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Particle Approximations of Feynman-Kac representations

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A. F-K representations for solutions of linear PDEs
- classical approximations (Euler)
- Kusuoka-Lyons-Victoir cubature methods

B. F-K representations for solutions of linear SPDEs (Zakai)
- particle filters (Sequential MCMs)
- applications of the K-L-V cubature method

C F-K representations for solutions of non-linear PDEs
- Malliavin weights Monte-Carlo method
- Kusuoka-Lyons-Victoir cubature based method

Joint work with S. Ghazali, K. Manolarakis

References:

- D.C., S. Ghazali, On the convergence rates of a general class of weak approximations of SDEs, in Stochastic differentia equations: theory and applications, Eds P. Baxendale and S. Lototsky, 221-248, 2007.
- D.C., K. Manolarakis, N. Touzi, On the Monte-Carlo simulation of BSDEs: An improvement on the Malliavin weights, submitted 2008
- D.C., K Manolarakis, Numerical solution for a BSDE using the Cubature methods, preprint 2008
- D.C., J. Xiong An approximate McKean-Vlasov model for the stochastic filtering problem, Conference Oxford sur les méthods de Monte-Carlo sequentielles, 18-21 ESAIM Proc., 19 EDP Sci., Les Ulis, 2007
- D.C., J. Xiong Numerical solutions for a class of SPDEs over bounded domains. Conference Oxford sur les méthods de Monte-Carlo sequentielles 121-125, ESAIM Proc. 19, EDP Sci., Les Ulis, 2007

Equivalently : $L = \sum_{i=1}^{d} f_i \frac{\partial}{\partial x_i} + \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$

where $a_{ij} = \frac{1}{2} \sum_{k=1}^{d} V_k^i V_k^j$, $f_i = V_0^i + \sum_{k=1}^{d} \sum_{j=1}^{d} V_k^j \frac{\partial V_k^i}{\partial x_j}$

(3) $X^{s,x}(t) = x + \int_s^t f(X^{s,x}(u))du + \sum_{i=1}^{d} \int_s^t V_i(X^{s,x}(u)) dW_u^i \quad s \le t \le T$

- The stochastic integral in (2) is a Stratonovitch integral, whilst the one in (3) is Itô integral.
- In the following, wlog $s=0$

$$u(0,x) = E[\varphi(X^x(T))] = P_t \varphi(x),$$

where $\begin{cases} dX^x(t) = \sum_{i=0}^{d} V_i(X^x(t)) \circ dW_t^i \\ X^x(0) = x \end{cases}$ $\quad W_t^0 = t \quad \circ dW_t^0 = dt$

$t \in [0,T]$ $\quad (W^1, \dots W^d)$ - d-dimensional BM

Classical approximations

$0 = t_0 < t_1 < \dots < t_N = T$ equidistant partition of $[0,T]$ $\quad t_{n+1} - t_n = \delta = \frac{T}{N}$.

Euler approximation $\quad Y(t_n) = Y_n$

$$Y_{n+1} = Y_n + f(Y_n)\delta + \sum_{i=1}^{d} V_i(Y_n)\xi^{i,n} \quad \text{where } \xi^{i,n}$$

— mutually independent

— $|E[\xi^{i,n}]|, |E[(\xi^{i,n})^3]|, |E[(\xi^{i,n})^2] - \delta| \le K\delta^2$ $\quad (*)$

$Y_0 = x$

Theorem If (a_{ij}) is strictly elliptic, then there exists K independent of δ s.t.

$$|E[\varphi(X_T^x)] - E[\varphi(Y_T)]| \le K\delta$$

Remark $\quad \xi^{i,n} \sim N(0,\delta)$ satisfies $(*)$

$$\xi^{i,n} = \begin{cases} \sqrt{\delta} & \text{prob } 1/2 \\ -\sqrt{\delta} & \text{prob } -1/2 \end{cases} \quad \text{satisfies } (*)$$

