1. Find the Maclaurin series for the function $x^2/(1+x)$. Does the series converge when $x = 2$? (Give a brief reason.)
   Answer:
   $$x^2 - x^3 + x^4 - x^5 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{n+2} = \sum_{n=2}^{\infty} (-1)^n x^n$$
   The series converges when $|x| < 1$, so does not converge when $x = 2$.

2. Obtain the Maclaurin series, through the term in x^6, for $\ln(\cos x)$ by substituting the series for $y = 1 - \cos x$ into the series for $\ln(1 - y)$.
   Answer:
   We have $\ln(1-y) = -\sum_{n=1}^{\infty} \frac{y^n}{n} = -y - y^2/2 - y^3/3 - \cdots$, and
   $$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - x^2/2 + x^4/24 - x^6/720 + \cdots$$
   So now
   $$\begin{aligned} y &= 1 - \cos x \\ &= 1 - (1 - x^2/2 + x^4/24 - x^6/720 + \cdots) \\ &= x^2/2 - x^4/24 + x^6/720 + \cdots \end{aligned}$$
   So now we can say
   $$\begin{aligned} \ln(\cos x) &= \ln(1 - (1 - \cos x)) = \ln(1 - y) \\ &= -y - y^2/2 - y^3/3 - \cdots \\ &= -(x^2/2 - x^4/24 + x^6/720 + \cdots) - (x^2/2 - x^4/24 + x^6/720 + \cdots)^2/2 \\ &\qquad - (x^2/2 - x^4/24 + x^6/720 + \cdots)^3/3 - \cdots \\ &= -(x^2/2 - x^4/24 + x^6/720 + \cdots) - (x^4/4 - x^6/48 - x^6/48 + \cdots)/2 - (x^6/8)/3 \\ &= -x^2/2 - x^4/12 - x^6/45 + \cdots \end{aligned}$$

3. Expand the function $f(x) = \sqrt{1+x^2}$ in powers of $(x-1)$.
   Answer: Note: for this problem, it is difficult to find a pattern, so just find the first few terms. Here is the answer up to degree 5:
   $$\sqrt{2} + \frac{1}{\sqrt{2}}(x-1) + \frac{1}{4\sqrt{2}}(x-1)^2 - \frac{1}{8\sqrt{2}}(x-1)^3 + \frac{3}{64\sqrt{2}}(x-1)^4 - \frac{1}{128\sqrt{2}}(x-1)^5 + \cdots$$

4. Expand the function $f(x) = 1/(1-x)$ in powers of $(x-2)$.
   Answer:
   $$\sum_{n=0}^{\infty} (-1)^{n+1} (x-2)^n = -1 + (x-2) - (x-2)^2 + (x-3)^2 + \cdots$$

5. Expand the function $f(x) = 1/(x+1)$ in powers of $(x-3)$.
Answer:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}}(x-3)^n = \frac{1}{4} - \frac{1}{16}(x-3) + \frac{1}{64}(x-3)^2 - \frac{1}{256}(x-3)^3 + \cdots$$

6. Expand $\cos x$ in powers of $x - \pi/3$.
Answer:
We use the formula

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

with $f(x) = \cos x$ and $a = \pi/3$. Taking derivatives we have:

$$f(x) = \cos x, f'(x) = -\sin x, f''(x) = -\cos x, f'''(x) = \sin x, f^{(4)}(x) = \cos x, \ldots$$

Plugging in $\pi/3$, we have

$$f(\pi/3) = 1/2, f'(\pi/3) = -\sqrt{3}/2, f''(\pi/3) = -1/2, f'''(\pi/3) = \sqrt{3}/2, f^{(4)}(\pi/2) = 1/2, \ldots$$

Since the formula is different in the even and odd cases, we break it up into two pieces:

$$\text{Even case: } f^{(2k)}(\pi/3) = (-1)^k/2$$

$$\text{Odd case: } f^{(2k+1)}(\pi/2) = (-1)^{k+1}\sqrt{3}/2$$

So now the final answer is:

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2(2k)!}\left(x - \frac{\pi}{3}\right)^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}\sqrt{3}}{2(2k+1)!}\left(x - \frac{\pi}{3}\right)^{2k+1}$$

7. Find the first three terms of the Taylor series for $f(x) = 1/x$ about the point π.
Answer:

$$\frac{1}{\pi} - \frac{1}{\pi^2}(x-\pi) + \frac{1}{\pi^3}(x-\pi)^2 + \frac{1}{\pi^4}(x-\pi)^3 + \cdots$$

8. Find the first four terms (up to x^3) of $f(x) = e^{e^x}$.
Answer:

$$e + ex + ex^2 + \frac{5}{6}ex^3 + \cdots$$

9. Use series to find $\lim_{x \to 0} \dfrac{\ln(1-x) - \sin x}{1 - \cos^2 x}$.
Answer:

$$\lim_{x\to 0}\frac{\ln(1-x)-\sin x}{1-\cos^2 x} = \lim_{x\to 0}\frac{(-x-x^2/2-x^3/3-\cdots)-(x-x^3/6+x^5/120-\cdots)}{1-(1-x^2/2+x^4/24-\cdots)^2}$$
$$= \lim_{x\to 0}\frac{-2x-x^2/2-x^3/6+\cdots}{x^2/2-x^4/24+\cdots}$$
$$= \lim_{x\to 0}\frac{x(-2-x/2-x^2/6+\cdots)}{x^2(1/2-x^2/24+\cdots)}$$
$$= \lim_{x\to 0}\frac{(-2-x/2-x^2/6+\cdots)}{x^(1/2-x^2/24+\cdots)}$$

This last limit, when you plug in $x = 0$ is of the form $-2/0$. Since the denominator blows up, but the numerator goes to a nonzero number, the limit blows up as well, and so does not exist.

