Math 142: Review Sheet for Third Test

4.9 Antiderivatives
   (a) Find antiderivatives by guessing and checking.
   (b) Determine unknown constants by initial data.

5.1 Areas and distances [We did not talk about distance.]
   (a) Write area under a curve as a limit of areas of rectangles.
   (b) Estimate area using finitely many rectangles.

5.2 The Definite Integral
   (a) Similar to section 5.1, but now use the name “definite integral” instead of just “area under a curve.”
   (b) Comparison and other miscellaneous properties of integrals

5.3 Evaluating Definite Integrals
   (a) Evaluation theorem
   (b) Table of elementary integrals. (Memorize table on p.369)

5.4 The Fundamental Theorem of Calculus
   (a) What is $\frac{d}{dx}\int_a^x f(t)\,dt$?

5.5 The Substitution Rule

5.6 Integration by Parts

5.7 Additional Techniques of Integration
   (a) Trigonometric integrals
   (b) Trigonometric substitutions
   (c) Partial Fractions

Practice Problems:

(1) Find F such that $F''(x) = e^{2x} + 3\cos x$, and $F'(0) = 1$ and $F(0) = 2$.
Solution:

$$F'(x) = \frac{1}{2}e^{2x} + 3\sin x + C$$
$$F(x) = \frac{1}{4}e^{2x} - 3\cos x + Cx + D$$

Now $2 = F(0) = \frac{1}{4} - 3 + D$, so $D = \frac{19}{4}$. And $1 = F'(0) = \frac{1}{2} + C$, so $C = \frac{1}{2}$.
So the final answer is

$$F(x) = \frac{1}{4}e^{2x} - 3\cos x + \frac{1}{2}x + \frac{19}{4}$$

□

(2) Find a function f such that $f'(x) = x^3$ and the line $x + y = 0$ is tangent to the graph of f.
Solution:
Since $x + y = 0$ is tangent to f, their slopes are the same at the point where they meet. The slope of $x + y = 0$ is -1, so we must have the slope $f'(x) = -1$. So $x^3 = -1$ and $x = -1$. Hence the tangency between $x + y = 0$ and $f(x)$ must occur at $x = -1$. And since $y = -x$, y must be 1. Now, we have

$$f'(x) = x^3$$
$$f(x) = \frac{x^4}{4} + C$$

Since $y + x = 0$ meets $f(x)$ at $(-1, 1)$, the point $(-1, 1)$ must lie on the graph of $f(x)$:

$$1 = \frac{(-1)^4}{4} + C$$
$$C = \frac{3}{4}$$

So the final answer is

$$f(x) = \frac{x^4}{4} + \frac{3}{4}$$

□

(3) Calculate $\int_{-2}^{2} \sqrt{4-x^2}\,dx$ by interpreting the function geometrically.
Solution:
This is the area of a semicircle of radius 2. So the area is $\frac{1}{2}\pi(2)^2 = 2\pi$. $\square$

(4) Calculate $\int_1^4 3x\,dx$ by interpreting the function geometrically.
Solution:
This is the area of a trapezoid with base $b = 3$ and two heights $h_1 = 3$ and $h_2 = 12$. The formula for the area of a trapezoid is $\frac{h_1+h_2}{2}b = \frac{3+12}{2} * 3 = \frac{45}{2}$.
$\square$

(5) Write an expression for $\displaystyle\int_1^2 \frac{dx}{x^2}$ as a limit of Riemann sums using the left endpoint.
Solution:
As mentioned in class, the formulas for the left endpoint Riemann sum and right endpoint Riemann sum are as follows, for $\int_a^b f(x)\,dx$

$$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x$$
$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = (b-a)/n$ and $x_i = a + i\Delta x$. Then we have

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n.$$

So for this particular function, we have $f(x) = 1/x^2$, $\Delta x = 1/n$ and $x_i = 1 + \frac{i}{n}$. So, plugging in for L_n we have

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{(1+\frac{i}{n})^2}\frac{1}{n}$$

