

## Math 142: Review Sheet for Final

### True or False?

- (1)  $\sqrt{x^2 + 36x^4} = x + 6x^2$  **False**
- (2)  $\int xe^x dx = \frac{x^2}{2}e^x + C$  **False:** Integrals do not respect multiplication.
- (3) All functions that are antiderivatives must have derivatives. **True.**
- (4) If  $\int_a^b f(x) dx = 0$ , then  $f(x) = 0$  for all  $x$ . **False** It just means the area below the  $x$ -axis equals the area above.
- (5)  $\int_0^{2\pi} |\sin x| dx = 4 \int_0^{\pi/2} \sin x dx$  **True.**
- (6)  $\lim_{n \rightarrow \infty} \sum_{i=1}^n \sin\left(\frac{2i}{n}\right) \cdot \frac{2}{n} = \int_0^2 \sin x dx$  **True** This is the right endpoint formula  $R_n$ .
- (7) The midpoint rule is exact for linear functions. **True** Use the error formula, or argue geometrically that the part of the approximating rectangle above the line equals the part below.
- (8)  $\frac{d}{dx} \int_0^{x^2} \frac{1}{1+t^2} dt = \frac{1}{1+x^4}$ . **False:** The correct formula is  $\frac{2x}{1+x^4}$ . (In general  $\frac{d}{dx} \int_a^{u(x)} f(t) dt = f(u(x)) \frac{du}{dx}$ .)
- (9) If  $f'$  is continuous on  $[1, 3]$  then  $\int_1^3 f'(v) dv = f(3) - f(1)$ . **True:**  $f(v)$  is an antiderivative for  $f'(v)$ .
- (10)  $\int_1^\infty \frac{1}{x^\pi} dx$  is divergent. **False:**

$$\begin{aligned} \lim_{h \rightarrow \infty} \int_1^h x^{-\pi} dx &= \lim_{h \rightarrow \infty} \left[ \frac{x^{-\pi+1}}{-\pi+1} \right]_1^h \\ &= \lim_{h \rightarrow \infty} \frac{h^{-\pi+1}}{-\pi+1} - \frac{1}{-\pi+1} \\ &= \frac{1}{1-\pi} \end{aligned}$$

where the last line follows since  $-\pi + 1 < 0$ .

- (11) All continuous functions have derivatives. **False:** Consider  $f(x) = |x|$ .

- (12) All continuous functions have antiderivatives. **True:** But they may not have nice formulas.
- (13) The Midpoint rule is always more accurate than the Trapezoid rule. **False:** You can always cook up examples where one rule works better than the other two. For example, make a function which is linear except it has narrow spikes at the midpoints of the subdivided intervals. Then the approximating rectangles for the midpoint rule will rise up to the level of the spikes, and be a huge overestimate. The trapezoid rule will basically not count the spikes.
- (14) The area of a circle of radius  $a$  is  $4 \int_0^a \sqrt{a^2 - x^2} dx$ . **True**
- (15) If two solids with bases in the same plane have cross sections of the same area in all planes parallel to their bases, then they have the same volume. **True:** In both cases, one can calculate the integral as  $\int_a^b A(y) dy$  where  $A(y)$  is the area of a cross-section in the plane at height  $y$  above the base plane.
- (16) The area of the region bounded by  $y = \sqrt{x}$ ,  $y = 0$  and  $x = 9$  is  $\int_0^3 (9 - y^2) dy$ . **True.**
- (17) If  $\lim_{n \rightarrow \infty} a_n = 0$  then  $\sum_{n=0}^{\infty} a_n$  is convergent. **False:** The Harmonic Series  $\sum 1/n$  does not converge.
- (18) The series  $\sum_{n=1}^{\infty} n^{-\sin 1}$  is convergent. **False:**  $\sin 1 \approx .841471$ . According to the  $p$ -test, this diverges since  $.841471 \leq 1$ .
- (19) If  $\sum c_n 6^n$  is convergent, then  $\sum c_n (-2)^n$  is convergent. **True:** The power series  $\sum c_n x^n$  has radius of convergence at least as big as 6. So we definitely know it converges for all  $x$  satisfying  $-6 < x \leq 6$ . In particular for  $x = -2$ .
- (20) If  $\sum c_n 6^n$  is convergent, then  $\sum c_n (-6)^n$  is convergent. **False:**  $-6$  could be right on the edge of the interval of convergence, and so might not converge.
- (21) If  $\sum c_n x^n$  diverges when  $x = 6$ , then it diverges when  $x = 10$ . **True:** The radius of convergence is  $\leq 6$ , and so 10 definitely lies outside the interval of convergence.
- (22) The ratio test can be used to determine whether  $\sum 1/n^3$  converges. **False:**