2nd order approximation $(d=1)$

$$Y_{n+1} = Y_n + f(Y_n)\delta + V(Y_n)\xi^n + \frac{1}{2}V(Y_n)V'(Y_n)\left((\xi^n)^2 - \delta\right)$$

$$+ \frac{1}{2}\left(f(Y_n)f'(Y_n) + \frac{1}{2}f''(Y_n)(V(Y_n))^2\right)\delta^2$$

$$+ \left(f'(Y_n)V(Y_n) + f(Y_n)V'(Y_n) + \frac{1}{2}V''(Y_n)V(Y_n)\right)\xi^n \cdot \delta$$

where ξ^n mutually independent

$$(**) \quad |E[\xi^n]| + |E[(\xi^n)^3] + |E[(\xi^n)^2] - \delta| + |E[(\xi^n)^4] - 3\delta^2| \le K\delta^3$$

then

$$|E[\varphi(X_T^*)] - E[\varphi(Y_T)]| \le K\delta^2$$

Remark $\xi^n \sim N(0,\delta)$ satisfies $(**)$

$$\xi^n = \begin{cases} \sqrt{3\delta} & \text{prob } 1/6 \\ -\sqrt{3\delta} & \text{prob } 1/6 \\ 0 & \text{prob } 2/3 \end{cases}$$

Monte Carlo approximation of $E[\varphi(Y_T)]$

Produce n independent realizations of Y : $Y^1, \dots, Y^n$, then

$$E\left[\left| E[\varphi(Y_T)] - \mu_n(\varphi)\right|\right] \leq \frac{C(\varphi)}{\sqrt{n}}$$

where

$$\mu_n(\varphi) = \frac{1}{n}\sum_{i=1}^{n} \varphi(Y_T^i) \quad , \quad \mu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{Y_T^i}$$

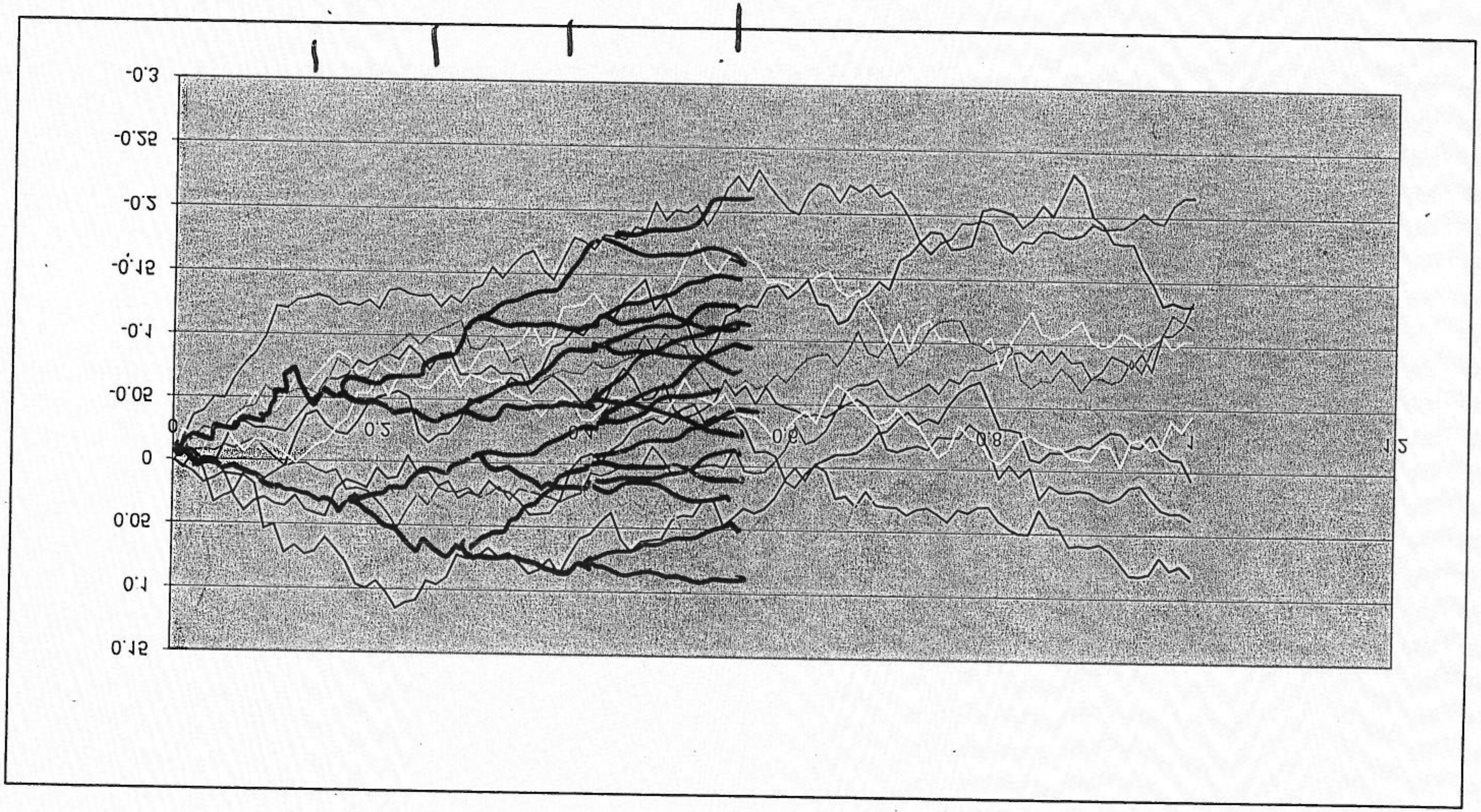
$$C(\varphi)^2 = Var(\varphi(Y_T)) = E[(\varphi(Y_T))^2] - (E[\varphi(Y_T)])^2$$

