Free groups and roses

1 Free groups

Consider the $2n$ symbols $x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}$. Collect these into a set $G$. There is a map from $G$ to itself which interchanges $x_i$ and $x_i^{-1}$. This map is denoted by the usual inversion symbol $g \mapsto g^{-1}$. Thus $x_i^{-1}$ is unambiguous and $(x_i^{-1})^{-1} = x_i$. The empty word will be denoted by 1.

A word is defined to be a finite ordered list of elements of $G$, depicted as exemplified by $x_1 x_2^{-1} x_1 x_1^{-1}$. Two words are said to be equivalent if they are related by a finite sequence of operations $wgg^{-1}v \mapsto vw$ or the reverse. Here $w$ and $v$ represent words in and $g \in G$.

Thus the word $x_1 x_2^{-1} x_1 x_1^{-1}$ is equivalent to $x_1 x_2^{-1}$.

The free group on $n$ generators, $F_n$, is defined as a set to be the set of equivalence classes of words. The group operation is defined by $v \cdot w = vw$. That is, placing the two words side-by-side. The inverse operation is defined by $(g_1 \cdots g_k)^{-1} = g_k^{-1} \cdots g_1^{-1}$.

Proposition 1 $F_n$ satisfies the axioms for being a group.

The proof is elementary.

Proposition 2 Every element of $F_n$ has a unique representative where no reductions $gg^{-1} \mapsto 1$ are possible. (This representative is said to be reduced.)

[Proof] The following argument is taken from Magnus-Karass-Solitar p.34. One can always reduce a word by repeatedly performing cancellations. Since the length of the word goes down by 2 after a cancellation, the process must terminate. However, we will define a specific process for reducing a word which will make proving uniqueness easier.

Let $\rho$ be defined on words by

$$
\rho(1) = 1 \\
\rho(x_i^{\pm 1}) = x_i^{\pm 1} \\
\rho(w) = x_{\mu_1}^{\epsilon_1} \cdots x_{\mu_q}^{\epsilon_q} \quad (\epsilon_i = \pm 1) \Rightarrow \rho(wx_{\mu}^{\epsilon}) = \begin{cases} 
  x_{\mu_1}^{\epsilon_1} \cdots x_{\mu_q}^{\epsilon_q} x_{\mu}^{\epsilon} & \mu \neq \mu \text{ or } \epsilon_q \neq -\epsilon \\
  x_{\mu_1}^{\epsilon_1} \cdots x_{\mu_{q-1}}^{\epsilon_{q-1}} & \mu_q = \mu \text{ and } \epsilon_q = -\epsilon
\end{cases}
$$

The function $\rho$ has the following properties:

1. $\rho(w)$ is reduced.
2. $\rho(w)$ is equivalent to $w$. 

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3. If \( v \) is a reduced word, then \( \rho(v) = v \).

4. \( \rho(w_1w_2) = \rho(\rho(w_1)w_2) \).

5. \( \rho(wx_\mu^\epsilon x_\mu^{-\epsilon}) = \rho(w) \) for \( \epsilon = \pm 1 \).

6. \( \rho(w_1x_\mu^\epsilon x_\mu^{-\epsilon}w_2) = \rho(w_1w_2) \) for \( \epsilon = \pm 1 \).

Properties 1, 2, 5 can be proved easily by induction on the length of \( w \). Property 3 uses induction on the length of \( v \). Property 4 follows from induction on the length of \( w_2 \), whereas Property 6 follows from 4 and 5.

Now suppose that two words \( u \) and \( v \) are equivalent. That means they are related by a finite sequence of insertions and deletions of canceling pairs from \( G \). But then property 6 implies that \( \rho \) is unchanged under each such modification. Hence \( \rho(u) = \rho(v) \).

Now we can show uniqueness of the reduced word. Suppose that \( w \) is equivalent to \( u \) and \( v \) where \( u \) and \( v \) are both reduced. Then \( \rho(w) = \rho(u) = \rho(v) \) by the preceding paragraph. By Property 3, \( \rho(u) = u \) and \( \rho(v) = v \). Hence \( u = v \).

Corollary 3 \( F_n \) is not abelian for \( n \geq 2 \).

Proof

The two reduced words \( x_1x_2 \) are \( x_2x_1 \) are distinct and hence inequivalent by the preceding proposition. Notice that \( F_1 \cong \mathbb{Z} \).

