Particle Approximations of Feynman-Kac representations
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A. F-K representations for solutions of linear PDEs
   - classical approximations (Euler)
   - Kusuoka-Lyons-Victoir cubature methods
B. F-K representations for solutions of linear SPDEs (Zakai)
   - particle filters (Sequential MCMC)
   - applications of the K-L-V cubature method
C. F-K representations for solutions of non-linear PDEs
   - Malliavin weights Monte-Carlo method
   - Kusuoka-Lyons-Victoir cubature based method

Joint work with S. Ghazali, K. Manolarakis
References:

Equivocally:  

\[ L = \sum_{i=1}^{d} f_i \frac{\partial}{\partial x_i} + \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \]

where \( a_{ij} = \frac{1}{2} \sum_{k=1}^{d} V_k V_k \), \( f_i = V_0 + \sum_{k=1}^{d} \sum_{j=1}^{d} V_k \frac{\partial V_k}{\partial x_j} \)

\( (3) \)

\[ X^{s,x}(t) = x + \int_{s}^{t} F(X^{s,x}(u)) du + \sum_{i=1}^{d} \int_{s}^{t} V_i(X^{s,x}(u)) dW_i^o, \quad s \leq t \leq T \]

- The stochastic integral in (2) is a Stratonovich integral, whilst the one in (3) is Itô integral.
- In the following, wlog \( s = 0 \)

\[ U(0,x) = \mathbb{E}[\gamma(X^x(T))] = P_{tT} \gamma(x), \]

where

\[ \begin{aligned}
    \frac{\partial}{\partial t} X^x(t) &= \sum_{i=0}^{d} V_i(X^x(t)) \, d W_i^o \\
    X^x(0) &= x \\
    t &\in [0,T]
\end{aligned} \]

\[ W^o_t = t, \quad d W^o_t = dt \]

\( (W^0, \ldots, W^d) \) - d-dimensional BM
Classical approximations

\[0 = z_0 < z_1 < ... < z_N = T\] equidistant partition of \([0, T]\) \[z_{n+1} - z_n = \frac{T}{N}\]

**Euler approximation** \[Y(z_n) = Y_n\]

\[Y_{n+1} = Y_n + f(Y_n) \Delta t + \sum_{i=1}^{d} \xi_i(z_n)\xi_i^{\text{in}}\] where \(\xi_i^{\text{in}}\) mutually independent

\[Y_0 = x\]

**Theorem** If \((a_{ij})\) is strictly elliptic, then there exists \(K\) independent of \(t\) s.t.

\[|E[Y(X_T)] - E[Y(Y_T)]| \leq KS\]

**Remark** \(\xi_i^{\text{in}} \sim N(0, 1)\) satisfies (*)

\[\xi_i^{\text{in}} = \begin{cases} \sqrt{\pi} \text{ prob } \frac{1}{2} \\ -\sqrt{\pi} \text{ prob } -\frac{1}{2} \end{cases}\] satisfies (*)
2nd order approximation \((d=1)\)

\[ Y_{n+1} = Y_n + f(Y_n) S + V(Y_n) \xi^n + \frac{1}{2} V(Y_n) V'(Y_n) (\xi^n)^2 - S \]
\[ + \frac{1}{2} \left( f(Y_n) f'(Y_n) + \frac{1}{2} ( f(Y_n) (V(Y_n))^2) \right) S^2 \]
\[ + \left( f'(Y_n) V(Y_n) + f(Y_n) V'(Y_n) - \frac{1}{2} V''(Y_n) V(Y_n) \right) \xi^n S \]

where \( \xi^n \) mutually independent

\((***) \quad |E[\xi^n]| + |E[(\xi^n)^3]| + |E[(\xi^n)^4]| - 5| + |E[(\xi^n)^5]| - 3S^2 | \leq KD^3 \)

then

\[ |E[Y(X_T^n)] - E[Y(Y_T^n)]| \leq K S^2 \]