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n \rightarrow \infty} \frac{n^3}{(n+1)^3} \\ &= 1 \end{aligned}$$

The ratio test gives no information when the limit is 1.

- (23) The ratio test can be used to determine whether  $\sum 1/n!$  converges. **True:**

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n+1} \\ &= 0\end{aligned}$$

Since  $0 < 1$ , the series converges.

- (24) If  $0 \leq a_n \leq b_n$  and  $\sum b_n$  diverges, then  $\sum a_n$  diverges. **False:**  $\sum a_n$  is smaller than a divergent series, so it could converge. Take  $a_n = 1/n^2$  and  $b_n = 1/n$  for a specific example.
- (25)  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}$ . **True:**  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ . Plug in  $x = -1$ .
- (26) If  $-1 < \alpha < 1$ , then  $\lim_{n \rightarrow \infty} \alpha^n = 0$ . **True.**
- (27) If  $\sum a_n$  is divergent, then  $\sum |a_n|$  is divergent. **True:** If  $\sum |a_n|$  were convergent, then the series  $\sum a_n$  is absolutely convergent and hence convergent.
- (28) If  $f(x) = 2x - x^2 + \frac{1}{3}x^3 - \dots$  converges for all  $x$ , then  $f'''(0) = 2$ . **True:**  $f'(x) = 2 - 2x + x^2 + \dots$ ,  $f''(x) = -2 + 2x + \dots$ ,  $f'''(x) = 2 + \dots$ . Plugging in  $x = 0$ , the higher order terms vanish, and we get  $f'''(0) = 2$ .
- (29) If  $\{a_n\}$  and  $\{b_n\}$  are divergent, then  $\{a_n + b_n\}$  is divergent. **False:** Let  $a_n = n, b_n = -n$ .
- (30) If  $\{a_n\}$  and  $\{b_n\}$  are divergent then  $\{a_n b_n\}$  is divergent. **False:** Let  $a_n = b_n = (-1)^n$ .
- (31) If  $\{a_n\}$  is decreasing and  $a_n > 0$  for all  $n$ , then  $\{a_n\}$  is convergent. **True:** See the monotone sequence test.
- (32) If  $a_n > 0$  and  $\sum a_n$  converges, then  $\sum (-1)^n a_n$  converges. **True:** The sequence  $\sum (-1)^n a_n$  converges absolutely.
- (33) If  $a_n > 0$  and  $\lim_{n \rightarrow \infty} (a_{n+1}/a_n) < 1$ , then  $\lim_{n \rightarrow \infty} a_n = 0$ . **True:** The first statement means that by the ratio test,  $\sum a_n$  converges. But then, by the Test for Divergence,  $a_n \rightarrow 0$ .

## Chapter 8:

- (1) You are stranded on a desert island without a calculator. All you have is as much sand as you can shake a stick at, and a stick. How would you calculate  $\sin(1)$  to 4 decimal places? **Answer:** Use the Taylor Series for  $\sin x$ , and the error formula to check how many terms you need.
- (2) Find the limit, or show it does not exist, for  $a_n = \frac{9n}{\sqrt{9n^2+1}}$ . **Answer:** Divide top and bottom by  $n = \sqrt{n^2}$  to get  $a_n = \frac{9}{\sqrt{9+1/n^2}}$ . This limits to  $\frac{9}{\sqrt{9+0}} = 3$ .
- (3) Same question as above for  $a_n = \frac{n!}{3^n}$ . **Answer:** We suspect that it limits to  $\infty$  since  $n!$  grows a lot more quickly than  $3^n$ .

$$\frac{n!}{3^n} = \frac{n}{3} \frac{n-1}{3} \cdots \frac{4}{3} \frac{3}{3} \frac{2}{3}$$

So we are multiplying together numbers which are increasing without bound. So the limit is indeed  $\infty$ .

