Written Solutions: (HW1)

#3 (a)

$$E_n := \int_0^1 x^n e^{x-1}\, dx$$

want to show that $E_n = 1 - n E_{n-1}$ for $n \geq 1$, n an integer

and $E_0 = \int_0^1 e^{x-1}\, dx = 1 - 1/e$

Proof: Using integration by parts, we show that

$$\int_0^1 x^n e^{x-1}\, dx = x^n e^{x-1}\Big|_0^1 - n\int_0^1 x^{n-1} e^{x-1}\, dx$$

let $u = x^n \qquad du = n x^{n-1} dx$

$dv = e^{x-1} \Rightarrow \quad v = e^{x-1}$

$$= (1 - 0) - n\int_0^1 x^{n-1} e^{x-1}\, dx$$

$$= 1 - n E_{(n-1)}$$

//

(c) Let E_n's be the resulting values from iterating

$$E_n = 1 - nE_{n-1}, \text{ with } E_0 \text{ given.}$$

and let $\tilde{E}_n$'s be the resulting values from iteration of

$$\tilde{E}_n = 1 - n\tilde{E}_{n-1}, \quad \text{where } \tilde{E}_0 = E_0 + \varepsilon_0$$ is a small perturbation of $\tilde{E}_0$.

$\Rightarrow \tilde{E}_1 = 1 - \tilde{E}_0 = 1 - E_0 - \varepsilon_0 = E_1 - \varepsilon_0$

$\tilde{E}_2 = 1 - 2\tilde{E}_1 = 1 - 2(E_1 - \varepsilon_0) = 1 - 2E_1 + 2\varepsilon_0 = E_2 + 2\varepsilon_0$

$\tilde{E}_3 = 1 - 3\tilde{E}_2 = 1 - 3(E_2 + 2\varepsilon_0) = E_3 - 3\cdot 2\,\varepsilon_0 \quad \ldots \quad \tilde{E}_n = E_n + (-1)^n n!\,\varepsilon_0$

So in other words, if $\tilde{E}_n = E_n + \varepsilon_n$, this implies $\varepsilon_n = (-1)^n n!\,\varepsilon_0$

What this says is that if we begin with some $\tilde{E}_0$ which is even slightly different from E_0, let's say

$$\varepsilon_0 = (\tilde{E}_0 - E_0) = \text{error in initial value of } E$$

then after n iterations,

$$\tilde{E}_n = E_n + (-1)^n n! \, \varepsilon_0$$

$$\Rightarrow (\tilde{E}_n - E_n) = (-1)^n n! \, \varepsilon_0 = \text{error in the } n^{th} \text{ approximation due to the error in the initial value.}$$

$n!$ grows quite quickly - in fact $50! = \mathcal{O}(10^{64})$.

So in our computation for (b), since e is an irrational number and so cannot be ~~expli~~ exactly represented by the computer, our initial value $E_0 = 1 - 1/e$ has (inherently) some small error in it due to roundoff error and possibly more. The above analysis says that even if this error is $\mathcal{O}(10^{-16})$ the resulting error by the n^{th} iteration is

$$\mathcal{O}(\varepsilon_n) = n! \, \mathcal{O}(\varepsilon)$$

so for $n = 50 \Rightarrow \mathcal{O}(10^{64}) \times \mathcal{O}(10^{-16}) = \mathcal{O}(10^{48})$

huge!

#3 part (b) - comments:

notice that

$$E_n := \int_0^1 x^n e^{x+1}\, dx \;<\; \int_0^1 e^{x+1}\, dx = 1 - 1/e$$

for all values of n, because $x \in [0,1] \Rightarrow x^n \leq 1$.

This says that for every n, the values of E_n should never be larger than $1 - 1/e$! Thus our numerical approximation must be far from reasonable for large values of n.

#4 (a) Analyze $\lim_{x\to\infty} \left|\frac{\sin(x)}{1}\right|$.

<u>First</u>, notice that this limit does not converge because $\sin(x)$ oscillates between -1 and 1 for all x. <u>However</u>, we do know that $\left|\frac{\sin(x)}{1}\right| \leq 1 \ \forall x$ regardless of how large x is $\Rightarrow$ "$\lim_{x\to\infty}$" $\left|\frac{\sin(x)}{1}\right| \leq C$

and so $\sin(x) = \Theta(1)$.

It is clearly <u>not</u> the case that

$$\lim_{x\to\infty} \frac{\sin(x)}{1} = 1, \text{ so } \sin(x) \not\sim 1 \text{ as } x\to\infty.$$

($\sin(x)$ is not asymptotic to 1 as $x\to\infty$)

(b) $\lim_{x\to\infty} \frac{\ln(x)}{x^m} = ?$ for $m>0$, m an integer.

