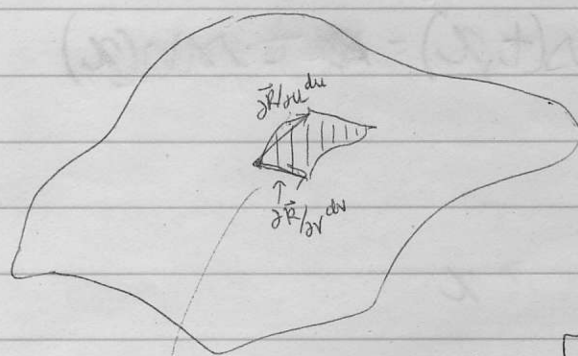
Why $\boxed{\iint_S \vec{F}\cdot\vec{n}\, dS \text{ is the same as } \iint_S \vec{F}\cdot d\vec{S}}$ ...

Remember $d\vec{S} = \underbrace{\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}\, du\,dv}_{=\text{ vector value!}}$ where $\vec{R}(u,v)$ parameterizes S (the surface)

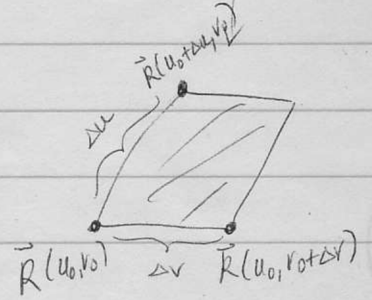
and $dS = \underbrace{|d\vec{S}|}_{\text{length!}} = \left|\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}\right| du\,dv$ = scalar value!



on a small patch of area on the surface S, we want to approximate it's area by finding two vectors (not parallel) that lie along the edges of the patch and take their cross product.

★ [This gives us a vector $\perp$ to S whose length is equal to the area of the patch!

i.e. $d\vec{S}$ = vector normal to S with length = area of infinitesmal patch on S

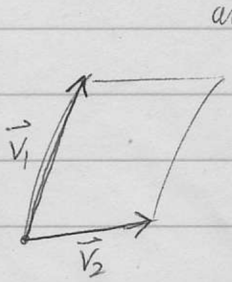


$\Rightarrow$ so a vector along the vertical side is obtained by:

$$\vec{V}_1 = \vec{R}(u_0+\Delta u, v_0) - \vec{R}(u_0, v_0) \cong \frac{\partial \vec{R}}{\partial u}\cdot \Delta u$$

and along the horizontal side by:

$$\vec{V}_2 = \vec{R}(u_0, v_0+\Delta v) - \vec{R}(u_0,v_0) \cong \frac{\partial \vec{R}}{\partial v}\cdot\Delta v$$



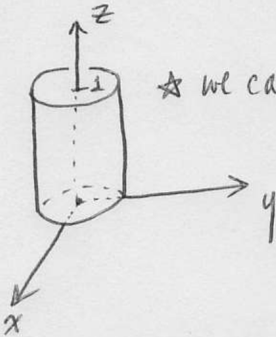
$\Rightarrow$ area of patch $= |\vec{V}_1 \times \vec{V}_2| = \left|\frac{\partial \vec{R}}{\partial u}\Delta u \times \frac{\partial \vec{R}}{\partial v}\Delta v\right| = \left|\frac{\partial \vec{R}}{\partial u}\times\frac{\partial \vec{R}}{\partial v}\right|\Delta u \Delta v$ (because $\Delta u, \Delta v$ are scalars!)

let $\Delta u, \Delta v \to 0$ $\Rightarrow$ infinitesmal area $= dS = \left|\frac{\partial \vec{R}}{\partial u}\times\frac{\partial \vec{R}}{\partial v}\right| du\,dv$

if $\vec{n}$ is a unit normal to S, then $\vec{n}$ and $d\vec{S}$ are parallel!

and since $dS = |d\vec{S}|$ $\Rightarrow$ $\vec{n}\,dS = d\vec{S}$ $\Rightarrow$ this is just two different ways to write the same thing!

So, for example in #5 of section 4.7, the problem asks to compute $\iint \vec{F}\cdot\vec{n}\, dS$ over the closed surface bounded by $z=1$, $z=1$ and $x^2+y^2=a^2$, where $\vec{n}$ is the unit outward normal. $\vec{F}(x,y,z) = \langle x, y, z^2-1\rangle$



★ we can parametrize the lateral surface by: (we look at the top + bottom at the end!)

$$\vec{R}(\theta,z) = \langle a\cos\theta,\ a\sin\theta,\ z\rangle = \langle x,y,z\rangle$$

we know $\iint \vec{F}\cdot\vec{n}\, dS = \iint \vec{F}\cdot d\vec{S}$ are the same as long as we make sure that $d\vec{S}$ points in the same direction as $\vec{n}$!

