0, Y_1, \ldots, Y_m \) are independent stable random variables, \( Y_0 \) is positive \( \alpha/2 \)-stable with distribution \( S_{\alpha/2}(\sigma_1, 1, 0) \), \( Y_1, \ldots, Y_m \) are i.i.d. symmetric \( \alpha \)-stable with distribution \( S_{\alpha}(\sigma_2, 0, 0) \).
3.3. Proof Ideas for Theorem 2

- The asymptotic behavior of \( \left( \frac{n}{\log n} \right)^{1/\alpha} h^{1/\alpha}(\hat{\theta}_n - \theta_0) \) will be determined by the asymptotic behavior of \( \Lambda_n, \Phi_1(n) \) and \( \Phi_2(n) \).

- It is obvious that \( \Lambda_n \to 0 \) as \( n \to \infty \) under condition (A1).

- **Proposition 1**: Under condition (A1), we have

\[
\Phi_2(n) \Rightarrow \frac{C_{\alpha}^2 \alpha Y_0}{2\theta_0},
\]

where \( Y_0 \) is a random variable with positively skewed stable distribution \( S_{\alpha/2}(\sigma_1, 1, 0) \) as specified in Theorem 2.

- Proof idea of Proposition 1:

  - Decomposition of \( \Phi_2(n) \):

\[
\Phi_2(n) = n^{-2/\alpha} h^{1-2/\alpha} \sum_{i=1}^{n} X_i^2 \\
= n^{-2/\alpha} h^{1-2/\alpha} X_0^2 + n^{-2/\alpha} h^{1-2/\alpha} \sum_{i=1}^{n-1} X_i^2 \\
:= \Phi_{2,1}(n) + \Phi_{2,2}(n).
\]
\[ \Phi_{2,2}(n) \]
\[ = n^{-2/\alpha} h^{1-2/\alpha} \sum_{i=1}^{n-1} \left[ c_{i,h} X_0 + \gamma_h \sum_{j=1}^{i} c_{j,h} U_{i-j} \right]^2 \]
\[ = n^{-2/\alpha} h^{1-2/\alpha} \sum_{i=1}^{n-1} c_{i,h}^2 X_0^2 \]
\[ + 2n^{-2/\alpha} h^{1-2/\alpha} \gamma_h \sum_{i=1}^{n-1} c_{i,h} X_0 \sum_{j=1}^{i} c_{j,h} U_{i-j} \]
\[ + n^{-2/\alpha} h^{1-2/\alpha} \gamma_h^2 \sum_{i=1}^{n-1} \sum_{j=1}^{i} c_{j,h}^2 U_{i-j}^2 \]
\[ + n^{-2/\alpha} h^{1-2/\alpha} \gamma_h^2 \sum_{i=1}^{n-1} \sum_{j=1}^{i} \sum_{k=1, k \neq j}^{i} c_{j,h} c_{k,h} U_{i-j} U_{i-k}. \] (14)

- By using some basic inequalities, truncation techniques, and Karamata’s theorem, we can prove that

\[ \Phi_2(n) - n^{-2/\alpha} h^{1-2/\alpha} \gamma_h^2 \sum_{i=1}^{n-1} \sum_{j=1}^{i} c_{j,h}^2 U_{i-j}^2 \to P 0. \] (15)
By using some basic inequalities and Lemma 1, we can show that
\[ n^{-2/\alpha} h^{(\alpha-2)/\alpha} \sum_{i=1}^{n-1} \sum_{j=1}^{i} c_{j,h}^{2} U_{i-j}^{2} \Rightarrow C_{\alpha}^{2/\alpha} Y_{0} / 2\theta_{0}, \] (16)
where \( Y_{0} \) is a random variable with positively skewed stable distribution \( S_{\alpha/2}(\sigma_{1}, 1, 0) \) as specified in Theorem 2.

