Math 147: Review Sheet for Third Test

2.7 Implicit Differentiation
   (a) Derivatives of inverse trigonometric functions. [See notes.]

2.8 Related rates

2.9 Differentials and Approximations
   (a) Linear approximation

3.1 Maxima and Minima
   (a) Critical Point Theorem

3.2 Monotonicity and Concavity
   (a) Monotonicity Theorem
   (b) Concavity Theorem
   (c) Inflection Points

3.3 Local Extrema
   (a) First derivative test
   (b) Second derivative test

3.4 Practical Problems

3.5 Graphing using Calculus

3.6 Mean Value Theorem

3.7 Solving Equations Numerically
   (a) Bisection Method
   (b) Newton's Method
   (c) Fixed Point Algorithm

Practice Problems:

(1) Let $x^3 + y^3 = 6xy$.

(a) Find y'.
Solution:

$$\begin{aligned} 3x^2 + 3y^2\frac{dy}{dx} &= 6y + 6x\frac{dy}{dx} \\ \frac{dy}{dx}(3y^2 - 6x) &= 6y - 3x^2 \\ \frac{dy}{dx} &= \frac{6y - 3x^2}{3y^2 - 6x} \end{aligned}$$

□

(b) Find the tangent line to the curve at the point $(3, 3)$.
Solution:
The slope of the tangent line at $(3, 3)$ is $\frac{6 \cdot 3 - 3 \cdot 3^2}{3 \cdot 3^2 - 6 \cdot 3} = -1$. So, using point slope form $y - 3 = (-1)(x - 3)$, or $y = -x + 6$. □

(c) At what points on the curve is the tangent line horizontal? **Solution:**
The derivative is zero when $6y = 3x^2$. Since $x^3 + y^3 = 6xy$, this gives us two equations in two unknowns to solve. Plugging $y = \frac{1}{2}x^2$ into $x^3 + y^3 = 6xy$, we get

$$\begin{aligned} x^3 + \frac{x^6}{8} &= 3x^3 \\ x^3(-2 + \frac{x^3}{8}) &= 0 \end{aligned}$$

So $x = 0$ or $x = \sqrt[3]{16}$ are the two solutions. Since $y = \frac{1}{2}x^2$, the two solutions are

$$(0, 0), (\sqrt[3]{16}, \frac{1}{2}(\sqrt[3]{16})^2)$$

This last point can be rewritten slightly:

$$(0, 0), (2\sqrt[3]{2}, 2\sqrt[3]{4})$$

□

(2) Find y' if

$$\sin(x + y) = y^2 \cos x.$$

Solution:

$$y' = \frac{-y^2 \sin x}{\cos(x+y) - 2y\cos x}$$

□

(3) Calculate $D_x(\sin^{-1}(2x+1))$.
Solution:
In class, we showed that $D_x(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$. So, by the chain rule we have

$$D_x(\sin^{-1}(2x+1)) = \frac{1}{\sqrt{1-(2x+1)^2}} \cdot 2$$

□

(4) A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?
Solution:
.75 ft/s □

(5) A runner sprints around a circular track of radius 100 m at a constant speed of 7 m/s. The runner's friend is standing at a distance 200 m from the center of the track. How fast is the distance between the two friends changing when the distance between them is 200 m?
Solution:
Center the track at the origin, so it has equation $x^2 + y^2 = 100^2$. Put the friend on the positive x-axis at $(200, 0)$. That the runner's speed is 7 is written as follows:

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 7.$$

We are trying to calculate $\dfrac{ds}{dt}$, where $s = \sqrt{(x-200)^2 + y^2}$ is the distance between the runner and friend. This derivative is

$$\frac{ds}{dt} = \frac{2(x-200)\frac{dx}{dt} + 2y\frac{dy}{dt}}{\sqrt{(x-200)^2 + y^2}}$$

So it looks like we need to calculate $x, y, dx/dt$, and dy/dt when $s = 200$. We'll begin by solving for x and y:

$$\begin{aligned}
200 &= \sqrt{(x-200)^2 + y^2} \\
200^2 &= (x-200)^2 + y^2 \\
200^2 &= x^2 - 400x + 200^2 + y^2 \\
0 &= x^2 + y^2 - 400x \\
0 &= 100^2 - 400x \\
x &= 25
\end{aligned}$$

Then

$$\begin{aligned}
25^2 + y^2 &= 100^2 \\
y &= \pm 25\sqrt{15}
\end{aligned}$$

So the points where $s = 200$ are $(25, \pm 25\sqrt{15})$.