$$\frac{\partial \vec{R}}{\partial z} = \langle 0, 0, 1\rangle \qquad \Rightarrow \quad \frac{\partial \vec{R}}{\partial z}\times\frac{\partial \vec{R}}{\partial \theta} = \langle -a\cos\theta, -a\sin\theta, 0\rangle$$

$$\frac{\partial \vec{R}}{\partial \theta} = \langle -a\sin\theta, a\cos\theta, 0\rangle \qquad = \langle -x, -y, 0\rangle$$

thus $\frac{\partial \vec{R}}{\partial z}\times\frac{\partial \vec{R}}{\partial \theta}$ gives an inward pointing normal to S!

so since we've already specified an orientation on S by choosing $\vec{n}$ to always be outward, we want

$$d\vec{S} = -\left(\frac{\partial \vec{R}}{\partial z}\times\frac{\partial \vec{R}}{\partial \theta}\right) = \langle a\cos\theta, a\sin\theta, 0\rangle\, d\theta\, dz$$

notice that regardless of orientation

$$dS = |d\vec{S}| = |-d\vec{S}| \quad \text{is always the same.}$$

method 1:

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^1\int_0^{2\pi} \underbrace{\langle a\cos\theta, a\sin\theta, z^2-1\rangle}_{\vec{F}(\vec{R}(\theta,z))}\cdot \overbrace{\langle a\cos\theta, a\sin\theta, 0\rangle\, d\theta\, dz}^{d\vec{S}} + \iint_{\text{top + bottom}} \vec{F}\cdot d\vec{S}$$

$$= \left(\int_0^1\int_0^{2\pi} a^2\cos^2\theta + a^2\sin^2\theta\, d\theta\, dz + \iint_{\text{top+bottom}} \vec{F}\cdot d\vec{S}\right) = \left(\int_0^1\int_0^{2\pi} a^2\, d\theta\, dz + \iint_{\text{top+bottom}} \vec{F}\cdot d\vec{S}\right) = a^2\cdot 2\pi + \iint_{\text{top + bottom}} \vec{F}\cdot d\vec{S}$$

Method 2:

$$\iint_S \vec{F}\cdot\vec{n}\, dS = ?$$

$d\vec{S} = \langle a\cos\theta, a\sin\theta, 0\rangle\, d\theta\, dz$

and $dS = |d\vec{S}| = a\, d\theta\, dz$

we get a unit normal to the lateral surface of S by

$$\vec{n} = \frac{d\vec{S}}{|d\vec{S}|} = \frac{\langle a\cos\theta, a\sin\theta, 0\rangle \cancel{d\theta dz}}{a\, \cancel{d\theta dz}} = \langle\cos\theta, \sin\theta, 0\rangle$$

so:

$$\iint_S \vec{F}\vec{n}\, dS = \int_0^1\int_0^{2\pi} \underbrace{\langle a\cos\theta, a\sin\theta, z^2-1\rangle}_{\vec{F}(\vec{R}(\theta,z))} \cdot \underbrace{\langle\cos\theta, \sin\theta, 0\rangle}_{\vec{n}}\, \underbrace{a\, d\theta\, dz}_{dS} + \iint_{top+bottom} \vec{F}\cdot\vec{n}\, dS$$

$$= \int_0^1\int_0^{2\pi} (a\cos^2\theta + a\sin^2\theta)\, a\, d\theta\, dz + \iint_{top+bottom} \vec{F}\cdot\vec{n}\, dS$$

$$= \int_0^1\int_0^{2\pi} a^2\, d\theta\, dz + \iint_{top+bottom} \vec{F}\cdot\vec{n}\, dS = a^2 2\pi + \iint_{top+bottom} \vec{F}\cdot\vec{n}\, dS$$

---

* As you can see which way is easier for the lateral part of S is a matter of opinion. However, the top and bottom are by far easier to compute when thinking in terms of $\iint_S \vec{F}\cdot\vec{n}\, dS$ rather than $\iint_S \vec{F}\cdot d\vec{S}$, because:

on top: $\vec{n} = \vec{k}$ and $\vec{F}\cdot\vec{k} = z^2 - 1$. But on the top of the can, $z = 1$

so $\vec{F}\cdot\vec{k} = 0$! thus $\iint_{top} \vec{F}\cdot\vec{n}\, dS = \iint_{top} \vec{F}\cdot\vec{k}\, dS = \iint_{top} 0\, dS = 0$

and I didn't even need to find a parametrization!

on bottom: $\vec{n} = -\vec{k} \Rightarrow \vec{F}\cdot\vec{n} = -(z^2-1)$. On the bottom, $z = 0 \Rightarrow \vec{F}\cdot\vec{n} = 1$

so $\iint_{bottom} \vec{F}\cdot\vec{n}\, dS = \iint_{bottom} 1\, dS$ = area of bottom $= \pi(a)^2$

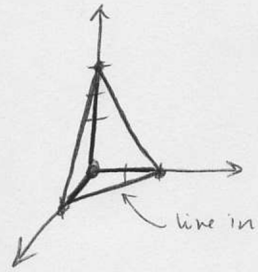
(again, I needed no parametrization!)