Remark \( \xi^n \sim \mathcal{N}(0, S) \) satisfies (***)

\[ \xi^n = \begin{cases} \sqrt{3S} & \text{prob } \frac{1}{6} \\ -\sqrt{3S} & \text{prob } \frac{1}{6} \\ 0 & \text{prob } \frac{1}{3} \end{cases} \]
Monte Carlo approximation of $E[Y(Y_T)]$

Produce $n$ independent realizations of $Y: Y^1, \ldots, Y^n$, then

$$E[|E[Y(Y_T)] - \mu_n(p)|] \leq \frac{c(p)}{\sqrt{n}}$$

where

$$\mu_n(p) = \frac{1}{n} \sum_{i=1}^{n} Y(Y_T^i) , \quad \mu_n = \frac{1}{n} \sum_{i=1}^{n} S_Y^i$$

$$C(p^2) = Var(Y(Y_T)) = E[(Y(Y_T))^2] - (E[Y(Y_T)])^2$$

Remarks:

- $\mu_n$ is an approximation of $E[Y(Y_T)]$
- Error = $E[\delta^n, n]$
- $|E[Y(Y_T)] - \mu_n(p)| = \text{Systematic Error + Statistical Error}$

- $\bar{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} \sum_{k=0}^{n} S(Y_{T_k}^i, Y_{T_k}^i, \ldots, Y_{T_k}^i)$ is an approximation of $E[r_{[0,1]}]$
Disadvantages:

- The methods require the ellipticity condition imposed on the matrix \((a_{ij})\) or the Hörmander condition imposed on \(\{V_0, V_1, \ldots, V_d\}\) to obtain the optimal convergence rates.
- The approximating processes can leave the SDE's reachability surface, thus leading to instability problems and poor results.
- There is equal computational effort allocated to each interval of the partition: wasteful at the beginning, not enough towards the end.
- A "tree-like" approximation would be desirable.
A new class of schemes have been introduced to answer these problems:

Kusuoka [2001, 2004]
Kusuoka and Ninomiya [2003]
Lyons and Victoir [2004]
Ninomiya [2003]
Ninomiya and Victoir [2004]
Lyons and Litterer [2008]
Ninomiya and Ninomiya [2009]

Major improvements in computational effort:

- Ninomiya [2003] reports a speed-up ratio of the Kusuoka-Ninomiya algorithm over Euler of "about 6580".
- Ninomiya and Victoir [2004] report that their algorithm combined with the Romberg extrapolation is "about 800" times faster than Euler.
The UFG Condition [Kusuoka & Stroock]

c1. \( A = \{ \emptyset \} \cup \bigcup_{m=1}^{\infty} \{ 0,1,\ldots,d \}^m \), \( A_1 = A \setminus \{ \emptyset, (0) \} \), \( A_1(m) = \{ x \in A_1, \| x \|_2 \leq m \} \)

c2. Two norms
   
   \( \| \emptyset \| = 0 \)
   
   \( \| x \| = \| x \|_2 + \sum_{i \in x} (\| x \|_2) \) if \( x = (x_1, \ldots, x_r) \)

c3. Concatenation on \( A \)
   
   \( (e_1, \ldots, e_r) \times (f_1, \ldots, f_s) = (e_1, \ldots, e_r, f_1, \ldots, f_s) \)

c4. \( V[\alpha] x \in A \) defined as:
   
   \( V[\emptyset] = 0, \ V[c_i] = 0 \)

   \( V[\alpha \times (i)] = [V[c_i] V[c_j]] = V[c_i] V[c_j] - V[c_j] V[c_i] \)

The UFG condition: there exists \( \epsilon > 0 \) such that for all \( x \in A_1 \)

\[
V[\alpha] = \sum_{\beta \in A_1(\alpha)} \gamma_{\alpha \beta} V[\beta] \quad \gamma_{\alpha \beta} \in C_0^\infty(\mathbb{R}^d)
\]
Remarks:

- The UFG condition is weaker than the Uniform Hörmander Condition.
- Algebra generated by \( \{ \mathcal{V}_{x}^{2} \}_{x \in \mathbb{R}^{d}} \) is finite dimensional as a \( \mathbb{C}_{b}^{\infty} \( \mathbb{R}^{d} \) \)-module.