Now we tackle finding dx/dt and dy/dt. Taking the derivative of $x^2 + y^2 = 100^2$ gives the equation $2x(dx/dt) + 2y(dy/dt) = 0$, and we also know that $49 = (dx/dt)^2 + (dy/dt)^2$. Plugging in the given values of x and y, we have two equations in two unknowns:

$$\begin{aligned}
50(dx/dt) \pm 25\sqrt{15}(dy/dt) &= 0 \\
(dx/dt)^2 + (dy/dt)^2 &= 49
\end{aligned}$$

Solving these equations gives two solutions:

$$\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \pm(7\sqrt{15/19}, -14/\sqrt{19}).$$

Finally we plug back in to the expression for ds/dt derived above, using the fact that $2x(dx/dt) + 2y(dy/dt) = 0$, and that $\sqrt{(x-200)^2 + y^2} = 200$:

$$\begin{aligned}
\frac{ds}{dt} &= \frac{(-400)(\pm 7\sqrt{\frac{15}{19}})}{200} \\
&= \pm 14\sqrt{\frac{15}{19}} \approx \pm 12.4393
\end{aligned}$$

This is our final answer. The signs indicate that at one point the runner is heading away from the friend, while at the other point where $s = 200$, the runner is heading toward the friend. □

(6) Find the linear approximation of the function $f(x) = x^{3/4}$ at $a = 16$.
Solution:
$L(x) = 8 + \frac{3}{8}(x - 16)$ □

(7) If $y = \sqrt{\sin^3(x^2) - 2x}$, calculate dy.
Solution:

$$dy = \frac{6x\sin^2(x^2)\cos(x^2) - 2}{2\sqrt{\sin^3(x^2) - 2x}}\,dx$$

□

(8) Use differentials to estimate the amount of paint needed to apply a coat of paint .05 cm thick to a hemispherical dome with diameter 50 m.
Solution:
The volume of the paint on the hemispherical dome is the difference in volumes between a hemisphere of radius 50.05 and 50.The volume of a hemisphere of radius r is $V = \frac{1}{2}\frac{4}{3}\pi r^3 = \frac{2}{3}\pi r^3$. Then $dV = 2\pi r^2 dr$. In our case $dr = .05$ and $\Delta V \approx dV = 2\pi(50^2)(.05) \approx 785.398$. □

(9) Find the absolute maximum and minimum of the function $f(x) = \cos x + \sin x$ on $[0, \pi/3]$.
Solution:
The critical points occur when $f'(x) = -\sin x + \cos x = 0$, which is the same as $\tan x = 1$, with $0 \leq x \leq \pi/3$, so $x = \pi/4$ is the interior critical point. Now, testing the function at this point and the two boundary points we see:

$$\begin{aligned} f(0) &= 1 \\ f(\pi/4) &= \sqrt{2} \approx 1.41 \\ f(\pi/3) &= \sqrt{3}/2 + 1/2 \approx 1.36 \end{aligned}$$

So the minimum is 1 at $x = 0$ and the maximum is $\sqrt{2}$ at $x = \pi/4$. □

(10) Find the absolute maximum and minimum of the function $f(x) = x^4 - 2x^2 + 3$ on $[-2, 3]$.
Solution:
The minimum value is 2 at $x = \pm 1$, and the maximum value is 66 at $x = 3$. □

(11) Determine a and b so that $f(x) = a\sqrt{x} + b/\sqrt{x}$ has $(4, 13)$ as an inflection point.
Solution:
$f(4) = 13$ means that $13 = 2a + b/2$. $f''(4) = 0$ means that $-(1/4)(4)^{-3/2}a + (3/4)(4^{-5/2})b = 0$, or $-4a + 3b = 0$. Solving these two equations simultaneously yields $a = 39/8$ and $b = 13/2$. □

(12) What conditions on a, b, c will ensure that $f(x) = ax^3 + bx^2 + cx + d$ is always increasing?
Solution:
$f'(x) = 3ax^2 + 2bx + c > 0$. So $f'(x)$ must be an upward-opening parabola which does not cross the x-axis. That is $f'(x)$ has no real roots. In the quadratic formula, one sees the term $\sqrt{(2b)^2 - 4(3a)c}$, and to have no real roots $(2b)^2 - 4(3a)c < 0$. So $4b^2 - 12ac < 0$. Dividing by 4, this is $b^2 - 3ac < 0$. To ensure the parabola opens upward, $a > 0$. So the conditions are:

$$b^2 - 3ac < 0 \text{ and } a > 0$$

□

(13) Suppose that $f'(x) = -(x-1)(x-2)(x-3)(x-4)$. What values of x are local minima, and what values are local maxima?
Solution:
We need to analyze the sign of $f'(x)$ for various x values.