2 Roses

Let \( R_n \) be the topological space formed by identifying \( n \) circles along a point. More formally let \( b_0 \in S^1 \). Define an equivalence relation on \( S^1 \times \{1, \ldots, n\} \) by \( x \sim x' \forall x \) and \( b_0 \times i \sim b_0 \times j \) for every \( 1 \leq i, j \leq n \). Then \( R_n = (S^1 \times \{1, \ldots, n\})/\sim \). \( R_n \) is called the rose with \( n \) petals. Abusing notation, denote the equivalence class \([b_0 \times 1]\) by \( b_0 \). Sometimes we will wish to think of the \( n \) circles as being labeled by the symbols \( x_1, \ldots, x_n \).

3 The main theorem

The main theorem I wish to prove is the following:

Theorem 4 \( \pi_1(R_n, b_0) \cong F_n \).

To do this I want to define a certain covering space for \( R_n \).

4 The tree \( T_n \).

We construct \( T_n \) in stages, \( T_n^{(1)} \subset \cdots \subset T_n^{(i)} \subset T_n^{(i+1)} \subset \cdots \). At the same time we define maps \( p^{(i)} : T_n^{(i)} \to R_n \) such that \( p^{(i)}|_{T_n^{(i-1)}} = p^{(i-1)} \).
The space $T_n^{(1)}$ is formed by identifying $2n$ intervals along an endpoint of each. More formally $T_n^{(1)}$ is the quotient space of $I \times \{1, \ldots, 2n\}$ formed by setting $0 \times i \sim 0 \times j$ for all $i, j \leq 2n$. Let $e_0$ denote the equivalence class $[0 \times 1]$. The map $p^{(1)}$ is defined by $p(e_0) = b_0$. On each copy of $I$, $p$ is defined by wrapping the interval once around a copy of $S^1$ inside of $R_n$, either in the clockwise or counterclockwise direction. There are $n$ copies of $S^1$ and 2 directions one can wrap, each choice being realized by one of the $2n$ the $2n$ intervals of $T_n^{(1)}$. A picture for the case $n = 2$ is given below. We can think of the edges of $T_n^{(1)}$ being labeled by the symbols $x_i$ depending on which circle of $R_n$ the edge gets mapped to. These edges are also directed according to how it wraps around the circle. The arrow chosen to agree with the circle’s counterclockwise orientation.

To define $T_n^{(2)}$ we notice that $T_n^{(1)}$ has $2n$ “endpoints.” Let $T_n^{(2)}$ be the quotient space gotten by gluing together $2n + 1$ copies of $T_n^{(1)}$ in the following way. Start with one of the copies of $T_n^{(1)}$ which we think of as being the original $T_n^{(1)}$. Each of the other copies of $T_n^{(1)}$ will have an edge identified with one of the edges of the original $T_n^{(1)}$. The edge of the copy is chosen so that its label and direction match with the edge of $T_n^{(1)}$, and so that the copy of $e_0$ does not get identified with the original $e_0$.

Note that there is still a map $p^{(2)} : T_n^{(2)} \to R_2$ which is defined by “following the instructions” of the directions and labels. An edge labeled $x_i$ gets wrapped around the $x_i$ circle, in the direction indicated by the edges direction. Clearly $p^{(2)}|_{T_n^{(1)}} = p^{(1)}$. 

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Similarly we can define $T_n^{(3)}$. The space $T_n^{(2)}$ has $(2n)(2n - 1)$ endpoints, and so $T_n^{(3)}$ can be formed by gluing $(2n)(2n - 1)$ copies of $T_n^{(1)}$ to $T_n^{(2)}$ along an edge from each in a manner consistent with the edge directions and labels as we did for the previous stage. $p^{(3)}$ is again defined.

Let $T_n = \bigcup_{i=1}^{\infty} T_n^{(i)}$. Let $p: T_n \to R_n$ be defined by $p(x) = p^{(i)}(x)$ if $x \in T_n^{(i)}$. The topology on $T_n$ is defined so that a set $U$ is open if and only if the intersection of $U$ with every $T_n^{(k)}$ is open in $T_n^{(k)}$. Since $p^{(i)}$ is an extension of $p^{(i-1)}$ this is well-defined. Here is a picture of $T_2$, where labels and edge directions have been suppressed.