Theorem [Kusuoka & Stroock]

Under the UFG condition, for \( p = 1, \ldots, \prod_{i=1}^{d} x_{i} \),

\[
\| V_{x,1} \cdots V_{x,d} P_{t} \phi \|_{\infty} \leq C_{p} \frac{1}{t^{\frac{1}{2}(r-p)}} \| \phi \|_{p} \quad t \in (0,1+\varepsilon)
\]

where

\[
\| \phi \|_{p} = \sum_{i=1}^{p} \| \nabla_{i} \phi \|_{\infty} \quad \phi \in \mathbb{C}_{b}^{\infty}(\mathbb{R}^{d})
\]

\[
\| \nabla_{i} \phi \|_{\infty} = \max_{d_{1}, \ldots, d_{i}} \| \frac{\partial^{d_{i}} \phi}{\partial x_{d_{1}} \cdots \partial x_{d_{i}}} \|_{\infty}.
\]
M-Perfect Families

Stochastic Taylor expansion of $\Psi(X_t)$

$$\Psi(X_t) = \Psi(X) + \sum_{\sum_{l=0}^{t} \tau_l = r} \mathbb{V}_{\sum_{l=0}^{t} \tau_l} \Psi(X) \int \ldots \int dW_{s_1} \ldots dW_{s_r} + R_{m,t} \Psi(X)$$

Let $\xi(x) = \{\xi_t(x), x \in \mathbb{R}^d\}$ be a family of (explicitly solvable) processes such that

$$\lim_{y \to x} \xi_t(y) = \xi_t(x) \quad \text{P.a.s. for any } (t,x) \in [0,T] \times \mathbb{R}^d$$

Let $Q_t \Psi(x) = E[\Psi(\xi_t(x))] \in L^1([0,T])$, $Q_t \Psi \in C_b(\mathbb{R}^d)$ for $\Psi \in C_b(\mathbb{R}^d)$

**Definition** The family $\xi(x) = \{\xi_t(x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ is m-perfect for $\Psi$

$$\sup_{x \in \mathbb{R}^d} \|Q_t \Psi(x) - E[\Psi_m(x)]\| \leq C \left[ \sum_{l=m+1}^{\infty} \frac{1}{l^{1/2}} \|\Psi_l\|_{\mathbb{R}^d} + t \|D\Psi\|_{\infty} \right]$$
where \[ \| \mathbf{v}_i \| = \sum_{u=1}^{i} \sum_{\alpha_1, \ldots, \alpha_u \in A_i} \| v_{\alpha_1} \ldots v_{\alpha_u} \|_{\infty} \]

Remarks:
- \( Q_t Y \) will have the same truncation as \( P_t Y \)
- \( P_t Y \) will be approximated by \( Q_{h_{n_m}} \ldots Q_{h} \), \( h_i = z_i - z_{i-1} \)

Lemma: let \( \{ R_t(x) \} \) be an \( m \)-perfect family, then
\[ \| P_t(P_s Y) - Q_t(P_s Y) \|_{\infty} \leq C \| Y \|_p \sum_{s=m+1}^{m} \frac{t^{\frac{3}{2}}}{s^{\frac{3}{2}-\frac{3}{2}}} \quad Y \in C^p_b(\mathbb{R}^d) \]
The Global Error corresponding to an $m$-Perfect Family

Let $T, \delta > 0$, $\Pi_n = \{Z_j = (\frac{j}{n}) T\}^n_{j=0}$, $h = Z_j - Z_{j-1} \in C_1([5])$

Theorem [Crisan and Ghazel 2007] Let $\text{Error}^\delta \in \beta = \|P_T \gamma - Q_{\alpha_1\alpha_{n-1}}...Q_{\alpha_1}\|_{\infty}$