$\square$

(6) What integral does the expression $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} \sqrt{x_i+1}\,\Delta x$ represent, on the interval $[2,3]$?
Solution:

$$\int_2^3 \sqrt{x+1}\,dx$$

$\square$

(7) Estimate $\int_0^2 \ln(x+1)\,dx$ using R_n and L_n, for $n=4$.
Solution:
We have $\Delta x = .5$, $x_0 = 0, x_1 = .5, x_2 = 1, x_3 = 1.5, x_4 = 2$. Then

$$\begin{aligned} L_4 &= (\ln(1+0)+\ln(1+.5)+\ln(1+1)+\ln(1+1.5)).5 \\ &= 1.00745 \\ R_4 &= (\ln(1+.5)+\ln(1+1)+\ln(1+1.5)+\ln(1+2)).5 \\ &= 1.55676 \end{aligned}$$

The exact answer turns out to be $\ln(27)-2 \approx 1.29584$. □

(8) Calculate $\int_1^2 (8x^3+3x^2)\,dx$.
Solution:
37 □

(9) Calculate $\int_0^1 (1-x^9)\,dx$.
Solution:
$\frac{9}{10}$ □

(10) Calculate $\int_1^9 \frac{\sqrt{u}-2u^2}{u}\,du$.
Solution:
-76 □

(11) Calculate $\int_0^1 \frac{x}{x^2+1}\,dx$.
Solution:
Let $u = x^2+1$. $du = 2x\,dx$. So we get

$$\begin{aligned} \frac{1}{2}\int_{u=0^2+1}^{u=1^2+1} \frac{1}{u}\,du &= \left[\frac{1}{2}\ln|u|\right]_{u=1}^{u=2} \\ &= \frac{1}{2}\ln 2 - \frac{1}{2}\ln 1 \\ &= \frac{1}{2}\ln 2 \end{aligned}$$

□

(12) Calculate $\int_0^1 v^2\cos(v^3)\,dv$.
Solution:

Let $u = v^3$ be your substitution. Then the final answer works out to be $\frac{1}{3}\sin 1$. $\square$

(13) Calculate $\int_0^1 e^{\pi t}\,dt$.
Solution:
$\left[\frac{1}{\pi}e^{\pi t}\right]_{t=0}^{t=1} = \frac{1}{\pi}(e^{\pi} - 1)$. $\square$

(14) Calculate $\int \frac{x+2}{\sqrt{x^2+4x}}\,dx$.
Solution:
Let $u = x^2 + 4x$. Then $du = (2x + 4x)dx = 2(x+2)dx$. So we get

$$\begin{aligned} \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du &= \frac{1}{2}\frac{u^{1/2}}{1/2} + C \\ &= \sqrt{u} + C \\ &= \sqrt{x^2+4x} + C \end{aligned}$$

$\square$

(15) Calculate $\int_0^5 \frac{x}{x+10}\,dx$.
Solution:
Use long division to write $\frac{x}{x+10} = 1 - \frac{10}{x+10}$. The final answer is $5 - 5\ln\left(\frac{9}{4}\right)$.
$\square$

(16) Calculate $\int_0^{\pi/2} \frac{\cos\theta}{1+\sin\theta}\,d\theta$.
Solution:
Let $u = 1 + \sin\theta$ be your substitution. Final answer is $\ln 2$. $\square$

(17) Calculate $\int_1^4 x^{3/2}\ln x\,dx$.
Solution:
This is an integration by parts problem.

$$\begin{array}{cc} u = \ln x & dv = x^{\frac{3}{2}}\,dx \\ du = \frac{1}{x}\,dx & v = \frac{2}{5}x^{\frac{5}{2}} \end{array}$$

$$\int x^{3/2}\ln x\,dx = \frac{2}{5}x^{5/2}\ln x - \frac{2}{5}\int \frac{1}{x}x^{5/2}\,dx$$
$$= \frac{2}{5}x^{5/2}\ln x - \frac{2}{5}\int x^{3/2}\,dx$$
$$= \frac{2}{5}x^{5/2}\ln x - \frac{4}{25}x^{5/2}$$

Plugging in the endpoints we have

$$\int_1^4 x^{3/2}\ln x\,dx = \left[\frac{2}{5}x^{5/2}\ln x - \frac{4}{25}x^{5/2}\right]_1^4$$
$$= \frac{64}{5}\ln 4 - \frac{124}{5} \approx 12.7846$$