\begin{center}
\begin{tikzpicture}
\fill[gray!30] (0,0) circle (0.5cm);
\draw (0,0) circle (0.5cm);
\draw (0,0) circle (0.3cm);
\draw (0,0) circle (0.1cm);
\end{tikzpicture}
\end{center}

**Proposition 5** The map $p: T_n \to R_n$ is a covering map.

**[Proof]**
Consider a circle in $R_n$ and remove the point $b_0$. Call this resulting set $U$. Then the preimage $p^{-1}(U)$ consists of the interiors of all of the edges in $T_n$ which are labeled by the same label as the circle. It is intuitively clear that the restriction of $p$ to any one of these edges is a homeomorphism onto $U$. Moreover these edges are disjoint open subsets of $T_n$. Thus $U$ is evenly covered.
It remains to show that a neighborhood of the point \( b_0 \) is evenly covered. This is also intuitively clear. A small neighborhood of \( b_0 \) looks like \( 2n \) half open intervals \([0, \epsilon)\) coming together at there \( 0 \) endpoints. Looking at the preimage, we see the same picture around all of the vertices of \( T_n \), and the map \( p \) restricts to a homeomorphism near each vertex. (A vertex is where several edges meet.)

**Proposition 6** The space \( T_n \) is simply connected. (In fact it is contractible, but I omit that proof here!)

[Proof]
Consider the space \( T_n^{(1)} = (S^1 \times \{1, \ldots, 2n\})/\sim \), and fix a subset, \( J \), of the set \( \{1, \ldots, 2n\} \). Then there is a homotopy \( id_{T_n^{(1)}} \simeq f \), where \( f(T_n^{(1)}) \) is contained in \((S^1 \times (\{1, \ldots, 2n\} \setminus J))/\sim \) and where the homotopy fixes \((S^1 \times (\{1, \ldots, 2n\} \setminus J))/\sim \). The homotopy can be defined by

\[
H([s \times i], t) = \begin{cases} 
[s(1-t) \times i] & \text{if } i \in J \\
[s \times i] & \text{if } i \not\in J
\end{cases}
\]

By definition \( H \) is fixed away from \((S^1 \times J)/\equiv \), and also \( f(x) = H(x, 1) \) has image contained in \((S^1 \times (\{1, \ldots, 2n\} \setminus J))/\sim \) since \( f([s \times i]) = [0 \times i] = e_0 \) for \( i \in J \).

(In particular, this shows that \( T_n^{(1)} \) is contractible, by letting \( J = \{1, \ldots, 2n\} \).)

Next we show that \( T_n^{(k)} \) is contractible. We prove this by induction. Suppose that \( T_n^{(k-1)} \) is contractible. Let \( F: T_n^{(k-1)} \times I \to T_n^{(k-1)} \) be a homotopy from \( id_{T_n^{(k-1)}} \) to a constant map \( e_{e_0} \). Now \( T_n^{(k)} \) is formed by gluing several copies of \( T_n^{(1)} \) onto the edges adjacent to the endpoints of \( T_n^{(k-1)} \). In each of these copies of \( T_n^{(1)} \) let \( J \) index the subset of \( 2n - 1 \) edges that were not identified with an edge of \( T_n^{(k-1)} \). (Let these edges be called “new.”) Then we have a homotopy between the identity and a map \( f \) whose image is contained in the remaining edge. Let the homotopy on the ith copy of \( T_n^{(1)} \) be denoted \( H_i \). Since each of these homotopy fix everything away from the new edges they can be extended to a homotopy \( G \) on the entire space \( T_n^{(k)} \) defined by

\[
G(x, t) = \begin{cases} 
H(x, t) = H_i(x, t) & \text{if } x \in \text{ith copy of } T_n^{(1)} \\
x & \text{if } x \in T_n^{(k-1)}
\end{cases}
\]

Now we can define a homotopy, \( K \), of \( id_{T_n^{(k)}} \) with the constant map \( e_{e_0} \).

\[
K(x, t) = \begin{cases} 
G(x, 2t) & 0 \leq t \leq .5 \\
F(G(x, 1), 2t - 1) & .5 \leq t \leq 1
\end{cases}
\]

This is continuous by the gluing lemma, since

\[
F(G(x, 1), 2(.5) - 1) = F(G(x, 1), 0) = id_{T_n^{(k-1)}}(G(x, 1)) = G(x, 1) = G(x, 2(.5)).
\]

It is also well-defined since \( G(x, 1) \in T_n^{(k-1)} \), so that we are in the domain of \( F \).