- (4) Calculate  $\sum_{k=1}^{\infty} \left( \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right)$ . **Answer:** This is a telescoping series.

$$\begin{aligned} s_n &= (1 - 1/\sqrt{2}) + (1/\sqrt{2} - 1/\sqrt{3}) + \cdots + (1/\sqrt{n} - 1/\sqrt{n+1}) \\ &= 1 - 1/\sqrt{n+1} \end{aligned}$$

So the limit of the partial sums  $s_n$  is equal to 1.

- (5) Indicate whether the given series converges or diverges, giving a valid reason for your conclusion.

(a)  $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$ . **Answer:** Use the limit comparison test with  $a_n = n/(1+n^2)$  and  $b_n = 1/n$ . Then  $\lim_{n \rightarrow \infty} a_n/b_n = 1$ . Since this is a nonzero, noninfinite number, the limit comparison test says that  $\sum a_n$  and  $\sum b_n$  either both converge or both diverge. Since  $\sum b_n$  is the Harmonic Series, which diverges, the original series must also diverge.

(b)  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n}}$  **Answer:** This converges by the alternating series test.

(c)  $\sum_{n=2}^{\infty} (1 - 1/n)^n$ . **Answer:** This diverges by the Test for Divergence, since  $\lim_{n \rightarrow \infty} (1 - 1/n)^n = e^{-1} \neq 0$ .

- (6) Find the interval of convergence for the series  $\sum_{n=0}^{\infty} \frac{n!(x+1)^n}{3^n}$ . **Answer:**  
Always use the ratio test to determine intervals of convergence.

$$\begin{aligned}\lim_{n \rightarrow \infty} |a_{n+1}|/|a_n| &= \lim_{n \rightarrow \infty} \left| \frac{3^{n+1}}{(n+1)!(x+1)^{n+1}} \frac{n!(x+1)^n}{3^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{3}{(n+1)|x+1|} = 0\end{aligned}$$

Since  $0 < 1$ , this converges for all  $x$ . The interval of convergence is  $(-\infty, \infty)$ .

- (7) Find a power series that represents  $\frac{1}{(1+x)^3}$  on the interval  $(-1, 1)$ . **Answer:**  
 $f'(x) = -3(1+x)^{-4}$ ,  $f''(x) = 3 \cdot 4(1+x)^{-5}$ , ... and in general  $f^{(n)}(x) = (-1)^n 3 \cdot 4 \cdot 5 \cdots (n+2)(1+x)^{-n-3}$ . So for the Maclaurin series

$$\begin{aligned}a_n &= \frac{f^{(n)}(0)}{n!} \\ &= \frac{(-1)^n 3 \cdot 4 \cdots (n+2)}{1 \cdot 2 \cdot 3 \cdot 4 \cdots n} \\ &= \frac{(-1)^n (n+1)(n+2)}{2}\end{aligned}$$

So the answer is  $\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)}{2} x^n$ . Use the ratio test to verify that radius of convergence is 1.

- (8) Find the Maclaurin series for  $\sin^2 x$ . What is the interval of convergence?  
**Answer Sketch:** Verify that  $f^{(n)}(0)$  is 0 if  $n$  is odd. On the other hand  $f^{(2k)}(0) = (-1)^{k+1} 2^{2k-1}$  for  $k \geq 1$ . So the final answer is

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1}}{(2k)!} x^{2k}$$

- (9) Use the simplest method you can think of to find the first three nonzero terms of the Maclaurin series for  $e^{-x} \sin x$ . **Answer:**

$$\begin{aligned}e^{-x} \sin x &= (1 - x + x^2/2 + \cdots)(x - x^3/6 + x^5/120 + \cdots) \\ &= x - x^2 + x^3/3 + \cdots\end{aligned}$$

## Chapter 5:

- (1) Calculate  $\int_0^1 (1-x)^{2009} dx$ . **Answer:** Let  $u = 1-x$ ,  $du = -dx$ . Then we get

$$\begin{aligned} \int_{1-0}^{1-1} u^{2009} (-du) &= - \int_1^0 u^{2009} du \\ &= -[u^{2010}/2010]_1^0 = 1/2010. \end{aligned}$$

