(4) (a) Du Fort-Frankel for $u_t = u_{xx}$:

$$(1+2r)\,u_{m,n+1} = 2r\,(u_{m+1,n} + u_{m-1,n}) + (1-2r)\,u_{m,n-1}$$

letting $u_{m,n} = \xi^n e^{im\beta h}$

$$\Rightarrow \quad (1+2r)\,u_{m,n+1} = 2r\,(e^{i\beta h} + e^{-i\beta h})\,u_{m,n} + (1-2r)\,u_{m,n-1}$$

$$\Rightarrow \quad u_{m,n+1} = \frac{4r\cos(\frac{\beta h}{2})}{1+2r}\,u_{m,n} + \frac{1-2r}{1+2r}\,u_{m,n-1}$$

$$\begin{bmatrix} u_{m,n+1} \\ u_{m,n} \end{bmatrix} = \begin{bmatrix} \frac{4r\cos(\frac{\beta h}{2})}{1+2r} & \frac{1-2r}{1+2r} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_{m,n} \\ u_{m,n-1} \end{bmatrix}$$

$\downarrow$

eigenvalues?

$$\lambda\left(-\frac{4r\cos(\frac{\beta h}{2})}{1+2r} + \lambda\right) - \left(\frac{1-2r}{1+2r}\right) = 0$$

$$\lambda^2 - \frac{4r\cos(\frac{\beta h}{2})}{1+2r}\,\lambda - \frac{1-2r}{1+2r} = 0$$

$$\Rightarrow (1+2r)\lambda^2 - 4r\cos(\tfrac{\beta h}{2})\,\lambda - (1-2r) = 0$$

$$\Rightarrow \quad \lambda = \frac{4r\cos(\frac{\beta h}{2}) \pm \sqrt{16r^2\cos^2(\frac{\beta h}{2}) + 4(1-2r)(1+2r)}}{2(1+2r)}$$

$$= \left(2r\cos(\tfrac{\beta h}{2}) \pm \sqrt{4r^2\cos^2(\tfrac{\beta h}{2}) - 4r^2 + 1}\right) \Big/ 1+2r$$

$$4r^2\cos^2\left(\frac{\beta h}{2}\right) - 4r^2 = -4r^2\left(1-\cos^2\left(\frac{\beta h}{2}\right)\right)$$

$$= -4r^2\sin^2\left(\frac{\beta h}{2}\right)$$

$$\Rightarrow \quad \lambda = \frac{2r\cos\left(\frac{\beta h}{2}\right) \pm \sqrt{1-4r^2\sin^2\left(\frac{\beta h}{2}\right)}}{1+2r}$$

If $1-4r^2\sin^2\left(\frac{\beta h}{2}\right) \geq 0 \Rightarrow \lambda$'s are real and

$$|\lambda| \leq \frac{2r + \left|\sqrt{1-4r^2\sin^2\left(\frac{\beta h}{2}\right)}\right|}{1+2r} < 1$$

Otherwise $1-4r^2\sin^2\left(\frac{\beta h}{2}\right) < 0$ and the λ's are complex:

$$\lambda = \frac{2r\cos\left(\frac{\beta h}{2}\right) \pm i\sqrt{4r^2\sin^2\left(\frac{\beta h}{2}\right)-1}}{1+2r}$$

$$|\lambda|^2 = \frac{4r^2\cos^2\left(\frac{\beta h}{2}\right) + 4r^2\sin^2\left(\frac{\beta h}{2}\right) - 1}{(1+2r)^2}$$

$$= \frac{4r^2-1}{1+4r+4r^2} < 1$$

and so regardless of the value of r, the eigenvalues are bounded above by 1 $\Rightarrow$ unconditionally stable.

(b) $u_t = i\, u_{xx}$

Leapfrog in time, central in space:

$$\frac{u_{m,n+1} - u_{m,n-1}}{2k} = i\left(\frac{u_{m+1,n} - 2u_{m,n} + u_{m-1,n}}{h^2}\right)$$

$$u_{m,n+1} = 2ri\left(u_{m+1,n} - 2u_{m,n} + u_{m-1,n}\right) + u_{m,n-1}$$

$$u_{m,n} = \xi^n e^{imh\beta}$$

$$\Rightarrow \quad u_{m,n+1} = 2ri\left(e^{ih\beta} - 2 + e^{-ih\beta}\right) u_{m,n} + u_{m,n-1}$$

$$= 2ri\left(2\cos(\beta h) - 2\right) u_{m,n} + u_{m,n-1}$$

$$= -8ri\sin^2\left(\tfrac{\beta h}{2}\right) u_{m,n} + u_{m,n-1}$$

$$\Rightarrow \begin{bmatrix} u_{m,n+1} \\ u_{m,n} \end{bmatrix} = \begin{bmatrix} -8ri\sin^2\left(\frac{\beta h}{2}\right) & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_{m,n} \\ u_{m,n-1} \end{bmatrix}$$

eigenvalues: $\lambda\left(8ri\sin^2\left(\tfrac{\beta h}{2}\right) + \lambda\right) - 1 = 0$

$$\lambda^2 + 8ri\sin^2\left(\tfrac{\beta h}{2}\right) - 1 = 0$$

$$\lambda = \frac{8ri\sin^2\left(\frac{\beta h}{2}\right) \pm \sqrt{-64r^2\sin^4\left(\frac{\beta h}{2}\right) + 4}}{2}$$