Finally we show that \( T_n \) is simply connected. Suppose that \([\alpha] \in \pi_1(T_n, e_0) \). Since \( I \) is compact, this implies that \( \alpha(I) \) is compact. The sets \( \text{int}(T_n^{(k)}) \) form an open covering of \( \alpha(I) \) so by compactness \( \alpha(I) \) is contained in a finite union of them. Since the open cover is nested, \( \alpha(I) \subseteq T_n^{(k)} \) for some \( k \). Since this is a contractible space \( \alpha \simeq_p e_{e_0} \), implying \([\alpha] \) is trivial. \( \square \)
5 Identifying the vertices of $T_n$ with the free group $F_n$.

Recall $e_0$ is a fixed lift of $b_0$. We identify this with the element $1$ of the free group. Given any other vertex $v \in T_n$, consider a sequence of edges $E_1, \ldots, E_k$ where $E_1$ is an edge starting at $e_0$, $E_i$ is an edge starting where $E_{i-1}$ finishes, and $E_k$ finishes at $v$. Assume also that there is no backtracking: $E_i$ is never $E_{i-1}$ traversed in the opposite direction. Each edge has a label $x_{E_i}$. Consider the word $w_v = x_{E_1}^{e_1} \cdots x_{E_k}^{e_k}$ where $e_i$ is positive if you traverse the edge with the same orientation, and negative if you traverse the edge against its orientation.

Then the vertex $v$ is assigned the word $w_v$.

**Proposition 7** The assignment $v \mapsto w_v$ is a well-defined bijection.

**[Proof]**
To show it is well-defined, it suffices to show that there is a unique chain of edges connecting $e_0$ with $v$ where backtracking is not allowed. If we delete $T_n^{(1)}$ from $T_n$, we are left with $2n$ components. Suppose $v$ lies in the $i$th such component. Then $E_1$ must be the edge that leads into this component, because otherwise the chain would have to go through $e_0$ again when it leaves the component it first entered, in order to get back to the component containing $v$. In particular, the chain would have to “turn around” at some point, traversing an edge and then traversing it backward on its way back to $e_0$.

Thus $E_1$ is uniquely determined. Repeating the argument at the vertex at the end of $E_1$, we see that $E_2$ is uniquely determined, etc.

To show this assignment is bijective we define a map in the other direction.

Given any reduced word $g_1 \cdots g_k$ Consider the chain where $E_1$ is the unique edge emanating from $e_0$ with label $g_1$. (Here a directed edge is “labeled $x_i$” if it is labeled $x_i$ and the edge direction is consistent with the direction of travel. It is labeled $x_i^{-1}$ if it is traversed in the order inconsistent with its direction.) $E_2$ is the unique edge emanating from the terminal point of $E_1$ labeled $g_2$ etc. The fact that the word is reduced ensures no backtracking. Then the word $g_1 \cdots g_k$ is assigned the vertex at the end of this chain of edges, denoted $v(g_1 \cdots g_k)$.

Now $w_{v(g_1 \cdots g_k)} = g_1 \cdots g_k$ and $v(w_v) = v$ are easily established and show that the assignment $v \mapsto w_v$ is a bijection. \hfill $\Box$

6 Some homeomorphisms of $T_n$.

Given any $g \in G$, I wish to define a homeomorphism

$$\phi_g : T_n \to T_n.$$ 

To avoid getting bogged down in technical details, I will just describe it using the following picture of $\phi_{x_1}$ operating on $T_2$. 


The map \( \phi x_1 \) moves everything one unit to the right. Thus it moves the vertex \( e_0 \) to the adjacent vertex by moving along the \( x_1 \) edge directed away from \( e_0 \). The two trees corresponding to \( x_2 \) get moved as well so that they now grow out of this adjacent vertex. Similarly all of the other pairs of trees growing from each point along the horizontal line get moved to the right.

The homeomorphism \( \phi x_{-1} \) is defined to be \( \phi x_1^{-1} \), and moves everything one to the left instead.

The case of \( T_n \) is handled similarly, the difference being that there are \( n - 1 \) pairs of trees sprouting out of each vertex on the horizontal line, and these all get moved by the homeomorphism.