- (2) Calculate  $\int_0^1 \frac{1}{x^2+1} dx$ . **Answer:**

$$\begin{aligned} \int_0^1 \frac{1}{x^2+1} dx &= [\tan^{-1} x]_0^1 \\ &= \tan^{-1} 1 - \tan^{-1} 0 = \pi/4 \end{aligned}$$

- (3) Calculate  $\int_0^1 \sin(3\pi t) dt$ . **Answer:**

$$\begin{aligned} \int_0^1 \sin(3\pi t) dt &= \left[ \frac{-1}{3\pi} \cos(3\pi t) \right]_0^1 \\ &= \frac{-1}{3\pi} \cos(3\pi) - \left( \frac{-1}{3\pi} \cos 0 \right) \\ &= \frac{2}{3\pi} \end{aligned}$$

- (4) Calculate  $\int_0^5 ye^{-.6y} dy$ . **Answer:** Integrate by parts

$u = y$	$dv = e^{-.6y} dy$
$du = dy$	$v = -\frac{1}{.6}e^{-.6y}$

$$\begin{aligned} \int ye^{-.6y} dy &= -\frac{1}{.6}ye^{-.6y} + \int \frac{1}{.6}e^{-.6y} dy \\ &= -\frac{5}{3}ye^{-(3/5)y} - \frac{25}{9}e^{-(3/5)y} \end{aligned}$$

Integrating from 0 to 5 yields

$$-\frac{5}{3}(5)e^{-3} - \frac{25}{9}e^{-3} - \left(-\frac{25}{9}\right) = -\frac{100}{9}e^{-3} + \frac{25}{9}$$

- (5) Calculate  $\int \frac{t^2}{t^2 + 4t - 5} dt$  **Answer:** Since the degree of the top is not less than the degree of the bottom, first we simplify with long division:

$$\begin{array}{r} X^2 + 4X - 5 \overline{) X^2} \\ \underline{-X^2 - 4X + 5} \\ -4X + 5 \end{array}$$

I replaced  $t$  with  $X$  since I have a computer program that typesets that automatically. So

$$\frac{t^2}{t^2 + 4t - 5} = 1 + \frac{-4t + 5}{t^2 + 4t - 5}$$

Use partial fractions to determine that

$$\frac{-4t + 5}{t^2 + 4t - 5} = \frac{(1/6)}{t - 1} + \frac{(-25/6)}{t + 5}$$

$$\int 1 + \frac{-4t + 5}{t^2 + 4t - 5} dt = t + \frac{1}{6} \ln |t - 1| - \frac{25}{6} \ln |t + 5| + C$$

- (6) Calculate  $\int x^p \ln x dx$  for any  $p \neq -1$ . **Answer:**  
Integrate by parts with

$u = \ln x$	$dv = x^p dx$
$du = x^{-1} dx$	$v = x^{p+1}/(p+1)$

So you get

$$\begin{aligned} (\ln x)x^{p+1}/(p+1) - \int x^{-1}(x^{p+1}/(p+1)) dx &= (\ln x)x^{p+1}/(p+1) - \int x^p/(p+1) dx \\ &= \frac{1}{p+1} x^{p+1} \ln x - \frac{x^{p+1}}{(p+1)^2} + C \end{aligned}$$

- (7) Use properties of integrals to verify that

$$0 \leq \int_0^1 x^4 \cos x dx \leq .2$$

**Answer:**

This could be integrated explicitly using integration by parts four times in

a row, but there's an easier way. Namely since  $0 \leq \cos(x) \leq 1$  for  $x$  in the range  $[0, 1]$ , we have

$$0 \leq x^4 \cos x \leq x^4$$

So

$$\int_0^1 0 \, dx \leq \int_0^1 x^4 \cos x \, dx \leq \int_0^1 x^4 \, dx$$

$$0 \leq \int_0^1 x^4 \cos x \, dx \leq 1/5$$

(8) Evaluate  $\int_1^\infty xe^{-x^2} \, dx$ . **Answer:** Let  $u = -x^2$ ,  $du = -2x \, dx$ .