10. Find $\lim_{x\to 0}[(\sin x)/x]^{1/x^2}$.

Answer:

As we saw in class this is the same as

$$e^{\lim_{x\to 0}\frac{1}{x^2}\ln\left(\frac{\sin x}{x}\right)},$$

so we calculate $\lim_{x\to 0}\frac{1}{x^2}\ln\left(\frac{\sin x}{x}\right)$ using series. First

$$(\sin x)/x = (x-x^3/6+x^5/120+\cdots)/x = 1-x^2/6+x^4/120+\cdots.$$

Now $\ln(1-u) = -u-u^2/2-u^3/3-\cdots$, so we have

$$\ln((\sin x)/x)) = \ln(1-x^2/6+x^4/120+\cdots)$$
$$= \ln(1-\underbrace{(x^2/6-x^4/120+\cdots)}_{u})$$
$$= -(x^2/6-x^4/120+\cdots)-(x^2/6-x^4/120+\cdots)^2/2-\cdots$$
$$= -x^2/6 + \text{ higher degree terms}$$

So we showed that

$$\ln((\sin x)/x) = -x^2/6 + \text{ higher degree terms}$$

So now

$$\lim_{x\to 0}\frac{1}{x^2}\ln\left(\frac{\sin x}{x}\right) = \lim_{x\to 0}\frac{1}{x^2}(-x^2/6+\cdots)$$
$$= \lim_{x\to 0} -1/6+\cdots$$
$$= -1/6$$

So the final answer is $e^{-1/6}$.

In the following problems, find the interval of convergence of the power series, being sure to test the endpoints if the interval is finite.

11. $1 + \frac{x+2}{3 \cdot 1} + \frac{(x+2)^2}{3^2 \cdot 2} + \cdots + \frac{(x+2)^n}{3^n \cdot n} + \cdots$
Answer:
$-5 \leq x < 1$

12. $1 + \frac{(x-1)^2}{2!} + \frac{(x-1)^4}{4!} + \cdots + \frac{(x-1)^{2n-2}}{(2n-2)!} + \cdots$
Answer:
Converges for all x. $-\infty < x < \infty$

13. $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$
Answer:
Converges for all x. $-\infty < x < \infty$

14. $\sum_{n=1}^{\infty} \frac{n! x^n}{n^n}$
Answer:
Calculating $\left|\frac{a_{n+1}}{a_n}\right|$, you should get $\frac{n^n}{(n+1)^n}|x|$. This is equal to $\frac{|x|}{(1+1/n)^n}$. You may remember if you covered continuously compounded interest in high school, that $\lim_{n\to\infty}(1+1/n)^n = e$. So we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{e} < 1$$

So $|x| < e$. In this case checking the endpoints is difficult, so we leave open the question of whether the series converges at the endpoints.

15. $\sum_{n=0}^{\infty} \frac{n+1}{2n+1} \frac{(x-3)^n}{2^n}$
Answer:

$$1 < x < 5$$

Neither endpoint converges!

16. $\sum_{n=0}^{\infty} \frac{n+1}{2n+1} \frac{(x-2)^n}{3^n}$
Answer:

$$-1 < x < 5$$

Neither endpoint converges!

17. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n^2}$
Answer:

$$0 \leq x \leq 2$$

18. $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$
Answer:

$$-1 \leq x < 1$$

19. $\sum_{n=1}^{\infty} \frac{(x-2)^{3n}}{n!}$
Answer:
Converges everywhere: $-\infty < x < \infty$.

In the following problems, find all x where these series converge.

20. $\sum_{n=1}^{\infty} \frac{2^n (\sin x)^n}{n^2}$
Answer:
All x such that $|\sin x| \leq 1/2$. This is as much as I want, but if you're interested, this occurs exactly when $n\pi - \pi/6 \leq x \leq n\pi + \pi/6$ for some integer n. Or another way of saying it is that $|x - n\pi| \leq \pi/6$ for some integer n.

21. $\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{x-1}{x} \right)^n$
Answer:
The ratio test gives you the inequality $\left| \frac{x-1}{x} \right| < 1$. Simplify this to get $\left| 1 - \frac{1}{x} \right| < 1$. This is the same as $-1 < 1 - \frac{1}{x} < 1$ which simplifies to $-2 < -\frac{1}{x} < 0$, and multiplying by -1, this is $2 > \frac{1}{x} > 0$. Taking reciprocals we get $1/2 < x < \infty$. Now we also have to check the endpoint when $x = 1/2$. Plugging in to the original series we get

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1/2 - 1}{1/2} \right)^n = \sum_{n=1}^{\infty} \frac{1}{n} (-1)^n,$$

which converges by the alternating series test. So our final answer is $x \geq 1/2$.

22. By multiplying the appropriate terms in the Maclaurin series for $\ln(1+x)$ and $\tan^{-1} x$ find the terms through x^5 in the Maclaurin series for their product $\ln(1+x)\tan^{-1} x$.
Answer:

$$x^2 - x^3/2 - x^5/12 + \cdots$$