1 (a)

$$u_{i,j+1} - ra\left(u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}\right) = u_{i,j}$$

$$-ra\,u_{i-1,j+1} + (1+2ra)u_{i,j+1} - ra\,u_{i+1,j+1} = u_{i,j}$$

→ if we assume $u(0,t) = g_L$, $u(\pi, t) = g_R$

then for $i = 1$:

$$-ra\,g_L + (1+2ra)u_{1,j+1} - ra\,u_{2,j+1} = u_{1,j}$$

and for $i = N-1$

$$-ra\,u_{N-2,j+1} + (1+2ra)u_{N-1,j+1} - ra\,g_R = u_{N-1,j}$$

so our matrix problem looks like:

$$\underbrace{\begin{bmatrix} 1+2ra & -ra & & & \\ -ra & 1+2ra & -ra & & \\ 0 & -ra & 1+2ra & -ra & \\ & & \ddots & & \\ & & -ra & 1+2ra & -ra \\ & & & -ra & 1+2ra \end{bmatrix}}_{A} \vec{u}_{j+1} = \vec{u}_j + \underbrace{\begin{bmatrix} ra\,g_L \\ 0 \\ \vdots \\ \vdots \\ 0 \\ ra\,g_R \end{bmatrix}}_{\vec{b}}$$

$$\Rightarrow \quad \vec{u}_{j+1} = A^{-1}\vec{u}_j + A^{-1}\vec{b}$$

and the stability is determined by whether or not $\|A^{-1}\| \le 1$.

The eigenvalues of A are:
(pg 59 in Smith)

$$\lambda_s = 1 + 2ra + 2ra\cos\left(\frac{s\pi}{N}\right) \quad \text{for } s = 1, 2, \ldots, N-1$$

so the eigenvalues of A^{-1} are

$$\frac{1}{\lambda_s} = \frac{1}{1+2ra+2ra\cos\left(\frac{s\pi}{N}\right)}$$

Notice that the smallest λ_s can be is

$$\lambda_s = 1 + 2ra\left(1+\cos\left(\frac{s\pi}{N}\right)\right)$$
$$= 1 + 4ra\cos^2\left(\frac{s\pi}{2N}\right) > 1$$

since $0 < \frac{s\pi}{2N} < \frac{\pi}{2}$ for all s

$\Rightarrow \frac{1}{\lambda_s} < 1$ for all s

$\Rightarrow$ since A is symmetric, then A^{-1} is symmetric

and $\|A^{-1}\|_2 = \max\left\{\frac{1}{\lambda_s}\right\} < 1 \Rightarrow$ the FDS is

unconditionally stable.

(b)

The discretization error is

$$e_i^j := U_{i,j} - \bar{u}_{i,j}$$

where $\bar{u}_{i,j}$ = approx solution = solution to FDE

$U_{i,j}$ = solution to PDE

$$\Rightarrow \bar{u}_{i,j} = U_{i,j} - e_i^j$$

subbing into the FDS, we get:

$$u_{i,j} - e_i^j = u_{i,j+1} - e_i^{j+1} - ra\left(u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}\right)$$
$$+ ra\left(e_{i-1}^{j+1} - 2e_i^{j+1} + e_{i+1}^{j+1}\right)$$

$\Rightarrow$

$$(\star) \quad e_i^j = u_{i,j} - u_{i,j+1} + ra\left(u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}\right)$$
$$+ e_i^{j+1} - ra\left(e_{i-1}^{j+1} - 2e_i^{j+1} + e_{i+1}^{j+1}\right)$$

The Local truncation error is defined to be the value of the FDS evaluated at the solution to the PDE:

$$T_{i,j} = \frac{u_{i,j+1} - u_{i,j}}{k} - a\left(\frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2}\right)$$

and using Taylor series we can show that

$$T_{i,j} = O(k) + O(h^2)$$

Subbing into $\star$, we see:

$$e_i^j = -kT_{i,j} + e_i^{j+1} - ra\left(e_{i-1}^{j+1} - 2e_i^{j+1} + e_{i+1}^{j+1}\right)$$

$$\Rightarrow \quad |e_i^j| \geq |e_i^{j+1}|(1+2ra) - ra|e_{i-1}^{j+1}| - ra|e_{i+1}^{j+1}| - k|T_{i,j}|$$

for a fixed j, take the max over all values in space on that time level and let $E^j = \max_i |e_i^j|$

$$\Rightarrow \quad k \max_i |T_{i,j}| + E^j \geq (1+2ra)E^{j+1}$$

$$\Rightarrow \quad k(O(k)+O(h^2)) + E^j \geq (1+2ra)E^{j+1} \geq E^{j+1}$$

$$\Rightarrow \quad E^j \leq \cancel{E^0} + jk\left(O(k)+O(h^2)\right)$$
$$= jk\left(O(k)+O(h^2)\right) \leq Jk\left(O(k)+O(h^2)\right)$$

where J is such that $Jk = T$ (some fixed time T)

$$\Rightarrow \quad E^j \leq T(\theta(k) + \theta(h^2))$$

and so regardless of our choice of j, we have that E^j (and so each e_i^j) goes to zero like $\theta(k) + \theta(h^2)$ as $h, k \to 0$.