Then

i. For any $\gamma \in C_0^M(\mathbb{R}^d)$ and $\delta > \frac{m-1}{M}$

$$\text{Error}^\delta \in \beta \leq c(\delta, M, T) \frac{1}{h^{m-1}} \|\gamma\|_M$$

ii. For any $\gamma \in C_0^1(\mathbb{R}^d)$ and $\delta > m-1$

$$\text{Error}^\delta \in \beta \leq \bar{c}(\delta, M, T) \frac{1}{h^{m-1}} \|\gamma\|_1$$

provided $\sup_{x \in \mathbb{R}^d} |x_+(t) - x_1| \leq \delta \sqrt{T}$ for $t$ small.
Example: The Lyons-Victoir cubature method:

A cubature of order $m$ for a finite measure $\mu$ is a set of points $\{x_i\}_{i=1}^N \in \mathbb{R}^d$ and a set of non-negative numbers $\lambda_i \geq 1, i = 1, \ldots, N$ such that

$$\int_{\mathbb{R}^d} x^k \mu(dx) = \sum_{i=1}^N \chi_i x^k \quad \forall k \leq m \quad \left[ \mu \sim \sum_{i=1}^N \lambda_i \delta_{x_i} \right]$$

Hence for any smooth function $f: \mathbb{R}^d \to \mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} f(x) \mu(dx) = \sum_{i=1}^N \lambda_i f(x_i)$$

The cubature method of Lyons and Victoir realises the same idea on the Wiener space.

Theorem (Lyons & Victoir (2004)): There exist paths $\omega_1, \ldots, \omega_N$ with bounded variation and $\lambda_1, \lambda_2, \ldots, \lambda_N$ such that

$$\mathbb{E} \left[ \left| \sum_{i=1}^N \lambda_i \int_0^1 \cdots \int_0^1 dW_{s_{i,1}} \cdots dW_{s_{i,1}} \cdots dW_{s_{i,r}} \right|^2 \right]$$
Remarks

- A 3-perfect family is required to obtain an approximation of the same order as the Euler method.
- A 5-perfect family is required to obtain an approximation of the same order as a classical 2nd order approximation.
- In general, the rougher \( Y \) is, the slower the rate of convergence becomes.
- By varying \( Y \), we can obtain the optimal rate regardless of the smoothness of \( Y \) (it needs to be at least Lipschitz).
- To limit the exponential increase in the computational effort, one can apply one of 4 methods:
  - Naive Monte-Carlo method
  - The Tree Based Branching Algorithm of Csan and Lyons
  - The recombination scheme of Schmeiser, Soreff, and Teichmüller
  - The recombining algorithm of Lyons and Lüerzer
\[
\sum_{j=1}^{N} \chi_j \int_{s_0}^{s_1} \cdots \int_{s_{l-2}} d\omega_{s_{l-1}} \cdots d\omega_{s_1} d\omega_{s_0}
\]

for any multi-index \( \gamma = (\epsilon_1, \ldots, \epsilon_r) \) such that \( \|\gamma\| \leq m \).

The measure \( \mathcal{Q}_t^m = \sum_{i=1}^{N} \lambda_i d\omega_i \) on \( C([0,T], \mathbb{R}^d) \) is called a cubature measure of order \( m \). In particular,

\[
E[Y_t^m(x)] = \mathcal{Q}_t[X_t^m(x)]
\]

Choose \( X_{t,x}^{3,x} \)

\[
dX_{t,x}^{3,x}(t) = \nu_0(X_{t,x}^{3,x}(t))dt + \sum_{j=1}^{d} \nu_j(X_{t,x}^{3,x}(t))d\omega_t^j
\]

Then \( \overline{X}_t(x) = \sum_{j=1}^{N} \lambda_j \overline{X}_t^j(x) \) is an \( m \)-perfect family.

\[ \text{Cubature of order 3} \quad N = 2^d, \quad \lambda_j = \frac{1}{2^d}, \quad \omega_t^j = t(Z_{j_1}^d, \ldots, Z_{j_r}^d) \]

\[ Z_{j_1}^d, \ldots, Z_{j_r}^d \in [-1,1]^r \]