$$\begin{aligned} \int_1^\infty xe^{-x^2} \, dx &= \lim_{h \rightarrow \infty} \int_1^h xe^{-x^2} \, dx \\ &= \lim_{h \rightarrow \infty} \int_{-1}^{-h^2} (-1/2)e^u \, du \\ &= \lim_{h \rightarrow \infty} [(-1/2)e^u]_{-1}^{-h^2} \\ &= \lim_{h \rightarrow \infty} (-1/2)e^{-h^2} - (-1/2)e^{-1} \\ &= \frac{1}{2e} \end{aligned}$$

(9) Evaluate  $\int_2^6 \frac{y}{\sqrt{y-2}} dy$ . **Answer:**

Make the substitution  $u = y - 2$ ,  $du = dy$ .

$$\begin{aligned} \int_2^6 \frac{y}{\sqrt{y-2}} dy &= \lim_{h \rightarrow 2^+} \int_h^6 \frac{y}{\sqrt{y-2}} dy \\ &= \lim_{h \rightarrow 2^+} \int_{h-2}^4 \frac{u+2}{\sqrt{u}} du \\ &= \lim_{h \rightarrow 2^+} \int_{h-2}^4 u^{1/2} + 2u^{-1/2} du \\ &= \lim_{h \rightarrow 2^+} \left[ \frac{2}{3} u^{3/2} + 4u^{1/2} \right]_{h-2}^4 \\ &= \lim_{h \rightarrow 2^+} \frac{2}{3} (4)^{3/2} + 4(4)^{1/2} - \frac{2}{3} (h-2)^{3/2} - 4(h-2)^{1/2} \\ &= 40/3 \end{aligned}$$

(10) If  $f$  is a continuous function such that

$$\int_0^x f(t) dt = xe^{2x} + \int_0^x e^{-t} f(t) dt$$

for all  $x$ , find an explicit formula for  $f(x)$ . **Answer:** Since we don't really know how to manipulate the integrals in the equation, we try taking the derivative to get rid of them. Taking the derivative of both sides we get

$$\begin{aligned} \frac{d}{dx} \int_0^x f(t) dt &= \frac{d}{dx} (xe^{2x}) + \frac{d}{dx} \int_0^x e^{-t} f(t) dt \\ f(x) &= e^{2x} + 2xe^{2x} + e^{-x} f(x) \\ f(x) - e^{-x} f(x) &= e^{2x} + 2xe^{2x} \\ f(x)(1 - e^{-x}) &= e^{2x}(1 + 2x) \\ f(x) &= e^{2x} \frac{1 + 2x}{1 - e^{-x}} \end{aligned}$$

## Chapter 6:

(1) Find the area of the region bounded by the parabola  $y = x^2$  and the line  $y = mx + b$ , for  $b \geq 0$ . **Answer:**

The first step is to find where the parabola and line intersect.  $x^2 = mx + b$   
Using the quadratic formula, we get the solutions  $x = \frac{1}{2}(m \pm \sqrt{m^2 + 4b})$ .

Then we integrate the upper function minus the lower function between these two points of intersection:

$$\begin{aligned} \int_{\frac{1}{2}(m-\sqrt{m^2+4b})}^{\frac{1}{2}(m+\sqrt{m^2+4b})} (mx+b) - x^2 dx &= \left[ m\frac{x^2}{2} + bx - \frac{x^3}{3} \right]_{x=\frac{1}{2}(m-\sqrt{m^2+4b})}^{x=\frac{1}{2}(m+\sqrt{m^2+4b})} \\ &= \frac{1}{6}(m^2+4b)^{3/2} \end{aligned}$$

This is a miraculous simplification!

- (2) The curve traced by the equations  $x = 2\theta - \sin\theta$ ,  $y = 2 - \cos\theta$  is called a *trochoid*. Find the area under one arch of the trochoid. ( $0 \leq \theta \leq 2\pi$ .)