□

(18) Calculate $\int x^2\cos x\,dx$.
Solution:
Use integration by parts twice. The answer is

$$2x\cos x - 2\sin x + x^2\sin x + C.$$

□

(19) Calculate $\int \frac{dt}{t^2+6t+8}$.
Solution:
This is a partial fractions problem.

$$\frac{1}{t^2+6t+8} = \frac{1}{(t+2)(t+4)} \stackrel{?}{=} \frac{A}{t+2} + \frac{B}{t+4}$$

$$1 = A(t+4) + B(t+2)$$
$$1 = (A+B)t + (4A+2B)$$

So $A+B=0$ and $4A+2B=1$. Solving these two equations simultaneously, we get $A=1/2$, $B=-1/2$. So

$$\int \frac{dt}{t^2+6t+8} = \int \frac{1/2}{t+2} + \frac{-1/2}{t+4}\,dt$$
$$= \frac{1}{2}\ln|t+2| - \frac{1}{2}\ln|t+4| + C$$

This can be simplified to $\ln\sqrt{\left|\frac{t+2}{t+4}\right|} + C$ if you are so interested. □

(20) Calculate $\int e^{\sqrt[3]{x}}\,dx$.
Solution:
This one is tricky. First do a substitution $u = x^{1/3}$. Then $du = \frac{1}{3}x^{-2/3}\,dx$, so that $dx = 3x^{2/3}\,du = 3u^2\,du$. So the integral becomes

$$\int 3u^2 e^u\,du.$$

Now do integration by parts on this, twice, and get

$$\int 3u^2 e^u\,du = 3u^2e^u - 6ue^u + 6e^u + C$$

Plugging back in $x^{1/3}$ our final answer is

$$3x^{2/3}e^{x^{1/3}} - 6x^{1/3}e^{x^{1/3}} + 6e^{x^{1/3}} + C$$

□

(21) Calculate $\int \frac{\sec\theta\tan\theta}{1+\sec\theta}\,d\theta$.
Solution:
Let $u =! + \sec\theta$. Then $du = \sec\theta\tan\theta\,d\theta$, and we get

$$\int \frac{1}{u}\,du = \ln|u| + C = \ln|1+\sec\theta| + C.$$

□

(22) Calculate $\int \frac{1}{\sqrt{9-x^2}}\,dx$.
Solution:
Let $x = 3\sin\theta$. Then $dx = 3\cos\theta\,d\theta$. So we get

$$\begin{aligned}\int \frac{1}{\sqrt{9-x^2}}\,dx &= \int \frac{3\cos\theta}{\sqrt{9-9\sin^2\theta}}\,d\theta \\ &= \int \frac{3\cos\theta}{\sqrt{9(1-\sin^2\theta)}} \\ &= \int \frac{3\cos\theta}{3\cos\theta} \\ &= \int 1\,d\theta \\ &= \theta + C\end{aligned}$$

But $x = 3\sin\theta$ implies $\theta = \sin^{-1}(x/3)$, so the final answer is

$$\sin^{-1}(x/3) + C.$$

□

(23) Find the derivative $g'(x)$ if $g(x) = \int_1^{\cos x} \sqrt[3]{1-t^2}\,dt$.

Solution:

Set $u = \cos x$. Then

$$\begin{aligned} \frac{d}{dx}\int_1^u \sqrt[3]{1-t^2}\,dt &= \left(\frac{d}{du}\int_1^u \sqrt[3]{1-t^2}\,dt\right)\left(\frac{du}{dx}\right) \\ &= \sqrt[3]{1-u^2}(-\sin x) \\ &= -\sin x\sqrt[3]{1-\cos^2 x} \end{aligned}$$

□

(24) Evaluate $\int_0^1 \frac{d}{dx} e^{\arctan x}\,dx$.

Solution:

An antiderivative for $\frac{d}{dx}e^{\arctan x}$ is $e^{\arctan x}$. So this problem is easy:

$$e^{\arctan x}\Big]_0^1 = e^{\arctan 1} - e^{\arctan 0} = e^{\pi/4} - 1$$

□