$$= 4ri\sin^2\left(\tfrac{\beta h}{2}\right) \pm \sqrt{1 - 16r^2\sin^4\left(\tfrac{\beta h}{2}\right)}$$

if $\quad 1-16r^2\sin^4\left(\frac{\beta h}{2}\right) \geq 0$

$\lambda = a + bi \quad$ with $\quad b = 4r\sin^2\left(\frac{\beta h}{2}\right)$

$$a = \pm\sqrt{1-16r^2\sin^4\left(\frac{\beta h}{2}\right)}$$

and so

$$|\lambda|^2 = a^2 + b^2 = 16r^2\sin^4\left(\frac{\beta h}{2}\right) + 1 - 16r^2\sin^4\left(\frac{\beta h}{2}\right)$$
$$= 1$$

if $\quad 1-16r^2\sin^4\left(\frac{\beta h}{2}\right) < 0$

$\Rightarrow \quad \lambda = bi \quad$ with $\quad b = 4r\sin^2\left(\frac{\beta h}{2}\right) \pm \sqrt{16r^2\sin^4\left(\frac{\beta h}{2}\right)-1}$

and $\quad |\lambda| = b$.

for $|\lambda| \leq 1 \Leftrightarrow \quad b \leq 1 \Leftrightarrow \pm\sqrt{16r^2\sin^4\left(\frac{\beta h}{2}\right)-1} \leq 1 - 4r\sin^2\left(\frac{\beta h}{2}\right)$

$$\Leftrightarrow \quad \cancel{16r^2\sin^4\left(\frac{\beta h}{2}\right)}-1 \leq 1 + \cancel{16r^2\sin^4\left(\frac{\beta h}{2}\right)} - 8r\sin^2\left(\frac{\beta h}{2}\right)$$

$$\Leftrightarrow \quad 8r\sin^2\left(\frac{\beta h}{2}\right) \leq 2$$

$$\Leftrightarrow \quad r \leq \frac{1}{4\sin^2\left(\frac{\beta h}{2}\right)} \quad (1)$$

<u>However</u>, notice that we are already requiring that $1-16r^2\sin^4\left(\frac{\beta h}{2}\right) < 0$ (2)
for this case, so combining (1) and (2) we get:

$$1 \overset{\text{equals}}{=} 1 - 16\left(\frac{1}{4\sin^2\left(\frac{\beta h}{2}\right)}\right)^2\sin^4\left(\frac{\beta h}{2}\right) \overset{\text{if } r \leq \frac{1}{4\sin^2(\frac{\beta h}{2})}}{\leq} 1 - 16r^2\sin^4\left(\frac{\beta h}{2}\right) < 0$$

$$\Rightarrow \quad 1 \leq 0 \; ?! \quad \text{impossible}.$$

So, this second case cannot be stable and for stability we must have

$$1 - 16 r^2 \sin^4\left(\frac{\beta h}{2}\right) \geq 0$$

$$\Rightarrow \quad r^2 \leq \frac{1}{16 \sin^4\left(\frac{\beta h}{2}\right)} \quad \Rightarrow \quad r \leq \frac{1}{4 \sin^2\left(\frac{\beta h}{2}\right)}$$

so $r \leq \frac{1}{4}$ will ensure stability.

(5) (a)

An implicit scheme of order $O(k^2) + O(h^2)$:

Crank-Nicolson type scheme - approximate PDE at $(x_i, t_{j+1/2})$:

$$(u_t)_{i,j+1/2} + \alpha (u_x)_{i,j+1/2} = \nu (u_{xx})_{i,j+1/2} + \underbrace{g(x_i, t_{j+1/2})}$$

introduce g in order to check for errors in code.

w/o the g:

$$\frac{u_{i,j+1} - u_{i,j}}{k} + \frac{\alpha}{2}\left(\frac{u_{i+1,j+1} - u_{i-1,j+1}}{2h} + \frac{u_{i+1,j} - u_{i-1,j}}{2h} \right)$$

$$= \frac{\nu}{2}\left(\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2} + \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \right)$$

$\Rightarrow$

$$u_{i,j+1} - u_{i,j} + \frac{k\alpha}{4h}\left(u_{i+1,j+1} - u_{i-1,j+1} + u_{i+1,j} - u_{i-1,j} \right)$$

$$= \frac{\nu r}{2}\left(u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \right)$$

$$\Rightarrow \underbrace{\left(\frac{k\alpha}{4h} - \frac{\nu r}{2}\right)}_{a} u_{i+1,j+1} + \left(1 + \nu r\right) u_{i,j+1} + \underbrace{\left(-\frac{k\alpha}{4h} - \frac{\nu r}{2}\right)}_{b} u_{i-1,j+1}$$

$$= \left(\frac{\nu r}{2} - \frac{k\alpha}{4h}\right) u_{i+1,j} + \left(1 - \nu r\right) u_{i,j} + \left(\frac{\nu r}{2} + \frac{k\alpha}{4h}\right) u_{i-1,j}$$

$$a\, u_{i+1,j+1} + (1+\nu r)\, u_{i,j+1} + b\, u_{i-1,j+1}$$

$$= -a\, u_{i+1,j} + (1-\nu r)\, u_{i,j} - b\, u_{i-1,j}$$