**Answer:** Recall that the formula for the area is  $\int_0^{2\pi} y(\theta)x'(\theta) d\theta$ .

$$\begin{aligned} \int_0^{2\pi} (2 - \cos\theta)(2 - \cos\theta) d\theta &= \int_0^{2\pi} 4 - 4\cos\theta + \cos^2\theta d\theta \\ &= \left[ 4\theta - 4\sin\theta + \frac{\theta}{2} + \frac{1}{4}\sin(2\theta) \right]_0^{2\pi} \\ &= 9\pi \end{aligned}$$

Here you can use the identity  $\cos^2 x = \frac{1}{2}(\cos(2x) + 1)$  to integrate  $\cos^2 x$ .

- (3) Consider the infinite region bounded above by the curve  $y = 1/x$ , to the left by  $x = 1$ , and below by  $y = 0$ . Find the volume of the solid obtained by rotating this region around the  $x$ -axis. **Answer:** Do this like a standard volume of rotation, but with an infinite extent. The radius at  $x$  is  $1/x$ .

$$\begin{aligned} \int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx &= \lim_{h \rightarrow \infty} \int_1^h \pi x^{-2} dx \\ &= \lim_{h \rightarrow \infty} \left[ -\pi x^{-1} \right]_1^h \\ &= \pi \end{aligned}$$

- (4) Describe the solid whose volume is given by the integral  $\int_0^1 \pi[(2-x^2)^2 - (2-\sqrt{x})^2] dx$ . **Answer:** It is the solid obtained by rotating the region bounded by  $y = \sqrt{x}$  and  $y = x^2$  around the line  $y = 2$ .
- (5) The height of a monument is 20 meters. A horizontal cross-section at a distance  $x$  meters from the top is an equilateral triangle with side length  $x/4$  meters. Find the volume of the monument. **Answer:** The volume is

the integral of the cross sectional areas:

$$\int_0^{20} A(x) dx$$

where  $A(x)$  is the area of the triangle at distance  $x$  from the top. The area of an equilateral triangle of side length  $a$  is given by  $\frac{\sqrt{3}}{4}a^2$  (use trigonometry or Heron's formula.) So  $A(x) = \frac{\sqrt{3}}{4}(x/4)^2$ .

$$\begin{aligned} \int_0^{20} A(x) dx &= \int_0^{20} \frac{\sqrt{3}}{4} (x/4)^2 dx \\ &= \int_0^{20} \frac{\sqrt{3}}{64} x^2 dx \\ &= \left[ \frac{\sqrt{3}x^3}{192} \right]_0^{20} \\ &= \frac{\sqrt{3}(20)^3}{192} \\ &\approx 72.1688m^3 \end{aligned}$$

- (6) Find the length of the curve  $y = \int_1^x \sqrt{\sqrt{t}-1} dt$   $1 \leq x \leq 16$ . **Answer:** Recall the formula for the length:  $\int_a^b \sqrt{1 + (dy/dx)^2} dx$ . In our case, by the fundamental theorem of calculus  $\frac{dy}{dx} = \sqrt{\sqrt{x}-1}$ , so we get

$$\begin{aligned} \int_1^{16} \sqrt{1 + (dy/dx)^2} dx &= \int_1^{16} \sqrt{1 + (\sqrt{\sqrt{x}-1})^2} dx \\ &= \int_1^{16} \sqrt{1 + \sqrt{x}-1} dx \\ &= \int_1^{16} \sqrt{\sqrt{x}} dx \\ &= \int_1^{16} x^{1/4} dx \\ &= \left[ \frac{4}{5} x^{5/4} \right]_1^{16} \\ &= \frac{124}{5} \end{aligned}$$

- (7) A. Square, a character from the book/movie *Flatland* has a 2 dimensional aquarium, which sits in the plane with one side at  $x = -1$  and the other side at  $x = +1$ . During a cold snap, he put it outside and the water froze so that its height is described by  $y = 2 + \cos x$ . After taking it back inside, what level did the water melt to? (Assume that the area of Flatland water doesn't change when frozen.) **Answer:** This is the average height of the water, or the average value of  $2 + \cos x$  between  $-1$  and  $1$ .

$$\begin{aligned} f_{ave} &= \frac{1}{1 - (-1)} \int_{-1}^1 2 + \cos x \, dx \\ &= 2 + \sin 1 \\ &\approx 2.84147 \end{aligned}$$