(5) (a)

An implicit scheme of order $O(k^2) + O(h^2)$:

Crank-Nicolson type scheme - approximate PDE at $(x_i, t_{j+1/2})$:

$$(u_t)_{i,j+1/2} + \alpha (u_x)_{i,j+1/2} = \nu (u_{xx})_{i,j+1/2} + g(x_i, t_{j+1/2})$$

introduce g in order to check for errors in code.

w/o the g:

$$\frac{u_{i,j+1} - u_{i,j}}{k} + \frac{\alpha}{2}\left(\frac{u_{i+1,j+1} - u_{i-1,j+1}}{2h} + \frac{u_{i+1,j} - u_{i-1,j}}{2h} \right)$$

$$= \frac{\nu}{2}\left(\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2} + \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \right)$$

$\Rightarrow$

$$u_{i,j+1} - u_{i,j} + \frac{k\alpha}{4h}\left(u_{i+1,j+1} - u_{i-1,j+1} + u_{i+1,j} - u_{i-1,j} \right)$$

$$= \frac{\nu r}{2}\left(u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \right)$$

$$\Rightarrow \underbrace{\left(\frac{k\alpha}{4h} - \frac{\nu r}{2} \right)}_{a} u_{i+1,j+1} + \left(1 + \nu r \right) u_{i,j+1} + \underbrace{\left(-\frac{k\alpha}{4h} - \frac{\nu r}{2} \right)}_{b} u_{i-1,j+1}$$

$$= \left(\frac{\nu r}{2} - \frac{k\alpha}{4h} \right) u_{i+1,j} + \left(1 - \nu r \right) u_{i,j} + \left(\frac{\nu r}{2} + \frac{k\alpha}{4h} \right) u_{i-1,j}$$

$$a\, u_{i+1,j+1} + (1 + \nu r) u_{i,j+1} + b\, u_{i-1,j+1}$$

$$= -a\, u_{i+1,j} + (1 - \nu r) u_{i,j} - b\, u_{i-1,j}$$

so with g:

$$a u_{i+1,j+1} + (1+\nu r) u_{i,j+1} + b u_{i-1,j+1} = -a u_{i+1,j} + (1-\nu r) u_{i,j} - b u_{i-1,j} + k g(x_i, t_{j+1/2})$$

<u>boundary conditions:</u>

$$u_x(-1,t) = 0 = u_x(1,t) \quad \text{for all } t$$

using central differences, we obtain:

@ $x = -1$:

$$u_x(-1,t) = (u_x)_{0,j} = \frac{u_{1,j} - u_{-1,j}}{2h} = 0$$

$$\Rightarrow \quad u_{1,j} = u_{-1,j}$$

@ $x = 1$:

$$u_x(1,t) = (u_x)_{N,j} = \frac{u_{N+1,j} - u_{N-1,j}}{2h} = 0$$

$$\Rightarrow \quad u_{N+1,j} = u_{N-1,j}$$

and so @ $i = 0$:

$$a u_{1,j+1} + (1+\nu r) u_{0,j+1} + b u_{1,j+1} = -a u_{1,j} + (1-\nu r) u_{0,j} - b u_{1,j}$$

$$(a+b) u_{1,j+1} + (1+\nu r) u_{0,j+1} = -(a+b) u_{1,j} + (1-\nu r) u_{0,j}$$

and at $i = N$:

$$a u_{N-1,j+1} + (1+\nu r) u_{N,j+1} + b u_{N-1,j+1} = -a u_{N-1,j} + (1-\nu r) u_{N,j} - b u_{N-1,j}$$

$$(a+b) u_{N-1,j+1} + (1+\nu r) u_{N,j+1} = -(a+b) u_{N-1,j} + (1-\nu r) u_{N,j}$$

and the matrix representation is:

$$a = \frac{k\alpha}{4h} - \frac{\nu r}{2} \qquad b = -\frac{k\alpha}{4h} - \frac{\nu r}{2}$$

$$\begin{bmatrix} 1+\nu r & a+b & 0 & \cdots & \cdots & 0 \\ b & 1+\nu r & a & 0 & \cdots & 0 \\ 0 & b & 1+\nu r & a & 0 \cdots & 0 \\ & & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & b & 1+\nu r & a \\ 0 & \cdots & \cdots & 0 & (a+b) & 1+\nu r \end{bmatrix} \vec{u}_{j+1}$$

$$= \begin{bmatrix} 1-\nu r & -(a+b) & 0 & 0 & \cdots & 0 \\ -b & 1-\nu r & -a & 0 & \cdots & 0 \\ & -b & 1-\nu r & -a & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & -b & 1-\nu r & -a \\ 0 & \cdots & \cdots & 0 & -(a+b) & 1-\nu r \end{bmatrix} \vec{u}_j + \begin{bmatrix} kg(x_0, t_{j+1/2}) \\ kg(x_1, t_{j+1/2}) \\ \vdots \\ \vdots \\ kg(x_N, t_{j+1/2}) \end{bmatrix}$$

In order to check the code, we can let

$$u(x,t) = \cos(\pi x)e^{-\nu\pi^2 t} \quad \Rightarrow \quad \text{initial condition is } u(x,0) = \cos(\pi x)$$

so:

$$u_t = -\nu\pi^2\cos(\pi x)e^{-\nu\pi^2 t} \qquad \Rightarrow \quad g(x,t) = u_t + \alpha u_x - \nu u_{xx}$$

$$u_x = -\pi\sin(\pi x)e^{-\nu\pi^2 t} \qquad = -\alpha\pi\sin(\pi x)e^{-\nu\pi^2 t}$$

$$\nu u_{xx} = -\pi^2\nu\cos(\pi x)e^{-\nu\pi^2 t}$$

notice that $u_x(-1,t) = 0$ and $u_x(1,t) = 0$ as needed.

For b and c, we can then eliminate g and set our initial conditions as either x^3-3x or $-(x^3-3x)$.