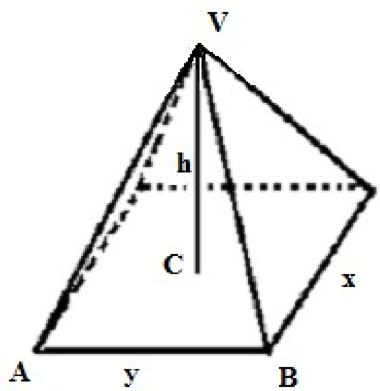
Math 241



Given a pyramid with a fixed volume, how do you minimize the lateral surface area? Assume the pyramid has a rectangular base and a vertex centered over the base.
Let the base be x by y, the altitude h and the volume = V.
The altitude of Δ ABV is: $\sqrt{h^2 + x^2/4}$ and its area is $\frac{y}{2}\sqrt{h^2 + x^2/4}$

The problem is: minimize $S = x\sqrt{h^2 + y^2/4} + y\sqrt{h^2 + x^2/4}$ subject to G(x, y, h) = xyh = 3V.

By the Lagrange multipliers, the minimum solves ∇S =λ ∇G.
The three equations are:

(1) $\sqrt{h^2 + y^2/4} + \dfrac{xy}{4\sqrt{h^2 + x^2/4}} = \lambda yh$ (2) $\sqrt{h^2 + x^2/4} + \dfrac{xy}{4\sqrt{h^2 + y^2/4}} = \lambda xh$

(3) $\dfrac{xh}{\sqrt{h^2 + y^2/4}} + \dfrac{yh}{\sqrt{h^2 + x^2/4}} = \lambda xy$ Note that (3) says $\lambda > 0$.

To clean this up write $Q(x) = \sqrt{h^2 + x^2/4}$ and clear denominators. The equations become:

(1') $4Q(x)Q(y) + xy = 4\lambda yhQ(x)$ (2') $4Q(x)Q(y) + xy = 4\lambda xhQ(y)$

(3') $xhQ(x) + yhQ(y) = \lambda xyQ(x)Q(y)$

The symmetry of (1') and (2') means x is probably y. To show this, subtract (1') from (2'). Then $0 = 4\lambda h[yQ(x) - xQ(y)]$ Since λ and h are positive, the term in brackets is zero and may be written as

$\dfrac{Q(x)}{x} = \dfrac{Q(y)}{y}$ or $\sqrt{\dfrac{h^2}{x^2} + \dfrac{1}{4}} = \sqrt{\dfrac{h^2}{y^2} + \dfrac{1}{4}}$ Thus x = y.

Next set x = y in eqs (1') and (3'); (1') becomes
(4) $4Q(x)^2 + x^2 = 4h\lambda xQ(x)$ and (3') becomes $2hxQ(x) = \lambda x^2 Q(x)^2$ or $2h = \lambda xQ(x)$
Substitute for 2h for $\lambda xQ(x)$ in (4); then $4Q(x)^2 + x^2 = 8h^2$ which expands to
$4\left(h^2 + x^2/4\right) + x^2 = 8h^2$ Solving for h yields $h = x/\sqrt{2}$ and $x = \left(3\sqrt{2}V\right)^{1/3}$

For minimum surface area the base is square and the height (CV) is equal to (BC) which is half the length of the diagonal of the base. Thus Δ ACV is an isoceles right triangle.