Remarks:

- μ_n is an approximation of $\mathcal{L}_{X_T}$ $\quad$ Error $= F(\delta, n)$

  $$| E[\varphi(X_T)] - \mu_n(\varphi)| = \text{Systematic Error} + \text{Statistical Error}$$

- $\bar{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{(Y^i_{t_0}, Y^i_{t_1}, \dots Y^i_{t_N})}$ is an approximation of $\mathcal{L}_{X_{[0,T]}}$



-0.3
-0.25
-0.2
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
0.2
0.4
0.6
0.8
1
1.2

Disadvantages:

- The methods require the ellipticity condition imposed on the matrix (a_{ij}) or the Hörmander condition imposed on $\{V_0, V_1, \dots V_d\}$ to obtain the optimal convergence rates.
- The approximating processes can leave the SDE's reachability surface, thus leading to instability problems and poor results.
- There is equal computational effort allocated to each interval of the partition: wasteful at the beginning, not enough towards the end.
- A "tree-like" approximation would be desirable.

A new class of schemes have been introduced to answer these problems:

Kusuoka [2001, 2004]

Kusuoka and Ninomiya [2004]

Lyons and Victoir [2004]

Ninomiya [2003]

Ninomiya and Victoir [2004]

Lyons and Litterer [2008]

Ninomiya and Ninomiya [2009]

Major improvements in computational effort:

- Ninomiya [2003] reports a speed-up ratio of the Kusuoka-Ninomiya algoritm over Euler of "about 6580"
- Ninomiya and Victoir [2004] report that their algorithm combined with the Romberg extrapolation is "about 800" times faster ~~that~~ Euler.

The UFG Condition [Kusuoka & Stroock]

i. $\mathcal{A} = \{\emptyset\} \cup \bigcup_{m=1}^{\infty} \{0,1,\dots d\}^m$, $\mathcal{A}_1 = \mathcal{A} \setminus \{\emptyset, (0)\}$, $\mathcal{A}_1(m) = \{\alpha \in \mathcal{A}_1 ; \|\alpha\| \le m\}$

ii. two norms $\quad |\emptyset| = 0 \quad |\alpha| = r$ if $\alpha = (i_1, \dots, i_r)$

$$\|\alpha\| = |\alpha| + \# \{ i \le j \le |\alpha| \quad i_j = 0 \}$$

iii. Concatenation on $\mathcal{A}$

$$(i_1, \dots, i_r) * (j_1, \dots, j_s) = (i_1, \dots, i_r, j_1, \dots j_s)$$

iv. $V_{[\alpha]}$ $\alpha \in \mathcal{A}$ defined as: $V_{[\emptyset]} = 0$, $V_{[(j)]} = V_j$

$$V_{[\alpha * (j)]} = [V_{[\alpha]}, V_j] = V_{[\alpha]} V_j - V_j V_{[\alpha]}$$

The UFG condition: there exists $\ell > 0$ such that for all $\alpha \in \mathcal{A}_1$

$$\boxed{V_{[\alpha]} = \sum_{\beta \in \mathcal{A}_1(\ell)} \varphi_{\alpha,\beta} V_{[\beta]}} \qquad \varphi_{\alpha,\beta} \in C_b^{\infty}(\mathbb{R}^d)$$

Remarks:

- The UFG condition is weaker than the Uniform Hörmander Condition
- Algebra generated by $\{V_{[\alpha]}\}_{\alpha \in \mathcal{A}}$ is finite dimensional as a $C_b^\infty(\mathbb{R}^d)$-module.

