MATH 606 - Homework #5 and #6
(due dates mixed, see below)

Problems on this page due: Monday May 23 in class

(1) Show that the Lax-Wendroff method is essentially the only explicit finite-difference method for $u_t = \alpha u_x$ that has the following two properties:

- (i) $u_{i,j+1}$ depends only on $u_{i-1,j}$, $u_{i,j}$ and $u_{i+1,j}$
- (ii) the global accuracy (the order of the discretization error) is $O(h^2 + k^2)$

You can show this by completing the following steps:

- (a) Let
$$u_{i,j+1} = A_1 u_{i+1,j} + A_0 u_{i,j} + A_{-1} u_{i-1,j}$$
and let U be the exact solution of the PDE and u be the solution of the FDE so that
$$U_{i,1} - u_{i,1} = kO(h^2 + k^2)$$
(So here I am asking that you assume $U_{i,0} = u_{i,0}$ for all i.)
- (b) Expand $U_{i,j+1}$ in a Taylor series about (i, j). Substitute this and the expression for u in part (a) into the expression for the discretization error in part (b).
- (c) Solve for A_1, A_0, and A_{-1}.

(2) Calculate the dissipation and dispersion relations for:

- (i) leapfrog
- (ii) Lax-Wendroff
- (iii) upwinding

Problems on this page are due by class time, Friday May 27

(3) Solve the hyperbolic PDE $u_t + \alpha u_x = 0$ for $x \in [-3, 3]$ for $\alpha = 1, -1$ with periodic boundary conditions using two different initial conditions:

(i) $u(x, 0) = 1 - |x|$ for $x \in [-1, 1]$, and $u(x, 0) = 0$ otherwise
(ii) $u(x, 0) = 2|x|^3 - 3x^2$ for $x \in [-1, 1]$, and $u(x, 0)$ otherwise

Do this using the two distinct methods

(a) Upwinding (FTBS or FTFS depending on if α is positive or negative). (note: write your code so that it easily switches automatically between the two depending on the sign of α)
(b) Lax-Wendroff

In all cases, computationally show that the method has the correct order of accuracy for fixed $\rho = 1/2$, varying h (you know the exact solution!). Plot the solutions at t=0,1,2,3,4,5.

Please write your code for a general α and show plots for $\alpha = 1$ and $\alpha = -1$, but you need only study the order of accuracy for $\alpha = 1$ for (a) and (c).

(4) Solve the hyperbolic PDE $u_t + u_x = 0$ for $x \in [0, 5]$ and $t \geq 5$ with the following initial/boundary values:

$$u(x, 0) = 1 - |x - 1| \text{ for } x \in [0, 2] \qquad \text{and} \qquad u(x, 0) = 0 \text{ for } x > 2$$

$$u(0, t) = 0 \qquad \text{for } t \geq 0$$

using Wendroff's implicit method. Study the error for $\rho = .5, 1$, and compare the results for this method with the results obtained via my code for Lax-Wendroff solving this same problem. Which combination(s) of ρ and difference scheme gives the best results? Please give a thorough explanation as to why. State which method, if any, seems to be advantageous and why.

Please zip all m-files into one file called "hw6" and email to me at heather@math.ohio-state.edu before class time Friday, May 27. Handwritten components and printed results for these problems should be turned in by the beginning of class the same day.