#7 – 4.7:

region – $\begin{cases}\text{solid} \\ \text{tetrahedron:}\end{cases}$ $x \geq 0, y \geq 0, z \geq 0$

$6x + 3y + 2z \leq 6$

(a) not a domain
because although it is connected, it is not open
(it includes it's boundary)



line in xy plane is
$6x + 3y = 6$
or
$x + \frac{1}{2}y = 1$
or
$y = 2 - 2x$

(b) It is simply connected. Any closed curve in the region can be contracted to it's center.

(c) $\vec{F} = \langle 2x, y, z \rangle$ find $\iint_S \vec{F} \cdot \vec{n} \, dS$

$\vec{n}$ = outward normal to ∂V

$S = \partial V = \partial$ of region above.

by the divergence thm: $2z \leq 6 - 6x - 3y \Rightarrow z \leq 3 - 3x - \frac{3}{2}y$

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iiint_V \vec{\nabla} \cdot \vec{F} \, dV = \int_0^1 \int_0^{2-2x} \int_0^{3-3x-\frac{3}{2}y} 2 + 1 + 1 \, dV$$

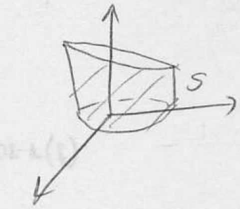
$$= \int_0^1 \int_0^{2-2x} \int_0^{3-3x-\frac{3}{2}y} 4 \, dz\,dy\,dx = \int_0^1 \int_0^{2-2x} 4\left(3 - 3x - \frac{3}{2}y\right) dy\,dx$$

$$= \int_0^1 12\underbrace{(1-x)(2-2x)}_{2 - 4x + 2x^2} - \frac{6}{2}\underbrace{(2-2x)^2}_{4 - 8x + 4x^2} \, dx$$

$$= \int_0^1 24 - 48x + 24x^2 - (12 - 24x + 12x^2) \, dx$$

$$= \int_0^1 12 - 24x + 12x^2 \, dx = 12 - 12 + 4 = 4$$

#9 - 4.7: Evaluate $\iint_S z^2\,dS$ where S = lateral part of surface bdd by $x^2+y^2=4$, between $z=0$ + $z=y+3$



Parametrize S using the parametrization of the cylinder of radius 2:

$$\vec{R}(\theta,z) = \langle 2\cos\theta,\ 2\sin\theta,\ z\rangle$$

with $0\le\theta\le 2\pi$ and $0\le z\le y+3 = 2\sin\theta+3$

$$\Rightarrow d\vec{S} = \left(\frac{\partial\vec{R}}{\partial\theta}\times\frac{\partial\vec{R}}{\partial z}\right)d\theta dz = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k}\\ -2\sin\theta & 2\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix} d\theta dz = \langle 2\cos\theta,\ 2\sin\theta,\ 0\rangle\, d\theta dz$$

$$\Rightarrow dS = |d\vec{S}| = \sqrt{4\cos^2\theta+4\sin^2\theta+0} = \sqrt{4} = \boxed{\underbrace{2\,d\theta dz} = dS}$$

$\Rightarrow \iint_S z^2\,dS =$ ($\star$ Note here $f(x,y,z)=z^2$, so $f(\vec{R}(\theta,z)) = f(2\cos\theta, 2\sin\theta, z) = z^2$ still!)

$$\int_0^{2\pi}\int_0^{2\sin\theta+3=y+3} z^2\cdot 2\,dz\,d\theta = \int_0^{2\pi}\frac{2}{3}z^3\Big|_0^{2\sin\theta+3} d\theta = \int_0^{2\pi}\frac{2}{3}(2\sin\theta+3)^3\,d\theta$$

$$= \frac{2}{3}\int_0^{2\pi}(4\sin^2\theta+12\sin\theta+9)(2\sin\theta+3)\,d\theta = \frac{2}{3}\int_0^{2\pi}(8\sin^3\theta+36\sin^2\theta+54\sin\theta+27)\,d\theta$$

$$= \frac{2}{3}\left[\int_0^{2\pi}8\sin\theta(1-\cos^2\theta)\,d\theta + \int_0^{2\pi}36\left(\frac{1-\cos 2\theta}{2}\right)d\theta + \int_0^{2\pi}54\sin\theta\,d\theta + 27(2\pi)\right]$$

$$= \frac{2}{3}\left[\cancelto{0}{-8\cos\theta\Big|_0^{2\pi}} - \int_0^{2\pi}8\cos^2\theta\sin\theta\,d\theta + \frac{36}{2}(2\pi) - 18\cancelto{0}{\int_0^{2\pi}\cos(2\theta)\,d\theta} - \cancelto{0}{54\cos\theta\Big|_0^{2\pi}} + 27(2\pi)\right]$$

let $u=\cos\theta \Rightarrow du = -\sin\theta\,d\theta$

$$= \frac{2}{3}\left[45(2\pi) + \int_0^{2\pi}8u^2\,du\right] = \frac{2}{3}\left[90\pi + \cancelto{0}{\frac{8}{3}\cos^3\theta\Big|_0^{2\pi}}\right] = 60\pi$$