Theorem [Kusuoka & Stroock]

Under the UFG condition, for $p = 1, \dots, \overbrace{\|\alpha_1 * \dots * \alpha_r\|}^{r}$

$$\| V_{[\alpha_1]} \dots V_{[\alpha_r]} P_t \varphi \|_\infty \leq C_p^T \frac{1}{t^{\frac{1}{2}(r-p)}} \|\varphi\|_p \qquad t \in [0, T]$$

where

$$\|\varphi\|_p = \sum_{i=1}^{p} \|\nabla^i \varphi\|_\infty \qquad \varphi \in C_b^p(\mathbb{R}^d)$$

$$\|\nabla^i \varphi\|_\infty = \max_{j_1, \dots j_i \in \{1, \dots d\}} \left\| \frac{\partial^i \varphi}{\partial x_{j_1}, \dots \partial x_{j_i}} \right\|_\infty .$$

M-Perfect Families

Stochastic Taylor expansion of $\varphi(X_t)$

$$\varphi(X_t) = \underbrace{\varphi(x) + \sum_{\substack{\alpha \in \mathcal{A}_0(m) \\ \alpha=(i_1,\dots,i_r)}} V_\alpha \varphi(x) \int_0^t \int_0^{s_0} \dots \int_0^{s_{r-2}} \circ dW_{s_{r-1}}^{i_1} \dots \circ dW_{s_1}^{i_{r-1}} \circ dW_{s_0}^{i_r}}_{\varphi_t^m(x)} + R_{m,t,\varphi}(x)$$

Let $\bar{X}(x) = \{\bar{X}_t(x)\}_{t\in[0,T],\, x\in\mathbb{R}^d}$ be a family of (explicitly solvable) processes such that

$$\lim_{y\to x} \bar{X}_t(y) = \bar{X}_t(x) \quad \mathbb{P}\text{ a.s. for any } (t,x)\in[0,T]\times\mathbb{R}^d$$

$$\Rightarrow$$

Let $Q_t\varphi(x) = E[\varphi(\bar{X}_t(x))]$ $t\in[0,T]$, $Q_t\varphi \in C_b(\mathbb{R}^d)$ for $\varphi \in C_b(\mathbb{R}^d)$

<u>Definition</u> The family $\bar{X}(x) = \{\bar{X}_t(x)\}_{(t,x)\in[0,T]\times\mathbb{R}^d}$ is m-perfect for X if

$$\sup_{x\in\mathbb{R}^d} |Q_t\varphi(x) - E[\varphi_t^m(x)]| \le C\left[\sum_{i=m+1}^{M} t^{i/2}\|\varphi\|_{V,i} + t^{m+1}\|\nabla\varphi\|_\infty\right]$$

where $\|\varphi\|_{V,i} = \sum_{v=1}^{i} \sum_{\substack{\alpha_1,\dots,\alpha_v \in \mathcal{A}_1 \\ \|\alpha_1 * \dots * \alpha_v\| = i}} \|V_{[\alpha_1]} \dots V_{[\alpha_v]} \varphi\|_\infty$

Remarks:

- $Q_t\varphi$ will have the same truncation as $P_t\varphi$
- $P_t\varphi$ will be approximated by $Q^m_{h_n} Q^m_{h_{n-1}} \dots Q^m_{h_1}$ $h_i = \tau_i - \tau_{i-1}$.

<u>Lemma</u>: Let $\{\bar{X}_t(x)\}$ be an m-perfect family. then

$$\|P_t(P_s\varphi) - Q_t(P_s\varphi)\|_\infty \le c\|\varphi\|_p \sum_{j=m+1}^{m} \frac{t^{j/2}}{s^{j-p/2}} \qquad \varphi \in C_b^p(\mathbb{R}^d)$$

The Global Error corresponding to an m-Perfect Family

Let $T, \gamma > 0$, $\Pi_n = \left\{ z_j = \left(\frac{j}{n}\right)^{\gamma} T \right\}_{j=0}^{n}$ $h_j = z_j - z_{j-1} \in [0,1]$