$\lim_{x\to\infty} \ln(x) = +\infty$ and $\lim_{x\to\infty} x^m = \infty$

So we can apply L'Hopital's rule:

$$\lim_{x\to\infty} \frac{\ln(x)}{x^m} \overset{L'}{=} \lim_{x\to\infty} \frac{1/x}{m x^{m-1}} = \lim_{x\to\infty} \frac{1}{m x^m} = 0$$ for all positive integers m!

$$\Rightarrow \ln(x) = \theta(x^m).$$

(c) First I show

$$\sinh(x) \sim e^{x}/2 \text{ as } x \to \infty : \quad \lim_{x\to\infty} \frac{\sinh(x)}{e^{x}/2} = \lim_{x\to\infty} \frac{e^{x} - e^{-x}}{e^{x}} = \lim_{x\to\infty} 1 - e^{-2x}$$

$$= 1.$$

$$\Rightarrow \sinh(x) \sim e^{x}/2.$$

Also:

$$\lim_{x\to\infty} \frac{\cosh(x)}{e^{x}/2} = \lim_{x\to\infty} \frac{e^{x} + e^{-x}}{e^{x}} = \lim_{x\to\infty} 1 + e^{-2x} = 1$$

$$\Rightarrow \cosh(x) \sim \frac{e^{x}}{2}$$

★ In general, if $f \sim g$ and $h \sim g$ as $x \to \infty$ then $f \sim h$ as $x \to \infty$ as well.

proof:

$$f \sim g \text{ as } x\to\infty \Rightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$$

$$h \sim g \text{ as } x\to\infty \Rightarrow \lim_{x\to\infty} \frac{h(x)}{g(x)} = 1$$

$$\Rightarrow \lim_{x\to\infty} \frac{f(x)}{h(x)} = \lim_{x\to\infty} \frac{f(x)/g(x)}{h(x)/g(x)} = \frac{1}{1} = 1$$

$$\Rightarrow f \sim h. \quad //$$

So as a result $\sinh(x) \sim \cosh(x)$ automatically!

(d) Analyze asymptotics of $e^{1/x}$ for $x \to 0$.

First: Notice that

$$\lim_{x \to 0^+} e^{1/x} = \lim_{\hat{x} \to \infty} e^{\hat{x}} = \infty \quad \left(\text{consider a change of variables } \hat{x} = 1/x\right)$$

and $$\lim_{x \to 0^-} e^{1/x} = \lim_{\hat{x} \to -\infty} e^{\hat{x}} = 0$$

$\Rightarrow$ So $\lim_{x \to 0} e^{1/x}$ does not exist!

Look at one-sided limits: since $\frac{e^{\hat{x}}}{2} \sim \sinh(\hat{x}) \sim \cosh(\hat{x})$ as $\hat{x} \to \infty$ by part (c)

$$\Rightarrow \quad e^{1/x}/2 \sim \sinh(1/x) \sim \cosh(1/x) \quad \text{as } x \to 0^+$$

looking at $$\lim_{x \to 0^-} e^{1/x} = \lim_{\hat{x} \to -\infty} e^{\hat{x}} = 0$$

and since:
$$\lim_{x \to 0^-} \frac{\sinh(1/x)}{e^{1/x}/2} = \lim_{x \to 0^-} \frac{e^{1/x} - e^{-1/x}}{e^{1/x}}$$

$$= \lim_{x \to 0^-} 1 - e^{-2/x} = +\infty$$

$$\sinh(1/x) \not\sim e^{1/x}/2 \quad \text{as } x \to 0^-$$

Moral: Even though it may look as though you can just do a "change of variables"

from $$\lim_{x \to \infty} \frac{e^{x} - e^{-x}}{e^{x}} = \lim_{x \to 0} \frac{e^{1/x} - e^{-1/x}}{e^{1/x}}$$

doesn't always work! Be very careful!

#6: First $p(x)$ is a degree 2 polynomial, so the Taylor series for $p(x)$ only has 3 terms ($p^{(n)}(x) = 0$ for $n \geq 3$), and

$$p(x+h) = p(x) + hp'(x) + h^2 \frac{p''(x)}{2}$$

$$p(x-h) = p(x) - hp'(x) + h^2 \frac{p''(x)}{2}$$

↑ exactly equal!

Subtracting ⇒ $\dfrac{p(x+h) - p(x-h)}{2h} = p'(x)$ (again this is an exact equality)

now since p is defined so that $p(x) = f(x)$, $p(x+h) = f(x+h)$ and $p(x-h) = f(x-h)$

$$\Rightarrow p'(x) = \frac{f(x+h) - f(x-h)}{2h} \cong f'(x)$$

Similarly, adding ⇒ $p(x+h) + p(x-h) - 2p(x) = p''(x)h^2$

$$\Rightarrow p''(x) = \frac{p(x+h) - 2p(x) + p(x-h)}{h^2}$$

$$= \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \cong f''(x)$$

Thus, there is no difference between using the central diff. approx to f' and f'' or approximating f first by p and then computing $p'(x) \cong f'$ and $p'' \cong f''$ directly!

Moreover, the central difference approximation is exact if $f(x)$ is a polynomial of degree ≤ 2.