Given a word \( g_1 \ldots g_k \) representing an element, \( w \), of \( F_n \), define \( \phi w \circ T_n \rightarrow T_n \) by the equation \( \phi w = \phi g_1 \circ \ldots \circ \phi g_k \). This is well defined because \( \phi g_1^{-1} \circ \phi g = id_T \) for any \( g \), so that it is independent of the particular word chosen to represent \( w \).

Note that these homeomorphisms preserve edge directions and labels.

**Proposition 8** \( \phi w(e_0) \) *is the vertex which is assigned to* \( w \in F_n \).

**Proof**

Let us proceed by induction on the length of \( w \). If the length of \( w \) is 1, then this is clear. Suppose now that \( \phi g_{2 \ldots g_k}(e_0) \) is the vertex corresponding to \( g_2 \ldots g_k \), which we had denoted \( v(g_2 \ldots g_k) \). Then we must show that \( \phi g_1(v(g_2 \ldots g_k)) = v(g_1 \ldots g_k) \). To see this note that by definition, there is a chain of edges from \( e_0 \) to \( v(g_2 \ldots g_k) \) traversing the labels \( g_2, \ldots, g_k \). Now that chain of edges gets moved by the homeomorphism \( \phi g_1 \) to a chain of edges starting at the \( g_1 \) vertex and terminating at \( \phi g_1 = \phi g_1(v(g_2 \ldots g_k)) \), which by the inductive hypothesis is equal to \( \phi g_{1 \ldots g_k}(e_0) \). Adding the edge between \( e_0 \) and the \( g_1 \) vertex to the beginning of the chain of edges, we get a chain of edges from \( e_0 \) to \( \phi g_{1 \ldots g_k}(e_0) \) which has the edge labels \( g_1, \ldots, g_k \), thus demonstrating that \( \phi g_{1 \ldots g_k}(e_0) = v(g_1 \ldots g_k) \) as desired. \( \square \)
7 The proof of the main theorem

Recall that the lifting correspondence is a map

\[ \phi: \pi_1(R_n, b_0) \to p^{-1}(b_0) \]

which is a bijection because \( T_n \) is simply connected. Now \( p^{-1}(b_0) \) is the set of vertices of \( T_n \), which we have identified with the free group \( F_n \). Thus the lifting correspondence can be viewed as a bijection \( \phi: \pi_1(R_n, b_0) \to F_n \). It suffices to check that this is a homomorphism. Let \([\alpha]\) and \([\beta]\) be two elements of \( \pi_1(R_n, b_0) \). Let \( \tilde{\alpha} \) be a lift of \( \alpha \) to \( T_n \) starting at \( e_0 \) and let \( \tilde{\beta} \) be a lift of \( \beta \) to \( T_n \) starting at \( e_0 \). Suppose \( \tilde{\alpha}(1) = a \in F_n \) and \( \tilde{\beta}(1) = b \in F_n \). (By definition, this means \( \phi([\alpha]) = a, \phi([\beta]) = b \).) Let \( \tilde{\beta} = \phi_a \circ \tilde{\beta} \). I claim that \( \tilde{\beta} \) is a lift of \( \beta \) beginning at the \( a \) vertex. This is because the homeomorphism \( \phi_a \) preserves labels, implying that \( p \circ \phi_a = p \). The fact that this lift begins at the \( a \) vertex follows since \( \phi_a(\tilde{\beta}(0)) = \phi_a(e_0) = a \). Thus \( \tilde{\alpha} \ast \tilde{\beta} \) is a lift of \( \alpha \ast \beta \) beginning at \( e_0 \). By definition of the lifting correspondence \( \phi([\alpha] \ast [\beta]) = (\tilde{\alpha} \ast \tilde{\beta})(1) = \tilde{\alpha} \ast \tilde{\beta}(1) \). But this is equal to \( \phi_a(b) \). To see that this is \( ab \), note that \( b = \phi_b(e_0) \) so that \( \phi_a(b) = \phi_a \circ \phi_b(e_0) \). Writing \( a = g_1 \ldots g_k \) and \( b = h_1 \ldots h_l \) in terms of generators, we have, by definition

\[
\begin{align*}
\phi_a \circ \phi_b(e_0) &= \phi_{g_1} \circ \cdots \circ \phi_{g_k} \circ \phi_{h_1} \circ \cdots \circ \phi_{h_l} \\
&= \phi_{ab}(e_0) \\
&= ab
\end{align*}
\]

Thus \( \phi([\alpha] \ast [\beta]) = ab = \phi([\alpha]) \cdot \phi([\beta]) \). \qed