Theorem [Crisan & Ghazali 2007] Let $\text{Error}^{\gamma,n}(\varphi) = \| P_t \varphi - Q_{h_n} Q_{h_{n-1}} \cdots Q_{h_1} \varphi \|_{\infty}$

Then

i. For any $\varphi \in C_b^M(\mathbb{R}^d)$ and $\gamma > \frac{m-1}{M}$

$$\boxed{\text{Error}^{\gamma,n}(\varphi) \leq c(\gamma, M, t) \frac{1}{n^{\frac{m-1}{2}}} \|\varphi\|_M}$$

ii. For any $\varphi \in C_b^1(\mathbb{R}^d)$ and $\gamma > m-1$

$$\boxed{\text{Error}^{\gamma,n}(\varphi) \leq \bar{c}(\gamma, M, t) \frac{1}{n^{\frac{m-1}{2}}} \|\varphi\|_1}$$

provided $\sup_{x \in \mathbb{R}^d} |\bar{X}_t(x) - x| \leq c\sqrt{t}$ for t small.

Example : The Lyons-Victoir cubature method:

A cubature of order m for a finite measure μ is a set of points $\{x_i\}_{i=1}^{N} \in \mathbb{R}^d$ and a set of non-negative numbers λ_i, $i=1,\dots,N$ such that

$$\int_{\mathbb{R}^d} x^k \mu(dx) = \sum_{i=1}^{N} \lambda_i x_i^k \quad \forall k \le m \quad \left[\mu \sim \sum_{i=1}^{N} \lambda_i \delta_{\{x_i\}}\right]$$

Hence for any smooth function $f: \mathbb{R}^d \to \mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} f(x)\, \mu(dx) \simeq \sum_{i=1}^{N} \lambda_i f(x_i)$$

the cubature method of Lyons and Victoir realises the same idea on the Wiener space.

<u>Theorem</u> (Lyons & Victoir (2004)) : there exists paths $\omega_1, \dots \omega_N$ with bounded variation and $\lambda_1, \lambda_2, \dots \lambda_N$ such that

$$E\left[\int_0^t \int_0^{s_0} \dots \int_0^{s_{r-2}} \circ\, dW_{s_{r-1}}^{i_1} \circ \dots \circ dW_{s_1}^{i_{r-1}} \circ dW_{s_0}^{i_r}\right]$$

Remarks

- A 3-perfect family is required to obtain an approximation of the same order as the Euler method.
- A 5-perfect family is required to obtain an approximation of the same order as a classical 2nd order approximation
- In general, the rougher φ is, the slower the rate of convergences becomes.
- By varying γ, we can obtain the optimal rate regardless of the smoothness of φ (it needs to be at least Lipschitz)
- to limit the exponential-increase in the computational effort one can apply one of 4 methods
  - naive Monte-Carlo method
  - the Tree Based Branching Algorithm of Crisan and Lyons
  - the recombination scheme of Schmeiser, Soreff and Teichmann
  - the recombining algorithm of Lyons and Litterer

$$= \sum_{j=1}^{N} \lambda_j \int_0^t \int_0^{s_0} \cdots \int_0^{s_{r-2}} \circ dw^{i_1}_{s_{r-1}} \cdots dw^{i_{r-1}}_{s_1} dw^{i_r}_{s_0}$$

for any multi-index $\gamma = (i_1, \dots i_r)$ such that $\|\gamma\| \le m$.

The measure $\mathbb{Q}^m_t = \sum_{i=1}^{N} \lambda_i \delta_{\omega_i}$ on $C([0,T], \mathbb{R}^d)$ is called a cubature measure of order m. In particular

$$E[\varphi^m_t(x)] = \mathbb{Q}^m_t[\varphi^m_t(x)]$$

Choose $X^{j,x}$ such that

$$dX^{j,x}(t) = V_0(X^{j,x}(t))dt + \sum_{j=1}^{d} V_j(X^{j,x}(t))\, d\omega^j_t .$$

then $\bar{X}_t(x) = \sum_{j=1}^{N} \lambda_i \bar{X}^{j,x}_t$ is an m-perfect family.

<u>Cubature of order 3</u> $N = 2^d$, $\lambda_j = \frac{1}{2^d}$ $\omega^j_t = t(z^1_j, \dots z^d_j)$

$$z^1_j, \dots z^d_j \in \{-1, 1\}$$