- Proposition 2: Under the condition (A1), we have
  \[ \Phi_{1}(n) \Rightarrow C_{\alpha}^{2/\alpha} Y \] (17)
as \( n \to \infty \), where \( Y \) is a random variable with stable distribution \( S_{\alpha}((\alpha\theta_{0})^{-1/\alpha} \sigma_{2}, 0, 0) \).

- Proof idea is similar to that of Proposition 1, i.e., using some decomposition technique, basic inequalities, Lemma 1 and Skorohod’s representation theorem.

- Finally, the proof of Theorem 2 is completed by combining Propositions 1 and 2.
4. Simulations

- We apply our estimator to the following stochastic differential equation:

\[ dX_t = -\theta_0 X_t dt + dZ_t, \quad X_0 = 1, \]

where \( \theta_0 = 2 \) and \( Z_t \) is a stable process with index \( \alpha = 1.8 \). We simulate the process on the interval \([0, T]\) with \( T = 200 \).

- We plot \( \hat{\theta}_T = \hat{\theta}_n \) (where \( T = nh \)) as a function of \( T \) for \( h = 0.05 \) (Figure 1) and \( h = 0.01 \) (Figure 2).

- The following table describes \( \theta(25), \ldots, \theta(200) \) for different choice of \( h \).

<table>
<thead>
<tr>
<th>( T = )</th>
<th>25</th>
<th>50</th>
<th>75</th>
<th>100</th>
<th>125</th>
<th>150</th>
<th>175</th>
<th>200</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h = 0.1 )</td>
<td>1.3304</td>
<td>1.4764</td>
<td>1.4409</td>
<td>1.4593</td>
<td>1.5903</td>
<td>1.6104</td>
<td>1.4234</td>
<td>1.3817</td>
</tr>
<tr>
<td>( h = 0.05 )</td>
<td>1.7580</td>
<td>1.7326</td>
<td>1.7333</td>
<td>1.7051</td>
<td>1.4785</td>
<td>1.5509</td>
<td>1.5431</td>
<td>1.4053</td>
</tr>
<tr>
<td>( h = 0.033 )</td>
<td>2.3924</td>
<td>2.4681</td>
<td>2.4417</td>
<td>2.2373</td>
<td>2.2200</td>
<td>2.2094</td>
<td>2.1755</td>
<td>2.2002</td>
</tr>
<tr>
<td>( h = 0.025 )</td>
<td>1.8936</td>
<td>1.8320</td>
<td>1.8429</td>
<td>1.8323</td>
<td>1.8505</td>
<td>1.7615</td>
<td>1.7981</td>
<td>1.8109</td>
</tr>
<tr>
<td>( h = 0.02 )</td>
<td>2.0268</td>
<td>2.1199</td>
<td>2.1277</td>
<td>2.1162</td>
<td>2.1311</td>
<td>2.1529</td>
<td>2.1794</td>
<td>2.1734</td>
</tr>
<tr>
<td>( h = 0.0167 )</td>
<td>2.4107</td>
<td>2.5096</td>
<td>2.5188</td>
<td>2.4964</td>
<td>2.5093</td>
<td>2.5160</td>
<td>2.5260</td>
<td>2.5398</td>
</tr>
<tr>
<td>( h = 0.0143 )</td>
<td>2.2751</td>
<td>2.2514</td>
<td>2.1568</td>
<td>2.2191</td>
<td>2.0080</td>
<td>1.9516</td>
<td>1.6916</td>
<td>1.6516</td>
</tr>
<tr>
<td>( h = 0.0125 )</td>
<td>2.1116</td>
<td>1.9001</td>
<td>1.9310</td>
<td>1.9313</td>
<td>1.9313</td>
<td>1.9454</td>
<td>1.9353</td>
<td>1.9282</td>
</tr>
</tbody>
</table>

- We need to let both \( T \) goes to infinity and \( h \) goes to 0 to have the convergence of \( \theta_T \) to \( \theta_0 \).
Figure 1: LSE for $\theta_0 = 2$ when $h = 0.05$
Figure 2: LSE for $\theta_0 = 2$ when $h = 0.01$