$(-\infty, 1)$	$f'(x) < 0$
$(1, 2)$	$f'(x) > 0$
$(2, 3)$	$f'(x) < 0$
$(3, 4)$	$f'(x) > 0$
$(4, +\infty)$	$f'(x) < 0$

So $x = 1$ and $x = 3$ are local minima, since the derivative starts negative and switches to positive, while $x = 2$ and $x = 4$ are local maxima. □

(14) Find two positive numbers whose product is 100 and whose sum is minimum.
$x = y = 10$ as done in class. □

(15) A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimize the amount of material used.
Solution:
With dimensions x, x, z the volume is $V = x^2z = 32,000$. The surface

area, which is the amount of material used, is $SA = x^2 + 4xz$. Substituting $z = 32,000/x^2$, we have $SA = x^2 + 4(32,000)/x$. $\lim_{x\to 0^+} SA(x) = \infty$ and $\lim_{x\to\infty} SA(x) = \infty$. So there must be a minimum somewhere in the range $(0, +\infty)$. Now $SA'(x) = 2x - 128,000x^{-2} = 0$ implies $x = 40$. Then $z = 20$. So the box has dimensions 40 x 40 x 20. □

(16) Find the area of the largest rectangle that can be inscribed in the ellipse $(x/a)^2 + (y/b)^2 = 1$.
Solution:
The upper left corner of the rectangle is in position (x, y) on the ellipse. The area is then $A = 4xy$. Use the equation of the ellipse to eliminate y. $A = 4x(b\sqrt{1-(x/a)^2})$. This is valid over the range $0 \leq x \leq a$. At the endpoints $A = 0$. So the maximum must occur somewhere in the middle. Now

$$A' = -\frac{4bx^2}{a^2\sqrt{1-x^2/a^2}} + 4b\sqrt{1-x^2/a^2}.$$

$A' = 0$ has the solution $x = a/\sqrt{2}$. Plugging this into our equation for A, we get the solution $A = 2ab$. □

(17) Sketch the graph, using the procedure suggested in section 3.5, of the function

$$f(x) = \frac{|x| + x}{2}(3x + 2).$$

(18) Sketch the graph of a function f that

(a) Has a continuous first derivative.

(b) is decreasing and concave up for $x < 3$.

(c) has an extremum at $(3, 1)$.

(d) is increasing and concave up for $3 < x < 5$.

(e) has an inflection point at $(5, 4)$.

(f) is increasing and concave down for $5 < x < 6$.

(g) has an extremum at $(6, 7)$.

(h) is decreasing and concave down for $x > 6$.

(19) Show that if f is a quadratic function $f(x) = \alpha x^2 + \beta x + \gamma$, $\alpha \neq 0$, then the number c in the mean value theorem is always the midpoint of the interval.

Solution:
The mean value theorem states that there is a c such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Writing this out

$$\begin{aligned}
2\alpha c + \beta &= \frac{\alpha b^2 + \beta b + \gamma - (\alpha a^2 + \alpha a + \gamma)}{b - a} \\
2\alpha c + \beta &= \frac{\alpha(b^2 - a^2) + \beta(b - a)}{b - a} \\
2\alpha c + \beta &= \frac{\alpha(b + a)(b - a) + \beta(b - a))}{b - a} \\
2\alpha c + \beta &= \alpha(b + a) + \beta \\
c &= (b + a)/2
\end{aligned}$$

This was what was to be shown. □

(20) Use Newton's method to find a formula to calculate $\sqrt{a}$, where $a > 0$ through successive approximations. In other words find a formula to calculate x_{n+1} in terms of x_n.

Solution:
The equation that we want to solve with Newton's Method is $x^2 - a^2 = 0$. $f(x) = x^2 - a^2$ and $f'(x) = 2x$. So

$$x_{n+1} = x_n - \frac{x_n^2 - a^2}{2x_n}$$

which simplifies slightly to

$$x_{n+1} = \frac{x_n}{2} - \frac{a^2}{2x_n}$$